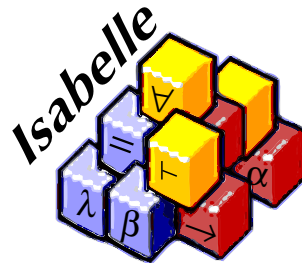


IJCAR 2004 — Tutorial 4



Introduction to the Isabelle Proof Assistant



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Tutorial Schedule

- ▶ Session I
 - ▶ Basics
- ▶ Session II
 - ▶ Specification Tools
 - ▶ Readable Proofs
- ▶ Session III
 - ▶ More on Readable Proofs
 - ▶ Modules
- ▶ Session IV
 - ▶ Applications
 - ▶ Q & A session with Larry Paulson

Session I

Basics

System Architecture

User can access all layers!

Proof General — User interface

HOL, ZF — Object-logics

Isabelle — Generic, interactive theorem prover

Standard ML — Logic implemented as ADT

Documentation

Available from <http://isabelle.in.tum.de>

- ▶ Learning Isabelle
 - ▶ Tutorial on Isabelle/HOL (LNCS 2283)
 - ▶ Tutorial on Isar
 - ▶ Tutorial on Locales
- ▶ Reference Manuals
 - ▶ Isabelle/Isar Reference Manual
 - ▶ Isabelle Reference Manual
 - ▶ Isabelle System Manual
- ▶ Reference Manuals for Object-Logics

Isabelle's Meta-Logic

- ▶ Intuitionistic fragment of Church's theory of simple types.
- ▶ With type variables.
- ▶ Can be used to formalise your own object-logic.
- ▶ Normally, use rich infrastructure of the object-logics HOL and ZF.
- ▶ This presentation assumes HOL.

Types

Syntax

Syntax:

| | |
|----------------------------|----------------------|
| $\tau ::= (\tau)$ | |
| $'a \mid 'b \mid \dots$ | type variables |
| $\tau \Rightarrow \tau$ | total functions |
| $bool \mid nat \mid \dots$ | HOL base types |
| $\tau \times \tau$ | HOL pairs (ascii: *) |
| $\tau \textit{ list}$ | HOL lists |
| \dots | user-defined types |

Parentheses: $T1 \Rightarrow T2 \Rightarrow T3 \equiv T1 \Rightarrow (T2 \Rightarrow T3)$

Introducing new Types: `typedecl`

`typedecl` *name*

Introduces new “opaque” type *name* without definition.

Example:

`typedecl` *addr* —An abstract type of addresses.

Terms

Syntax

Syntax: (curried version)

| | |
|-------------------|-----------------------------------|
| $term ::= (term)$ | |
| a | constant or variable (identifier) |
| $term\ term$ | function application |
| $\lambda x. term$ | function “abstraction” |
| \dots | lots of syntactic sugar |

Examples: $f (g\ x)\ y$ $h (\lambda x. f (g\ x))$

Parentheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$

Schematic variables

Three kinds of variables:

- ▶ bound: $\forall x. x = x$
- ▶ free: $x = x$
- ▶ **schematic**: $?x = ?x$ (“unknown”)
- ▶ Logically: free = schematic
- ▶ Operationally:
 - ▶ free variables are fixed
 - ▶ schematic variables are instantiated by substitutions and unification

Theorems

Connectives of the Meta-Logic

Implication \implies ($==>$)

For separating premises and conclusion of theorems.

Equality \equiv ($==$)

For definitions.

Universal quantifier \wedge ($!!$)

For parameters in goals.

Do not use *inside* object-logic formulae.

Notation

$$\llbracket A_1; \dots ; A_n \rrbracket \Longrightarrow B$$

abbreviates

$$A_1 \Longrightarrow \dots \Longrightarrow A_n \Longrightarrow B$$

; \approx “and”

Introducing New Theorems

- ▶ As axioms.
- ▶ Through definitions.
- ▶ Through proofs.

! Axioms should mainly be used when specifying object-logics. **!**

Definition (non-recursive)

Declaration:

consts

$sq :: nat \Rightarrow nat$

Definition:

defs

$sq_def: sq\ n \equiv n*n$

Declaration + definition:

constdefs

$sq :: nat \Rightarrow nat$

$sq\ n \equiv n*n$

Proofs

General schema:

```
lemma name: <goal>  
  apply <method>  
  apply <method>  
  ⋮  
done
```

- ▶ Sequential application of methods until all **subgoals** are solved.

The proof state

$$1. \bigwedge x_1 \dots x_p. [A_1; \dots ; A_n] \implies B$$

$$2. \bigwedge y_1 \dots y_q. [C_1; \dots ; C_n] \implies D$$

$x_1 \dots x_p$ Parameters

$A_1 \dots A_n$ Local assumptions

B Actual (sub)goal

Isabelle Theories

Theory = Source file

Syntax:

```
theory MyTh = ImpTh1 + ... + ImpThn :  
(declarations, definitions, theorems, proofs, ...)*  
end
```

- ▶ *MyTh*: name of theory. Must live in file *MyTh.thy*
- ▶ *ImpTh*_{*i*}: name of *imported* theories. Import transitive.

Unless you need something special:

```
theory MyTh = Main:
```

X-Symbols

Input of funny symbols in Proof General

- ▶ via menu (“X-Symbol”)
- ▶ via ascii encoding (similar to $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$): `\<and>`, `\<or>`,
...
- ▶ via abbreviation: `/\`, `\/`, `-->`, ...

| | | | | | | | | |
|-----------|------------------------------|------------------------------|------------------------------|---------------------------|-----------------|-----------------|---------------------|--------------------|
| x-symbol | \forall | \exists | λ | \neg | \wedge | \vee | \longrightarrow | \Rightarrow |
| ascii (1) | <code>\<forall></code> | <code>\<exists></code> | <code>\<lambda></code> | <code>\<not></code> | <code>/\</code> | <code>\/</code> | <code>--></code> | <code>=></code> |
| ascii (2) | ALL | EX | % | ~ | & | | | |

(1) is converted to x-symbol, (2) stays ascii.

Demo: Isabelle theories

Natural Deduction

Rules

$$\frac{A \quad B}{A \wedge B} \text{ conjI}$$

$$\frac{A \wedge B \quad \llbracket A; B \rrbracket \Longrightarrow C}{C} \text{ conjE}$$

$$\frac{A}{A \vee B} \quad \frac{B}{A \vee B} \text{ disjI1/2}$$

$$\frac{A \vee B \quad A \Longrightarrow C \quad B \Longrightarrow C}{C} \text{ disjE}$$

$$\frac{A \Longrightarrow B}{A \longrightarrow B} \text{ impl}$$

$$\frac{A \longrightarrow B \quad A \quad B \Longrightarrow C}{C} \text{ impE}$$

Proof by assumption

apply assumption

proves

$$1. \llbracket B_1; \dots ; B_m \rrbracket \Longrightarrow C$$

by unifying C with one of the B_i (backtracking!)

How to prove it by natural deduction

- ▶ **Intro** rules decompose formulae to the right of \implies .

apply(rule <intro-rule>)

Applying rule $\llbracket A_1; \dots ; A_n \rrbracket \implies A$ to subgoal C :

- ▶ Unify A and C
- ▶ Replace C with n new subgoals $A_1 \dots A_n$
- ▶ **Elim** rules decompose formulae on the left of \implies .

apply(erule <elim-rule>)

Like *rule* but also

- ▶ unifies first premise of rule with an assumption
- ▶ eliminates that assumption

Demo: natural deduction

Safe and unsafe rules

Safe rules preserve provability

conjI, impI, conjE, disjE,
notI, iffI, refl, ccontr, classical

Unsafe rules can turn provable goal into unprovable goal

disjI1, disjI2, impE,
iffD1, iffD2, notE

Apply safe rules before unsafe ones

Predicate Logic: \forall and \exists

Scope

- ▶ Scope of parameters: whole subgoal
- ▶ Scope of \forall, \exists, \dots : ends with ; or \implies

$$\wedge x y. \llbracket \forall y. P y \longrightarrow Q z y; Q x y \rrbracket \implies \exists x. Q x y$$

means

$$\wedge x y. \llbracket (\forall y_1. P y_1 \longrightarrow Q z y_1); Q x y \rrbracket \implies (\exists x_1. Q x_1 y)$$

Natural deduction for quantifiers

$$\frac{\wedge x. P x}{\forall x. P x} \text{allI}$$

$$\frac{\forall x. P x \quad P ?x \implies R}{R} \text{allE}$$

$$\frac{P ?x}{\exists x. P x} \text{exI}$$

$$\frac{\exists x. P x \quad \wedge x. P x \implies R}{R} \text{exE}$$

- ▶ allI and exE introduce new parameters ($\wedge x$).
- ▶ allE and exI introduce new unknowns ($?x$).

Instantiating rules

`apply(rule_tac x = "term" in rule)`

Like *rule*, but *?x* in *rule* is instantiated by *term* before application.

Similar: *erule_tac*

! *x* is in *rule*, not in the goal **!**

Safe and unsafe rules

Safe all, exE

Unsafe allE, exI

Create parameters first, unknowns later

Forward proofs: frule and drule

apply(frule rulename)

Forward rule: $A_1 \implies A$

Subgoal: 1. $\llbracket B_1; \dots ; B_n \rrbracket \implies C$

Unifies: one B_i with A_1

New subgoal: 1. $\llbracket B_1; \dots ; B_n; A \rrbracket \implies C$

apply(drule rulename)

Like *frule* but also deletes B_i

Demo: quantifier proofs

Practical Session I

**In the cool morning
A man simplifies, a goal
A theorem is born.**

— Don Syme

Session II

HOL = Functional programming + Logic

Proof by Term Rewriting

Term rewriting means ...

Using equations $l = r$ from left to right
as long as possible

Terminology: equation \rightsquigarrow rewrite rule

Example

Example:

Equation: $0 + n = n$

Term: $a + (0 + (b + c))$

Result: $a + (b + c)$

Rewrite rules can be conditional: $\llbracket P_1 \dots P_n \rrbracket \implies l = r$
is used

- ▶ like $l = r$, but
- ▶ P_1, \dots, P_n must be proved by rewriting first.

Simplification in Isabelle

Goal: 1. $\llbracket P_1; \dots ; P_m \rrbracket \Longrightarrow C$

apply(simp add: eq₁ ... eq_n)

Simplify $P_1 \dots P_m$ and C using

- ▶ lemmas with attribute *simp*
- ▶ additional lemmas $eq_1 \dots eq_n$
- ▶ assumptions $P_1 \dots P_m$

Variations:

- ▶ *(simp ... del: ...)* removes *simp*-lemmas
- ▶ *add* and *del* are optional

Termination

Simplification may not terminate.

Isabelle uses *simp*-rules (almost) blindly from left to right.

Example: $f(x) = g(x), g(x) = f(x)$

$$\llbracket P_1 \dots P_n \rrbracket \Longrightarrow l = r$$

is suitable as a *simp*-rule only

if l is “bigger” than r and each P_i

$$n < m \Longrightarrow (n < \text{Suc } m) = \text{True} \quad \text{YES}$$

$$\text{Suc } n < m \Longrightarrow (n < m) = \text{True} \quad \text{NO}$$

How to ignore assumptions

Assumptions sometimes cause problems, e.g. nontermination. How to exclude them from *simp*:

`apply(simp (no_asm_simp) ...)`

Simplify only conclusion

`apply(simp (no_asm_use) ...)`

Simplify but do not use assumptions

`apply(simp (no_asm) ...)`

Ignore assumptions completely

Tracing

Set trace mode on/off in Proof General:

Isabelle/Isar → Settings → Trace simplifier

Output in separate buffer:

Proof-General → Buffers → Trace

auto

- ▶ *auto* acts on all subgoals
- ▶ *simp* acts only on subgoal 1
- ▶ *auto* applies *simp* and more

Demo: simp

Type definitions in Isabelle/HOL

Keywords:

- ▶ **typedecl**: pure declaration (session 1)
- ▶ **types**: abbreviation
- ▶ **datatype**: recursive datatype

types

types *name* = τ

Introduces an *abbreviation* *name* for type τ

Examples:

types

name = *string*

('a, 'b)foo = "*a list* × *b list*"

Type abbreviations are expanded after parsing
Not present in internal representation and Isabelle output

datatype

datatype 'a list = Nil | Cons 'a "'a list"

Properties:

- ▶ **Types:** Nil :: 'a list
Cons :: 'a ⇒ 'a list ⇒ 'a list
- ▶ **Distinctness:** Nil ≠ Cons x xs
- ▶ **Injectivity:** (Cons x xs = Cons y ys) = (x = y ∧ xs = ys)

case

Every datatype introduces a *case* construct, e.g.

(case xs of Nil \Rightarrow ... | Cons y ys \Rightarrow ... y ... ys ...)

- ▶ one case per constructor
- ▶ no nested patterns (*Cons x (Cons y zs)*)
- ▶ but nested cases

apply(*case_tac xs*) \Rightarrow one subgoal for each constructor

xs = Nil \Longrightarrow ...

xs = Cons a list \Longrightarrow ...

Function definition schemas in Isabelle/HOL

- ▶ Non-recursive with **constdefs** (session 1)
No problem
- ▶ Primitive-recursive with **primrec**
Terminating by construction
- ▶ Well-founded recursion with **recdef**
User must (help to) prove termination

primrec

consts *app* :: "'a list \Rightarrow 'a list \Rightarrow 'a list"

primrec

"*app Nil ys = ys*"

"*app (Cons x xs) ys = Cons x (app xs ys)*"

- ▶ Each recursive call **structurally smaller** than lhs.
- ▶ Equations used automatically in simplifier

Structural induction

$P\ xs$ holds for all lists xs if

- ▶ $P\ Nil$
- ▶ and for arbitrary x and xs , $P\ xs$ implies $P\ (Cons\ x\ xs)$

Induction theorem `list.induct`:

$$\llbracket P\ Nil; \bigwedge a\ list. P\ list \implies P\ (Cons\ a\ list) \rrbracket$$
$$\implies P\ list$$

- ▶ General proof method for induction: *(induct x)*
 - ▶ x must be a free variable in the first subgoal.
 - ▶ The type of x must be a datatype.

Induction heuristics

Theorems about recursive functions proved by induction

consts *itrev* :: 'a list \Rightarrow 'a list \Rightarrow 'a list

primrec

itrev [] ys = ys

itrev (x#xs) ys = *itrev* xs (x#ys)

lemma *itrev* xs [] = *rev* xs

Demo: proof attempt

Generalisation

Replace constants by variables

lemma *itrev xs ys = rev xs @ ys*

Quantify free variables by \forall
(except the induction variable)

lemma $\forall ys.$ *itrev xs ys = rev xs @ ys*

Function definition schemas in Isabelle/HOL

- ▶ Non-recursive with `constdefs` (session 1)
No problem
- ▶ Primitive-recursive with `primrec`
Terminating by construction
- ▶ Well-founded recursion with `recdef`
User must (help to) prove termination

recdef — examples

```
consts sep :: "'a × 'a list ⇒ 'a list"
recdef sep "measure (λ(a, xs). size xs)"
  "sep (a, x # y # zs) = x # a # sep (a, y # zs)"
  "sep (a, xs) = xs"
```

```
consts ack :: "nat × nat ⇒ nat"
recdef ack "measure (λm. m) <*lex*> measure (λn. n)"
  "ack (0, n) = Suc n"
  "ack (Suc m, 0) = ack (m, 1)"
  "ack (Suc m, Suc n) = ack (m, ack (Suc m, n))"
```

recdef

- ▶ The definition:
 - ▶ one parameter
 - ▶ free pattern matching, order of rules important
 - ▶ termination relation
(*measure* sufficient for most cases)
- ▶ Termination relation:
 - ▶ must decrease for each recursive call
 - ▶ must be well founded
- ▶ Generates own induction principle.

Demo: recdef and induction

Sets

Notation

Type '*a set*: sets over type '*a*

- ▶ $\{\}, \{e_1, \dots, e_n\}, \{x. P x\}$
- ▶ $e \in A, A \subseteq B$
- ▶ $A \cup B, A \cap B, A - B, - A$
- ▶ $\bigcup_{x \in A} B x, \bigcap_{x \in A} B x$
- ▶ $\{i..j\}$
- ▶ $insert :: 'a \Rightarrow 'a set \Rightarrow 'a set$
- ▶ $f ' A \equiv \{y. \exists x \in A. y = f x\}$
- ▶ ...

Inductively defined sets: even numbers

Informally:

- ▶ 0 is even
- ▶ If n is even, so is $n + 2$
- ▶ These are the only even numbers

In Isabelle/HOL:

consts $Ev :: nat\ set$ — The set of all even numbers

inductive Ev

intros

$$0 \in Ev$$

$$n \in Ev \implies n + 2 \in Ev$$

Rule induction for Ev

To prove

$$n \in Ev \implies P n$$

by *rule induction* on $n \in Ev$ we must prove

▶ $P 0$

▶ $P n \implies P(n+2)$

Rule `Ev.induct`:

$$\llbracket n \in Ev; P 0; \bigwedge n. P n \implies P(n+2) \rrbracket \implies P n$$

An elimination rule

Demo: inductively defined sets

Isar

**A Language for Structured
Proofs**

Apply scripts

- ▶ unreadable
- ▶ hard to maintain
- ▶ do not scale

No structure!

A typical Isar proof

proof

assume $formula_0$

have $formula_1$ **by** *simp*

⋮

have $formula_n$ **by** *blast*

show $formula_{n+1}$ **by** ...

qed

proves $formula_0 \implies formula_{n+1}$

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = (*simp* ...) | (*blast* ...) | (*rule* ...) | ...

statement = **fix** variables (\wedge)
| **assume** proposition (\implies)
| [**from** name⁺] (**have** | **show**) proposition proof
| **next** (separates subgoals)

proposition = [name:] formula

Demo: propositional logic

Elimination rules / forward reasoning

- ▶ Elim rules are triggered by facts fed into a proof:
from \vec{a} have *formula* proof
- ▶ **from \vec{a} have *formula* proof (*rule* *rule*)**
 \vec{a} must prove the first n premises of *rule*
in the right order
the others are left as new subgoals
- ▶ **proof** alone abbreviates **proof *rule***
- ▶ ***rule***: tries elim rules first
(if there are incoming facts \vec{a} !)

Practical Session II

**Theorem proving and
sanity; Oh, my! What a
delicate balance.**

— Victor Carreno

Session III

More about Isar

Overview

- ▶ Abbreviations
- ▶ Predicate Logic
- ▶ Accumulating facts
- ▶ Reasoning with chains of equations
- ▶ Locales: the module system

Abbreviations

- this* = the previous proposition proved or assumed
- then* = **from** *this*
- with* \vec{a} = **from** \vec{a} *this*
- ?thesis* = the last enclosing **show** formula

Mixing proof styles

from ...

have ...

apply - make incoming facts assumptions

apply(...)

⋮

apply(...)

done

Demo: Abbreviations

Predicate Calculus

fix

Syntax:

fix variables

Introduces new arbitrary but fixed variables
(~ parameters)

obtain

Syntax:

obtain variables where proposition proof

Introduces new variables together with property

Demo: predicate calculus

moreover/ultimately

have *formula*₁ . . .

moreover

have *formula*₂ . . .

moreover

⋮

moreover

have *formula*_{*n*} . . .

ultimately

show . . .

— pipes facts *formula*₁ . . . *formula*_{*n*} into the proof

proof . . .

Demo: moreover/ultimately

General case distinctions

show *formula*

proof -

have $P_1 \vee P_2 \vee P_3 \dots$

moreover

{ **assume** $P_1 \dots$ **have** *?thesis* ... }

moreover

{ **assume** $P_2 \dots$ **have** *?thesis* ... }

moreover

{ **assume** $P_3 \dots$ **have** *?thesis* ... }

ultimately **show** *?thesis* **by blast**

qed

Chains of equations

- ▶ Keywords **also** and **finally**.
- ▶ `...`: predefined schematic term variable, refers to the **right hand side of the last expression**.
- ▶ Uses transitivity rule.

also/finally

have " $t_0 = t_1$ " ...

also

$$t_0 = t_1$$

have "... = t_2 " ...

also

$$t_0 = t_2$$

⋮

⋮

also

$$t_0 = t_{n-1}$$

have "... = t_n " ...

finally show ...

— pipes fact $t_0 = t_n$ into the proof

proof

⋮

More about also

- ▶ Works for all combinations of $=$, \leq and $<$.
- ▶ Uses rules declared as `[trans]`.
- ▶ To view all combinations in Proof General:
Isabelle/Isar \rightarrow Show me \rightarrow Transitivity rules

Demo: also/finally

Locales

Isabelle's Module System

Isar is based on contexts

theorem $\bigwedge x. A \implies C$

proof -

fix X

assume $Ass: A$

\vdots

from Ass **show** $C \dots$

qed

x and Ass are visible
inside this context

Beyond Isar contexts

Locales are extended contexts

- ▶ Locales are **named**
- ▶ Fixed variables may have **syntax**
- ▶ It is possible to **add** and **export** theorems
- ▶ Locale expression: **combine** and **modify** locales

Context elements

Locales consist of **context elements**.

| | |
|----------------|------------------------|
| fixes | Parameter, with syntax |
| assumes | Assumption |
| defines | Definition |
| notes | Record a theorem |

Declaring locales

locale *loc* =

loc1 + Import

fixes . . . Context elements

assumes . . .

Declares **named locale** *loc*.

Declaring locales

Theorems may be stated relative to a named locale.

lemma (in *loc*) *P* [simp]: *proposition*
proof

- ▶ Adds theorem *P* to context *loc*.
- ▶ Theorem *P* is in the simpset in context *loc*.
- ▶ Exported theorem *loc.P* visible in the entire theory.

Demo: locales 1

Parameters must be consistent!

- ▶ Parameters in **fixes** are distinct.
- ▶ Free variables in **assumes** and **defines** occur in preceding **fixes**.
- ▶ Defined parameters must neither occur in preceding **assumes** nor **defines**.

Locale expressions

Locale name: n

Rename: $e \ q_1 \ \dots \ q_n$

Change names of parameters in e .

Merge: $e_1 + e_2$

Context elements of e_1 , then e_2 .

- ▶ Syntax is lost after rename (**currently**).

Demo: locales 2

Normal form of locale expressions

Locale expressions are converted to flattened lists of locale names.

- ▶ With full parameter lists
- ▶ Duplicates removed

Allows for multiple inheritance!

Instantiation

Move from **abstract** to **concrete**.

instantiate *label* : *loc*

- ▶ From chained fact *loc t₁ ... t_n* instantiate locale *loc*.
- ▶ Imports all theorems of *loc* into current context.
 - ▶ Instantiates the parameters with *t₁ ... t_n*.
 - ▶ Interprets attributes of theorems.
 - ▶ Prefixes theorem names with *label*
- ▶ **Currently only works inside Isar contexts.**

Demo: locales 3

Practical Session III

**The sun spills darkness
A dog howls after midnight
Goals remain unsolved.**

— Chris Owens

Session IV

Case Studies

Case Study

Compiling Expressions

The Task

- ▶ develop a compiler
- ▶ from expressions
- ▶ to a stack machine
- ▶ and show its correctness

- ▶ expressions built from
 - ▶ variables
 - ▶ constants
 - ▶ binary operations

Expressions — Syntax

Syntax for

- ▶ binary operations
- ▶ expressions

Design decision:

- ▶ no syntax for **variables** and **values**

Instead:

- ▶ expressions generic in variable names,
- ▶ *nat* for values.

Expressions — Data Type

- ▶ Binary operations

datatype *binop* = *Plus* | *Minus* | *Mult*

- ▶ Expressions

datatype *'v expr* = *Const nat*
 | *Var 'v*
 | *Binop binop "'v expr" "'v expr"*

- ▶ *'v* = variable names

Expressions — Semantics

- ▶ Semantics for binary operations:

consts *semop* :: "*binop* \Rightarrow *nat* \Rightarrow *nat* \Rightarrow *nat*" (" $\llbracket _ \rrbracket$ ")

primrec " $\llbracket Plus \rrbracket = (\lambda x y. x + y)$ "

" $\llbracket Minus \rrbracket = (\lambda x y. x - y)$ "

" $\llbracket Mult \rrbracket = (\lambda x y. x * y)$ "

- ▶ Semantics for expressions:

consts *value* :: "'v *expr* \Rightarrow ('v \Rightarrow *nat*) \Rightarrow *nat*"

primrec

"*value* (*Const* *v*) *E* = *v*"

"*value* (*Var* *a*) *E* = *E* *a*"

"*value* (*Binop* *f* *e*₁ *e*₂) *E* = $\llbracket f \rrbracket$ (*value* *e*₁ *E*) (*value* *e*₂ *E*)"

Stack Machine — Syntax

Machine with 3 instructions:

- ▶ **push** constant value onto stack
- ▶ **load** contents of register onto stack
- ▶ **apply** binary operator to top of stack

Simplification: register names = variable names

```
datatype 'v instr = Push nat  
                | Load 'v  
                | Apply binop
```

Stack Machine — Execution

Modelled by a function taking

- ▶ list of instructions (program)
- ▶ store (register names to values)
- ▶ list of values (stack)

Returns

- ▶ new stack

exec

consts *exec* :: "'v instr list \Rightarrow ('v \Rightarrow nat) \Rightarrow nat list \Rightarrow nat list"

primrec

"*exec [] s vs = vs*"

"*exec (i#is) s vs = (case i of*

Push v \Rightarrow exec is s (v # vs)

| Load a \Rightarrow exec is s (s a # vs)

*| Apply f \Rightarrow let v₁ = hd vs; v₂ = hd (tl vs); ts = tl (tl vs) in
 exec is s ([f] v₁ v₂ # ts))"*

► *hd* and *tl* are head and tail of lists

The Compiler

Compilation easy:

- ▶ *Constants* \Rightarrow *Push*
- ▶ *Variables* \Rightarrow *Load*
- ▶ *Binop* \Rightarrow *Apply*

consts *comp* :: "*v expr* \Rightarrow '*v instr list*"

primrec

"*comp (Const v) = [Push v]*"

"*comp (Var a) = [Load a]*"

"*comp (Binop f e₁ e₂) = (comp e₂) @ (comp e₁) @ [Apply f]*"

Correctness

Executing compiled program yields value of expression

theorem "*exec (comp e) s [] = [value e s]*"

Proof?

Demo: correctness proof

Case Study

Commutative Algebra

Abstract Mathematics

- ▶ Concerns **classes** of objects specified by axioms, not concrete objects like the integers or reals.
- ▶ Objects are typically **structures**: $(G, \cdot, 1, ^{-1})$
 - ▶ Groups, rings, lattices, topological spaces
- ▶ Concepts are frequently combined and extended.
- ▶ Instances may be **concrete** or **abstract**.

Formalisation

- ▶ Structures are not theories of proof tools.
- ▶ Structures must be **first-class values**.
- ▶ Syntax should reflect **context**:
 - ▶ If G is a group, then $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ refers implicitly to G .
- ▶ Inheritance of syntax and theorems should be automatic.

Support for Abstraction

- ▶ **Locales**: portable contexts.
- ▶ λ (`\<index>`) arguments in syntax declarations.
- ▶ Extensible **records** (in HOL).
- ▶ Locale **instantiation**.

Index Arguments in Syntax Declarations

- ▶ One function argument may be `\<index>`.
- ▶ Works also for infix operators and binders:
$$x \otimes_G y \quad \bigoplus_{R} i \in \{0..n\}. f i$$
- ▶ Good for denoting record fields.
- ▶ Can declare default by **(structure)**.
- ▶ Yields a concise syntax for G while allowing references to other groups.
- ▶ Letter subscripts for `\<index>` only available in current development version of Isabelle.

Records

- ▶ Are used to represent **structures**.
- ▶ Fields are functions and can have special syntax.
- ▶ Records can be extended with additional fields.

```
record 'a monoid =  
  carrier :: "'a set"  
  mult    :: "[ 'a, 'a ] ⇒ 'a" (infixl "⊗/" 70)  
  one     :: 'a ("1/")
```

A Locale for Monoids

locale monoid = struct G +

assumes m_closed [intro, simp]:

"[[$x \in \text{carrier } G; y \in \text{carrier } G$]] $\implies x \otimes y \in \text{carrier } G$ "

and m_assoc:

"[[$x \in \text{carrier } G; y \in \text{carrier } G; z \in \text{carrier } G$]]

$\implies (x \otimes y) \otimes z = x \otimes (y \otimes z)$ "

and one_closed [intro, simp]: "1 \in carrier G"

and l_one [simp]: "x \in carrier G $\implies 1 \otimes x = x$ "

and r_one [simp]: "x \in carrier G $\implies x \otimes 1 = x$ "

A Locale for Groups

A **group** is a monoid whose elements have inverses.

locale group = monoid +
assumes **inv_ex**:

" $x \in \text{carrier } G \implies \exists y \in \text{carrier } G. y \otimes x = 1 \wedge x \otimes y = 1$ "

- ▶ Reasoning in locale group makes implicit the assumption that G is a group.
- ▶ Inverse operation is **derived**, not part of the record.

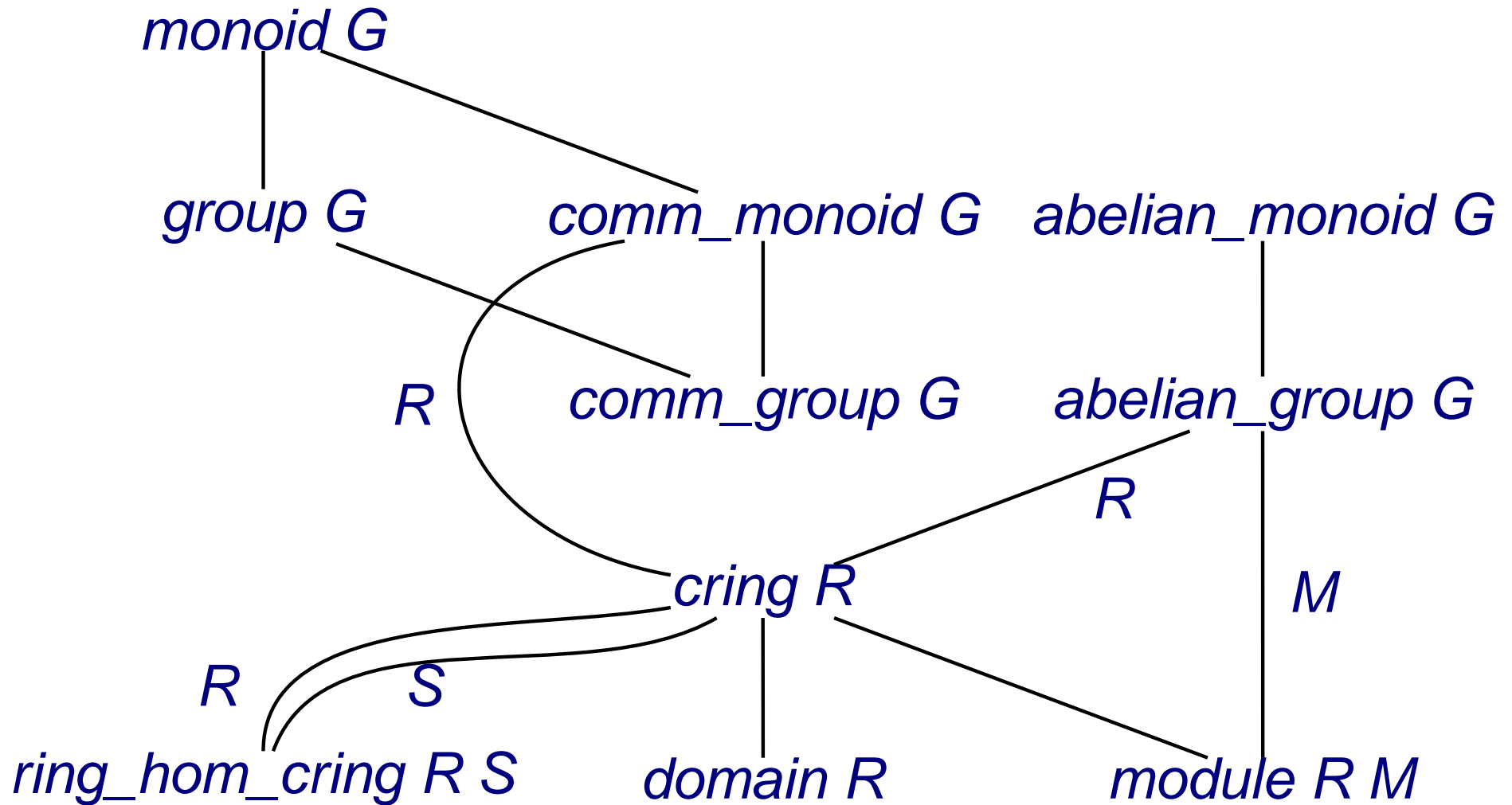
Hierarchy of Structures

```
record 'a ring = "'a monoid" +  
  zero :: 'a ("0/")  
  add :: "[ 'a, 'a ] => 'a" (infixl "⊕/" 65)
```

```
record ('a, 'b) module = "'b ring" +  
  smult :: "[ 'a, 'b ] => 'b" (infixl "⊙/" 70)
```

```
record ('a, 'p) up_ring = "('a, 'p) module" +  
  monom :: "[ 'a, nat ] => 'p"  
  coeff :: "[ 'p, nat ] => 'a"
```

Hierarchy of Specifications



Polynomials

Functor UP that maps ring structures to polynomial structures.

constdefs (structure R)

$UP :: \text{"('a, 'm) ring_scheme} \Rightarrow \text{'a, nat} \Rightarrow \text{'a) up_ring"$

$\text{"UP R} \equiv \text{(| carrier = up R,$

$\text{mult = } (\lambda p \in \text{up R. } \lambda q \in \text{up R. } \lambda n. \bigoplus_{i \in \{..n\}} p\ i \otimes q\ (n-i)),$

$\text{one = } (\lambda i. \text{ if } i=0 \text{ then } 1 \text{ else } 0),$

$\text{zero = } (\lambda i. 0),$

$\text{add = } (\lambda p \in \text{up R. } \lambda q \in \text{up R. } \lambda i. p\ i \oplus q\ i),$

$\text{smult = } (\lambda a \in \text{carrier R. } \lambda p \in \text{up R. } \lambda i. a \otimes p\ i),$

$\text{monom = } (\lambda a \in \text{carrier R. } \lambda n\ i. \text{ if } i=n \text{ then } a \text{ else } 0),$

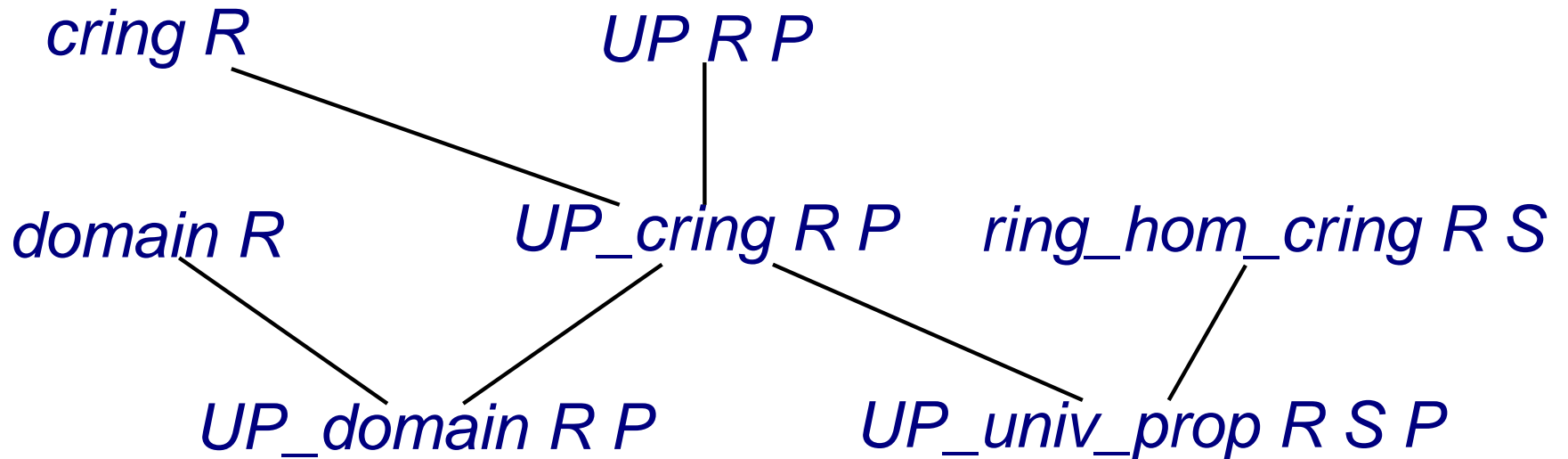
$\text{coeff = } (\lambda p \in \text{up R. } \lambda n. p\ n) \text{ |}"$

Locales for Polynomials

- ▶ Make the polynomial ring a locale parameter

locale UP = struct R + struct P +
defines P_def: "P ≡ UP R"

- ▶ Add information about base ring



Properties of *UP*

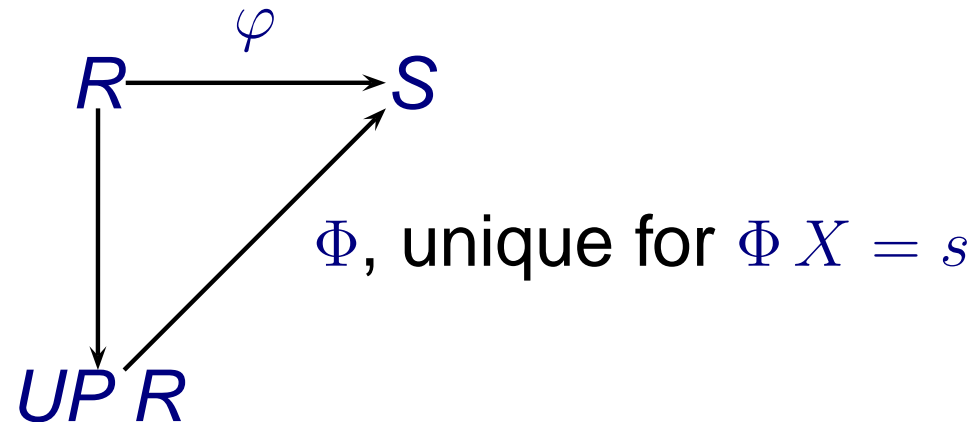
Polynomials over a ring form a ring.

theorem (in UP_cring) UP_cring: "cring P"

Polynomials over an integral domain form a domain.

theorem (in UP_domain) UP_domain: "domain P"

The Universal Property



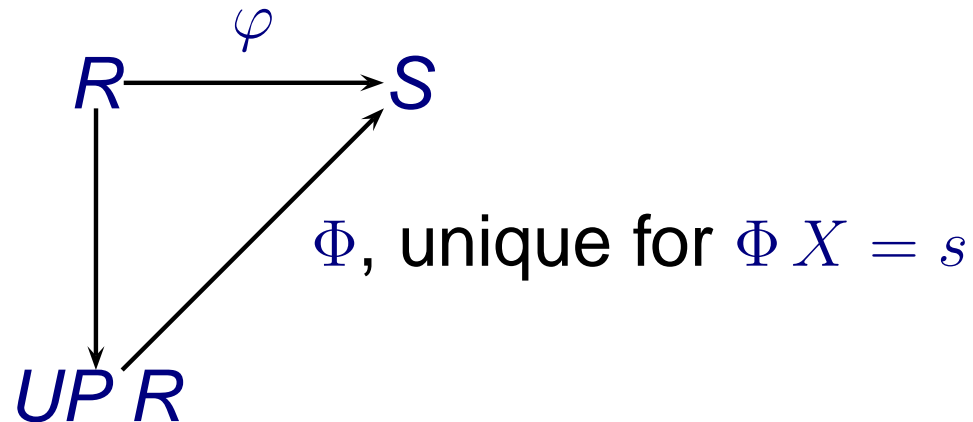
- Existence of Φ :

$$\text{eval } R \ S \ \text{phi } s \equiv \lambda p \in \text{carrier } (UP \ R).$$

$$\bigoplus_{i \in \{..deg \ R \ p\}} \text{phi } (\text{coeff } (UP \ R) \ p \ i) \otimes s \ (\wedge) \ i$$

Show that $\text{eval } R \ S \ \text{phi}$ is a homomorphism.

The Universal Property



- Uniqueness of Φ :

Show that two homomorphisms $\Phi, \Psi : UP R \rightarrow S$ with $\Phi X = \Psi X = s$ are identical.

Demo: uniqueness

Questions answered by Larry Paulson

**Hah! A proof of False
Your axioms are bogus
Go back to square one.**

— Larry Paulson