Axioms for Logics of Knowledge and Past Time: Synchrony and Unique Initial States

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ABSTRACT.

Sound and complete axiomatizations are provided for two different logics involving modalities for knowledge and both past and future time modalities. The logics considered allow for multiple agents with unique initial state and synchrony. The synchrony restriction gives every agent access to a system clock. Such semantic restrictions are of particular interest in the context of past time modalities since both synchrony and unique initial state restrictions are not expressible using future time modalities.

1 Introduction

There has been significant interest in multi-modal logics combining operators for knowledge and time in recent years [1, 3, 4, 7]. With only a few exceptions [3], this literature deals with *future time* temporal operators. In this paper we consider the effect of adding past time operators to such logics.

There are some compelling reasons to consider this extension. One of the topics of interest in the literature has been the interaction between knowledge and time when a variety of semantic properties are assumed, such as uniqueness of initial states, synchrony, perfect recall and no learning (a dual of perfect recall) [8]. These properties lead to interaction axioms, which involve both epistemic and temporal operators. Halpern, van der Meyden and Vardi [7] provide complete axiomatizations of logics of knowledge and linear future time for all the axiomatizable cases arising out of combinations of these assumptions. However, their results indicate that in some cases, some of the properties have no impact on the axiomatization. For example, [7] obtains identical complete axiomatizations for the cases of no assumptions, synchrony alone, unique initial states alone, and for both synchrony and unique initial states. This indicates that the logic with future time axioms is too weak to fully express the unique initial states and synchrony prop-

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erties. It has also been noted that past time operators allow for a much cleaner axiom for perfect recall in an asynchronous setting [13].

Another reason to consider knowledge in combination with past time operators is that *knowledge-based programs* [2] are better behaved with past-time operators than with future time operators. A knowledge-based program is like a standard program with formulas expressing the knowledge of the agent allowed to occur as conditions in conditional statements. A concrete implementation of such a program replaces the knowledge conditions by concrete conditions of the agent's local state. Knowledge-based programs behave somewhat like specifications, and in general, may have zero, one, or many different implementations. However, it is possible to provide conditions under which there is guaranteed to be a unique implementation [2]. One of these conditions is when the system is synchronous, and all knowledge tests involve only past time operators.

We would like to have an interaction axiom for each of the properties mentioned above, such that combinations of properties can be handled by combining their corresponding axiom. In this paper, we take a step in this direction by providing axioms which individually characterize the properties of unique initial states and synchrony. (We will deal with combinations in future work.) As already remarked, a past time axiom for perfect recall is already given in [13]. The property of no learning is best captured by the future time axiom in [7].

The synchrony restriction is particularly interesting since the axiomatization appears to require a complex automaton-based rule. We sketch the rather interesting completeness proof here, based on completeness proofs given in [7]. We show we can construct a canonical model for any consistent formula by induction over the nestings of knowledge operators. To enforce the synchrony constraint we introduce transducers to represent sufficiently detailed information about the time. This transducer can be encoded in a characteristic formula, and we use the new rule, SYNC, to show that the synchrony constraint is maintained.

2 Syntax and Semantics

The language is given by the abstract syntax:

$$\alpha = x \mid \neg\alpha \mid \alpha \wedge \alpha' \mid \bigcirc\alpha \mid \alpha\,\mathcal{U}\,\alpha' \mid \circledw\alpha \mid \alpha\,\mathcal{S}\,\alpha' \mid K_i\alpha$$

where $x \in \mathcal{V}$ is some propositional atom, and $1 \leq i \leq k$ is the index of an agent. The operators are respectively *not, and, tomorrow, until, weak yesterday, since and i-knows*, and have their usual meaning. Along with the usual propositional abbreviations (*true*, *false*, $\vee$, $\rightarrow$) we will also use the temporal abbreviations: $\ominus\alpha = \neg\circledw\neg\alpha$; $\Diamond\alpha = true\,\mathcal{U}\,\alpha$; $\Diamond\!\!\!\!-\,\alpha = true\,\mathcal{S}\,\alpha$;

$\Box\alpha = \neg\Diamond\neg\alpha$; and $\boxminus\alpha = \neg\diamondminus\neg\alpha$, and the epistemic operator, $L_i\alpha = \neg K_i\neg\alpha$.

For the semantics, we suppose a model is given by a set of runs, and each formula is evaluated with respect to some time in some run. Given sets $\mathcal{L}_i$ representing the possible local states of agent i, for $i = 1 \ldots k$, we define a *run* to be an element of the set

(1) $\mathcal{R} = \{r \mid r : \omega \longrightarrow \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_k\}.$

We define a model to be a subset of $\mathcal{R}$.

Given, $M \subseteq \mathcal{R}$ we give the semantic interpretation of formulas with respect to one run $r \in M$ and one moment of time, $n \in \omega$. We inductively define $M, r, n \models \alpha$ as follows:

$$\begin{array}{rcl}
M, r, n \models x & \Leftrightarrow & x \in r(n)_0 \\
M, r, n \models \neg\alpha & \Leftrightarrow & M, r, n \not\models \alpha \\
M, r, n \models \alpha \wedge \alpha' & \Leftrightarrow & M, r, n \models \alpha \text{ and } M, r, n \models \alpha' \\
M, r, n \models \bigcirc\alpha & \Leftrightarrow & M, r, n+1 \models \alpha \\
M, r, n \models \alpha\,\mathcal{U}\,\alpha' & \Leftrightarrow & \exists m \geq n,\ M, r, m \models \alpha' \text{ and } n \leq j < m \Rightarrow M, r, j \models \alpha \\
M, r, n \models \text{ⓦ}\alpha & \Leftrightarrow & n = 0 \text{ or } M, r, n-1 \models \alpha \\
M, r, n \models \alpha\,\mathcal{S}\,\alpha' & \Leftrightarrow & \exists m \leq n,\ M, r, m \models \alpha' \text{ and } m < j \leq n \Rightarrow M, r, j \models \alpha \\
M, r, n \models K_i\alpha & \Leftrightarrow & \forall r' \in M, \forall m \in \omega,\ r(n)_i = r'(m)_i \Rightarrow M, r', m \models \alpha
\end{array}$$

for each agent i.

This gives the most general description of a language that describes knowledge and past time. However there are several useful restrictions we will consider:

- We say a model has *unique initial states* if for all runs $r, r' \in M$, for all $i \in \{1, \ldots, k\}$, we have $r(0)_i = r'(0)_i$;
- We say a model is *synchronous* if for all runs $r, r' \in M$, for all $n, m \in \omega$, for all $i \in \{1, \ldots, k\}$, we have $r(n)_i = r'(m)_i \Longrightarrow n = m$;

There are several other semantic restrictions that can be applied to combinations of temporal and modal logic, including *perfect recall* and *no learning* [1]. We have chosen to focus on the synchrony and unique initial state restrictions in this paper as they are especially relevant to temporal logics with past. The synchrony and unique initial state restrictions have little effect in logics without past operators, as these restrictions do not alter the set of valid formulas.

The unique initial state restriction requires that no agent can initially distinguish between the possible initial states of the system. This restriction can have significant consequences for the language. In [8] it was shown that when the unique initial state restriction is combined with the no learning restriction the resulting language is highly undecidable. We also note that in the presence of past operators the unique initial state restriction allows us to express a *universal modality* [12, 6]. Specifically, for any formula α, we can define the formula $\diamondminus(\textcircled{w}\mathit{false} \wedge L_i \Diamond \alpha)$ which is satisfied by a model M with the unique initial state restriction if and only if there is some run $r \in M$ and some n such that $M, r, n \models \alpha$.

Once past operators are added to the language, the synchrony restriction has a dramatic affect on the set of valid formulas. Since every agent *knows* the time, an axiomatization must allow reasoning about which formulas can be true at which times. For example, if there is some formula, α, that is true at only even times, then, if at some time an agent even suspects that α might be true, then that agent should know that every formula that is true at only odd times must be false. This situation is captured in the following formula, which is a validity in the synchronous semantics.

(2) $\quad L_i(x \wedge \boxminus(x \leftrightarrow \textcircled{w}\neg x)) \rightarrow K_i(\boxminus(y \leftrightarrow \textcircled{w}\neg y) \rightarrow y)$

Here, $L_i(x \wedge \boxminus(x \leftrightarrow \textcircled{w}\neg x))$ means that agent i *considers it possible* that x is currently true, and up to (and including) now, x has only been true at even moments of time. Since agent i knows the time, agent i knows that the time must be even. Hence agent i *knows* that if up to and including now y has only been true at even moments of time, then y must currently be true, or $K_i(\boxminus(y \leftrightarrow \textcircled{w}\neg y) \rightarrow y)$.

3 Axioms

In this section, we describe the axioms and inference rules that we need for reasoning about knowledge and time for various classes of systems, and state the completeness results.

For reasoning about knowledge alone, the following system, with axioms K1–K5 and rules of inference R1–R2, is well known to be sound and complete [1, 9]:

K1. All propositional tautologies
K2. $K_i\varphi \wedge K_i(\varphi \rightarrow \psi) \rightarrow K_i\psi,\ i = 1, \ldots, k$
K3. $K_i\varphi \rightarrow \varphi,\ i = 1, \ldots, k$
K4. $K_i\varphi \rightarrow K_iK_i\varphi,\ i = 1, \ldots, k$
K5. $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi,\ i = 1, \ldots, k$
R1. From φ and $\varphi \rightarrow \psi$ infer ψ
R2. From φ infer $K_i\varphi,\ i = 1, \ldots, k$

This axiom system is known as $S5_m$.

For reasoning about pure temporal formulas (or formulas not containing knowledge operators), the following axioms and rules (together with K1 and R1), can be shown to be sound and complete [10]:

F1. $\bigcirc(\varphi \to \psi) \to \bigcirc\varphi \to \bigcirc\psi$
F2. $\bigcirc(\neg\varphi) \leftrightarrow \neg\bigcirc\varphi$
F3. $\varphi\,\mathcal{U}\,\psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi\,\mathcal{U}\,\psi))$
P1. $ⓦ(\varphi \to \psi) \to ⓦ\varphi \to ⓦ\psi$
P2. $\ominus\neg\varphi \to \neg\ominus\varphi$
P3. $\varphi\,\mathcal{S}\,\psi \leftrightarrow \psi \vee (\varphi \wedge \ominus(\varphi\,\mathcal{S}\,\psi))$
P4. $\mathit{true}\,\mathcal{S}\,ⓦ\mathit{false}$
FP. $\varphi \to \bigcirc\ominus\varphi$
PF. $\varphi \to ⓦ\bigcirc\varphi$

RT1. From φ infer $\bigcirc\varphi$
RT2. From $\varphi' \to \neg\psi \wedge \bigcirc\varphi'$ infer $\varphi' \to \neg(\varphi\,\mathcal{U}\,\psi)$
RP1. From φ infer $ⓦ\varphi$.
RP2. From $\varphi' \to \neg\psi \wedge ⓦ\varphi'$ infer $\varphi' \to \neg(\varphi\,\mathcal{S}\,\psi)$

This set of axioms gives a sufficient axiomatization of knowledge with past time. To allow for the unique initial states restriction, we add the following axiom:

UIS. $\boxminus(ⓦ\mathit{false} \to K_i\alpha) \to K_j \boxminus(ⓦ\mathit{false} \to \alpha),\ i,j = 1,\dots k.$

For the rules required for synchrony, we must first define a characteristic formula for a *transducer*. A transducer is deterministic finite automaton over a one letter alphabet, and can be described by the tuple (Q, q_0, δ) where Q is a finite set of states, $q_0 \in Q$ is the initial state, and $\delta : Q \longrightarrow Q$ is the transition function.

DEFINITION 1. A *characteristic formula* is a formula of the form

$$\diamondminus \left(ⓦ\mathit{false} \wedge \overline{a_0} \wedge \Box \bigwedge_{a \subseteq X} (\overline{a} \to \bigcirc\overline{\delta(a)}) \right),$$

where: X is an arbitrary finite set of propositional atoms; $a_0 \subseteq X$; for each $a \subseteq X$, $\overline{a}$ is the formula $\bigwedge_{x \in a} x \wedge \neg \bigvee_{x \in X \setminus a} x$; and $\delta : \wp(X) \longrightarrow \wp(X)$ is some function.

A characteristic formula describes the operation of a transducer in linear temporal logic. Specifically we can take $\wp(X)$ to be the set of states, a_0 to be the initial state and δ to be the transition function. It should be clear that a characteristic formula is always satisfiable. It simply declares which atoms should be true at which times in a deterministic manner.

In the case of synchronous systems we require two new rules:

AUT. From $\chi \to \beta$ infer β, where $var(\chi) \cap var(\beta) = \emptyset$ and χ is a characteristic formula.

SYNC. From $\alpha \to \beta$ infer $\alpha \to K_i\beta$, where $var(\alpha) \cap var(\beta) = \emptyset$ and $i = 1, \ldots, k$.

where given a formula α, we let $var(\alpha)$ be the set of propositional atoms appearing in α.

We require the rule, AUT, to add extra propositions into a proof when the propositions already in the proof do not yield sufficient information about the system clock. It should be clear that any transducer, (Q, q_0, δ) will generate a unique sequence of states, $s_0, s_1, \ldots$, where $s_0 = q_0$ and $s_{i+1} = \delta(s_i)$. Furthermore for every transducer there is a unique $k \geq 0$ and a unique $n \geq 1$ such that

1. $\forall i < k$, $\forall j \neq i$, $s_i \neq s_j$, and
2. $\forall i \geq k$, $\forall j > i$ $s_i = s_j$ if and only if n divides $j - i$.

In this respect a transducer is simply a clock which tells the time up to k, and then reports the time modulo n. The rule, AUT, is interesting in that it does not use any knowledge operators. Such a rule is valid in temporal logics with past but has rarely been used in proof systems (although it is similar to the AA rule of [11]).

Just as the AUT rule does not use knowledge operators, the SYNC rule does not explicitly use any temporal operators. In fact the complete axiomatization for synchronous systems has no axiom or rule that uses both temporal and epistemic operators. This may be surprising since the language clearly has a strong interaction between time and knowledge. However the SYNC rule allows an implicit interaction between time and knowledge. Suppose that $\vdash \alpha \to \beta$ and $var(\alpha) \cap var(\beta) = \emptyset$. Since α and β do not share any propositional atom, we can only infer β from α if α describes some *structural property* (i.e., some property that is independent of propositions, like “this is an initial state”). By examining the language we can see that the only structural information that can be expressed are conditions on the time, such as $x \wedge \boxminus(x \leftrightarrow \circledw\neg x)$ (“the time is even”). Thus we can only infer β from α if for every time that α could be true in some model, β must be true in every model. Since agents know the time, and are logically omniscient, if α is true then any agent will be able to deduce β. Thus the SYNC rule captures the interaction between knowledge and time in a synchronous system.

For an example of the effectiveness of the SYNC rule, consider the formula (2). Since it is clear that

$$\vdash (x \wedge \boxminus(x \leftrightarrow \circledw \neg x)) \rightarrow (\boxminus(y \leftrightarrow \circledw \neg y) \rightarrow y),$$

by the completeness of the temporal rules and axioms, the provability of (2) follows directly from the SYNC rule and S5 reasoning.

4 Soundness for unique initial states

Suppose the axiom, UIS, was not sound. Then there would be some model M with unique initial states, such that for some $r \in M$ and some j,

$$M, r, j \models \boxminus(\circledw \mathit{false} \rightarrow K_i\alpha) \wedge \neg K_i \boxminus(\circledw \mathit{false} \rightarrow \alpha).$$

Therefore there must be some $r' \in M$ such that $r(j)_i = r'(j)_i$ such that $M, r', j \models \neg \boxminus(\circledw \mathit{false} \rightarrow \alpha)$. Thus $M, r', 0 \models \neg\alpha$, and $M, r, 0 \models K_i\alpha$ contradicting the unique initial states requirement of the model.

5 Completeness for unique initial states

To prove the axiom system augmented with UIS is complete we use a standard Henkin-style construction with finite sets of formulas. Given a consistent formula, ψ, we show that ψ has a model generated from the maximal consistent subsets of some closure set (see, for example [5]). We define the closure set in two stages. Given ψ, let $\Gamma_\psi = \{\alpha, \neg\alpha, \circledw \mathit{false} \mid \alpha \subseteq \psi\}$, where $\alpha \subseteq \psi$ if and only if α is a subformula of ψ. As usual we let Σ be the set of maximally consistent sets of formulas, and $S_\psi = \{\Delta \cap \Gamma_\psi \mid \Delta \in \Sigma\}$. We let $S^0_\psi = \{s \in S_\psi \mid \circledw \mathit{false} \in s\}$.

For the next stage, we let $\Gamma = \Gamma_\psi \cup \{\diamondminus \hat{s} \mid s \in S^0_\psi\}$ where $\widehat{s}$ is the conjunction of the formulas in s. We define $S = \{\Delta \cap \Gamma \mid \Delta \in \Sigma\}$ and define the relations $\rightsquigarrow, \sim_i \subseteq S \times S$ as:

- $s \rightsquigarrow t$ if and only if there exists $\Delta, \Delta' \in \Sigma$ such that $s = \Delta \cap \Gamma$, $t = \Delta' \cap \Gamma$ and for all $\alpha \in \Delta'$, $\bigcirc\alpha \in \Delta$;
- $s \sim_i t$ if and only if there exists $\Delta, \Delta' \in \Sigma$ such that $s = \Delta \cap \Gamma$, $t = \Delta' \cap \Gamma$ and for all $K_i\alpha \in \Delta$, $K_i\alpha \in \Delta'$.

It can be seen using the S5 axioms that $\sim_i$ is an equivalence relation. The following lemma gives an alternative characterization of the relation $\sim_i$.

LEMMA 2. *For all s and t in S, $s \sim_i t$ if and only if $\widehat{s} \wedge L_i\widehat{t}$ is consistent.*

Proof. If $s \sim_i t$ then there exists maximally consistent sets, Δ and Δ' such that $s \subset \Delta$, $t \subset \Delta'$ and $K_i\alpha \in \Delta$ if and only if $K_i\alpha \in \Delta'$. Since $t \subset \Delta'$ it follows that $K_iL_i\widehat{t} \in \Delta'$ so $\widehat{s} \wedge L_i\widehat{t} \in \Delta$, hence $\widehat{s} \wedge L_i\widehat{t}$ is consistent.

Conversely, if $\widehat{t} \wedge L_i\widehat{s}$ is consistent then it is contained in some maximally consistent set, Δ. We define $\Lambda = \{\alpha \mid K_i(\widehat{t} \rightarrow \alpha) \in \Delta\}$. It should be clear that Λ is consistent, and $t \subset \Lambda$. Furthermore, $K_i\alpha \in \Lambda$ if and only if $K_i\alpha \in \Delta$ and $L_i\alpha \in \Lambda$ if and only if $L_i\alpha \in \Delta$. Since Λ is consistent it can be extended to a maximally consistent set which is sufficient to show $s \sim_i t$. ■

For all $s \in S$, we let $[s]_i$ be the corresponding equivalence class of $\sim_i$ and let R be the set of functions $r : \omega \longrightarrow S$ such that:

1. $ⓦ\mathit{false} \in r(0)$;
2. for all n, $r(n) \rightsquigarrow r(n+1)$; and
3. for all n, if $\alpha \mathcal{U} \beta \in r(n)$ then there exists $m \geq n$ such that $\beta \in r(m)$.

From R we can derive a model $M = \{\pi_r : \omega \rightarrow \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_k \mid r \in R\}$ where $\pi_r(j)_0 = r(j) \cap \mathcal{V}$, and $\pi_r(j)_i = [r(j)]_i$. Finally, for every $r \in R$ we let $M_r \subset M$ be defined to be the smallest set such that $\pi_r \in M_r$, and for every $\pi_t \in M_r$, and $j \in \omega$, $\{\pi_u \in M \mid \exists i, j' \text{ s.t. } t(j) \sim_i u(j')\} \subseteq M_r$.

The standard approach here is to extend ψ to a maximal consistent set and use this to find a run r with a state containing ψ. We then prove a truth lemma on M_r, i.e. for every j we show $\alpha \in r(j)$ if and only if $M, \pi_r, j \models \alpha$. Therefore to complete the proof all we have to do is show that the resulting model satisfies the unique initial states constraint. We use the following tautology:

LEMMA 3. $\vdash ⓦ\mathit{false} \rightarrow (\varphi \rightarrow \Box(K_i \boxminus(ⓦ\mathit{false} \rightarrow \neg K_j\neg\varphi)))$

Proof. We define the formula γ as

(3) $\gamma = ⓦ\mathit{false} \wedge (\varphi \wedge \Diamond(L_i \diamondminus (ⓦ\mathit{false} \wedge K_j\neg\varphi)))$.

By taking the contrapositive of UIS we can derive the tautology

(4) $\vdash \neg K_i \neg \diamondminus (ⓦ\mathit{false} \wedge \neg\alpha) \rightarrow \diamondminus (ⓦ\mathit{false} \wedge \neg K_j\alpha)$.

Let $\neg K_j \neg\varphi$. Applied to γ we have

$$
\begin{array}{lll}
UIS & \vdash \gamma \to (@\mathit{false} \wedge \varphi \wedge \Diamond\ominus\!\!\!\!\Diamond (@\mathit{false} \wedge \neg K_j \neg K_j \neg\varphi)) & (5)\\
K5 & \vdash \neg K_j \neg \varphi \to K_j \neg K_j \neg \varphi & (6)\\
K1 & \vdash \neg K_j \neg K_j \neg\varphi \to K_j \neg\varphi & (7)\\
LTL & \vdash \gamma \to (@\mathit{false} \wedge \varphi \wedge \Diamond\ominus\!\!\!\!\Diamond (@\mathit{false} \wedge K_j \neg\varphi)) & (8)\\
K1 & \vdash \gamma \to \varphi \wedge \neg\varphi & (9)\\
K1 & \vdash \neg\gamma & (10)
\end{array}
$$

Since $\neg\gamma$ is equivalent to $@\mathit{false} \to (\varphi \to \Box(K_i \boxminus(@\mathit{false} \to \neg K_j \neg\varphi)))$, the proof is complete. ■

We note that this lemma is the only place where we are required to use the UIS axiom, so this tautology could be used instead of the UIS axiom.

COROLLARY 4. *The model M_r satisfies the unique initial states constraint.*

Proof. If this were not true there would be some runs with non-unique initial states. Thus there would be some $s(0), t(0), s(u), t(v) \in S$ (where $s, t \in R$) such that $s(u) \sim_i t(v)$, but $s(0) \not\sim_j t(0)$. Since $@\mathit{false} \in s(0)$ we can use the Lemma 3 to derive

(11) $\vdash \widehat{s(0)} \to \Box K_i \boxminus(@\mathit{false} \to L_j \widehat{s(0)})$.

Let $a = s(0) \cap \Gamma_\psi$. Given the definition of the closure, Γ, it follows that $\vdash \widehat{s(u)} \to \ominus\!\!\!\!\Diamond(@\mathit{false} \wedge \widehat{a})$). Furthermore, it is clear that $\widehat{s(0)}$ is equivalent to $\widehat{a}$, since $s(0) = a \cup \{\ominus\!\!\!\!\Diamond \widehat{a})\}$. Combining this with (11) we derive $\vdash \widehat{s(u)} \to K_i \boxminus(@\mathit{false} \to L_j \widehat{s(0)})$, and since $s(u) \sim_i t(v)$ it follows using Lemma **??** and S5 reasoning that $\widehat{t(v)} \wedge \boxminus(@\mathit{false} \to L_j \widehat{s(0)})$ is consistent. It follows that $\widehat{t(0)} \wedge L_j \widehat{s(0)}$ must be consistent, contradicting the assumption $s(0) \not\sim_j t(0)$, again by Lemma **??**. ■

6 Soundness for synchronous systems

The soundness of the rule AUT is straightforward, and is left to the reader. To show SYNC is sound, suppose that α and β do not share propositional atoms, $\alpha \to \beta$ is a validity, but $\alpha \wedge L_i \neg\beta$ has some model, M. Therefore there are runs $r_\alpha, r_\beta \in M$ and some j such that $M, r_\alpha, j \models \alpha$, and $M, r_\beta, j \models \neg\beta$, and $r_\alpha(j)_i = r_\beta(j)_i$. Note that the interpretation of α, and the interpretation of β can only depend on the propositional atoms that appear in α or β (this can be seen by the recursive definition of the $\models$ relation).

Now let M^+ be a new model defined by $M^+ = \{r \cdot s | r, s \in M\}$, where the run $r \cdot s$ is defined by $r \cdot s(u) = (a, l_1, \ldots, l_k)$ where

- $a = (r(u)_0 \cap var(\alpha)) \cup (s(u)_0 \cap var(\beta))$
- $l_m = (r(u)_m, s(u)_m)$

Note that M^+ is synchronous if M is synchronous.

We can show that $M, r, j \models \alpha$ if and only if $M^+, r \cdot s, j \models \alpha$ for all runs s of M, and $M, s, j \models \beta$ if and only if $M^+, r \cdots, j \models \beta$ for all runs r of M. (This is done by induction over the complexity of formulas, using the semantic descriptions given, and is left to the reader). If we let $r = r_\alpha$ and $s = r_\beta$, it follows that $M^+, r_a^b, j \models \alpha \wedge \neg\beta$, contradicting the fact that $\alpha \rightarrow \beta$ is a validity.

7 Completeness for synchronous systems

We use the strategy used in [7] to construct the model as a series of levels, where each level defines the depth of nestings of knowledge operators in a formula. Given any consistent formula, ψ, we will create a model (a set of runs) by taking sequences of maximally consistent subsets of a closure set of ψ. We will then show that any formula that appears in a maximal consistent subset will be true at the corresponding state in the model. To create such a model we need to find a sequence of maximally consistent sets, where one of the sets contains ψ. We then need to provide additional runs to ensure that if $L_i\gamma$ appears in some set, γ appears in some other run. However these additional runs can be defined over a smaller closure set since we are only interested in the formulas that appear in the scope of a knowledge operator. We can apply this process recursively until we only have to add runs defined over a closure containing no knowledge operators.

When we add additional runs, we have to ensure the knowledge relations conform to the synchrony constraint as well as the normal rules for epistemic logic. To do this, at each level of the construction we include the characteristic formula of a transducer. The state of the transducer at a given level provides sufficient information about the time for us to be able to deduce which of the sets in a lower level will be inconsistent with the current time. The SYNC rule allows us to use this information to ensure the model satisfies the synchrony constraint. The following definitions contribute to this construction.

Each level in the model is represented by a string of agent indexes, (that is, an element of $\{1, \ldots, k\}^*$). We call such strings *knowledge sequences*, and use the following notation:

- we let λ refer to the empty string;

- we let τi be the string τ, concatenated with the index i and let $(\tau i)^- = \tau$;
- we let $\tau \backslash i$ be the largest string μ such that μi is a prefix of τ, or λ if such a string does not exist; and
- we define $\tau \leq \sigma$ ($\tau < \sigma$) to mean τ is a (proper) prefix of σ.

To construct a model of a consistent formula we will use the following hierarchy of languages. We let $\mathcal{L}$ be the language defined above (for k agents), and define the hierarchy over knowledge sequences.

1. $\mathcal{L}_\lambda = \{\alpha \in \mathcal{L} \mid \forall \beta \in \mathcal{L}, \forall i\ K_i\beta \not\subseteq \alpha\}$.
2. $\mathcal{L}_{\tau i} = \{\alpha \in \mathcal{L} \mid K_j\beta \subseteq \alpha \Rightarrow \text{ either } j = i \text{ and } \beta \in \mathcal{L}_\tau \text{ or } K_j\beta \in \mathcal{L}_\tau\}$.

We can see that $\mathcal{L}_\lambda$ is the set of all pure temporal formulas, and let σ be the smallest string such that $\psi \in \mathcal{L}_\sigma$.

We will now define the *closure* of a formula, ψ.

DEFINITION 5. Given a formula, ψ, we let Γ_ψ be the *closure* of ψ, defined recursively by:

- $\psi \in \Gamma_\psi$.
- $\alpha \subseteq \varphi$ implies $\alpha, \neg\alpha \in \Gamma_\psi$
- $\alpha \in \Gamma_\psi$ implies $\neg K_i\alpha \in \Gamma_\psi$ and $K_i\alpha \in \Gamma_\psi$ for $i = 1, \ldots, k$.

Given a knowledge sequence, τ, we define the τ*-closure* of ψ to be $\Gamma^\tau_\psi = \Gamma_\psi \cap \mathcal{L}_\tau$.

To be able to create a model we require that maximal consistent subsets of the closure contain sufficient information about the time. We do this as follows. Let Σ be the set of *maximally consistent sets* of formulas taken from the language (with respect to the axioms given and the two rules for synchrony), and given a set X of formulas, we let S_X be the set of maximally consistent subsets of X, (ie $S_X = \{\Delta \cap X \mid \Delta \in \Sigma\}$).

We define the temporal relation $\rightsquigarrow \subseteq S_X \times S_X$ by $s \rightsquigarrow t$ if and only if $\bigwedge_{\alpha\in s} \alpha \wedge \bigcirc \bigwedge_{\alpha\in t} \alpha$ is consistent.

The knowledge relations are quite complex, and will be constructed using the following definitions and lemmas. These constructions are given so that if we are considering formulas in $\mathcal{L}_{\tau i}$, then the closure includes an additional formula, χ_τ, that describes a transducer, A_τ. A run of this transducer associates a state with each moment of time and this state describes the set of maximal consistent subsets of $\mathcal{L}_\tau$ which are consistent with the given time. We do this by induction, where the base case is

$$X_\lambda = \Gamma^\lambda_\psi \cup \Gamma^\lambda_{\diamondminus ⓦ\mathit{false}}.$$

Given X_τ, for any τ, we can then define S_τ (the maximally consistent subsets of X_τ), A_τ (a transducer showing which subsets are consistent with which times), χ_τ (the characteristic formula of the transducer), and $X_{\tau i}$ (the inductive step). This is done as follows:

- $S_\tau = S_{X_\tau}$.
- For all τ, given S_τ and $\rightsquigarrow$ (defined above) we let A_τ be a transducer given by the tuple $(Q_\tau, p_\tau, \delta_\tau)$ where:
  - $Q_\tau = \wp(S_\tau)$ is the set of states;
  - $p_\tau = \{s \in S_\tau \mid ⓦ\mathit{false} \in s\}$
  - $\delta_\tau : Q_\tau \to Q_\tau$ is the transition function defined by $\delta_\tau(q) = \{t \mid \exists s \in q,\ s \rightsquigarrow t\}$.

  This transducer is defined to identify states which are reachable in the constructed model at a given time. The run of A_τ is the sequence from Q_τ, $(p_\tau, \delta_\tau(p_\tau), \delta^2_\tau(p_\tau), \ldots)$.
- χ_τ is the characteristic formula of A_τ. To define χ_τ, for each $s \in S_\tau$, let x_s be a propositional atom not appearing in Γ_τ, and for all $q \in Q_\tau$, let $\overline{q} = \bigwedge_{s\in q} x_s \wedge \neg \bigvee_{s \in S_\tau \setminus q} x_s$. Then

$$\chi_\tau = \diamondminus \left(ⓦ\mathit{false} \wedge \overline{p_\tau} \wedge \Box \bigwedge_{q \in Q_\tau} (\overline{q} \to \bigcirc \overline{\delta_\tau(q)}) \right).$$

- $X_{\tau i} = \Gamma^{\tau i}_\psi \cup \Gamma^\lambda_{\chi_\tau}$.

The proof of completeness will follow from the following lemmas. The first three are technical lemmas which contribute to the proof of the fourth.

LEMMA 6. *For all τ, and $q \in Q_\tau$, we have* $\vdash \chi_\tau \wedge \overline{q} \to K_i \bigvee_{s\in q} \hat{s}$.

Proof. By the construction the transducer A_τ the initial state p_τ is the set of all maximal consistent subsets which are consistent with ⓦ*false*, so the following is a validity

(12) $\vdash ⓦ\mathit{false} \wedge \chi_\tau \to \bigvee_{s \in p_\tau} \hat{s}.$

Since $\vdash \chi_\tau \to \diamondminus(\circleddash false \wedge \chi_\tau)$, we can derive

$$(13)\ \vdash \chi_\tau \wedge \overline{q} \to \diamondminus\left(\chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right)\right).$$

The transition function δ_τ is defined to map a set, q, of maximal consistent subsets to the set of all maximal consistent subsets that are consistent with $\ominus \bigvee_{s \in q} \hat{s}$. Since $\vdash \chi_\tau \wedge \overline{q} \to \bigcirc\overline{\delta(q)}$ and $\vdash \bigvee_{s \in q} \hat{s} \to \bigcirc \bigvee_{s \in \delta(q)} \hat{s}$ are tautologies we can derive the following:

$$(14)\ \vdash \chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right) \to \bigcirc\left(\chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right)\right).$$

Applying the rule, RT2, we can show

$$(15)\ \vdash \chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right) \to \square\left(\chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right)\right),$$

and the following tautology follows from (15) and (13):

$$(16)\ \vdash \chi_\tau \wedge \overline{q} \to \left(\chi_\tau \wedge \bigvee_{q \in Q_\tau}\left(\overline{q} \wedge \bigvee_{s \in q} \hat{s}\right)\right).$$

Since by definition, $\vdash \overline{q} \to \bigwedge_{r \neq q} \neg\overline{r}$, we have

$$(17)\ \vdash \chi_\tau \wedge \overline{q} \to \bigvee_{s \in q} \hat{s}.$$

Since the propositional atoms in χ_τ are defined to be disjoint from those in $\bigvee_{s \in q} \hat{s}$ the result follows from the SYNC rule. ■

This lemma shows that for any elements of S_τ which are not consistent with $\chi_\tau \wedge \overline{q}$, given $\chi_\tau \wedge \overline{q}$ we can prove an agent knows that those elements of S_τ are not true. The construction we will use requires us to also prove an agent knows which elements of $S_{\tau \setminus i}$ are not consistent with $\chi_\tau \wedge \overline{q}$. To do this we use the following lemma.

LEMMA 7. *For all knowledge sequences τ, for all $j \in \omega$ and for all $\mu \leq \tau$, $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \to \left(\chi_\mu \to \overline{\delta^j_\mu(p_\mu)}\right)$ where the propositional atoms in χ_τ are disjoint from the propositional atoms in χ_μ*

Proof. We will prove this by induction. The base case is the tautology

(18) $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \rightarrow \left(\chi'_\tau \rightarrow \overline{\delta^j_\tau(p_\tau)'}\right)$

where the sets of propositional atoms in χ'_τ and χ_τ are disjoint. The base case is a temporal validity (since transducers are deterministic) so we can assume this is provable from the temporal axioms and rules.

For the inductive step we will first show

(19) $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \rightarrow \left(\chi_{\tau^-} \rightarrow \overline{\delta^j_{\tau^-}(p_{\tau^-})}\right)$

where the sets of propositional atoms in χ_τ and χ_{τ^-} are disjoint. Since (19) is a pure temporal formula, it is sufficient for us to show it is valid using semantic reasoning. Since χ_τ is always satisfiable, if $p = \delta^n_\tau(p_\tau)$ then there must be some $s \in p$ and some $q \in Q_{\tau^-}$ such that $\{\chi_{\tau^-}, \overline{q}\} \subset s$. Furthermore, given such an s for all $t \in p$ if $\chi_{\tau^-} \in t$, then $\overline{q} \in t$ (since by definition $s, t \in p$ if and only if s and t can be reached in n steps from an initial set, and for every transducer there is always a unique state that can be reached in n steps from the initial state). Therefore (19) is valid.

To complete the induction, suppose that for some $\mu \leq \tau$

(20) $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \rightarrow \left(\chi_\mu \rightarrow \overline{\delta^j_\mu(p_\mu)}\right).$

Combining this with (19) we can derive

(21) $\vdash \chi_\mu \rightarrow \left(\chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \rightarrow \left(\chi_{\mu^-} \rightarrow \overline{\delta^j_{\mu^-}(p_{\mu^-})}\right)\right).$

Since we can assume that the sets of propositional atoms in χ_τ, χ_μ and χ_{μ^-} are all disjoint, the AUT rule gives us

(22) $\vdash \chi_\tau \wedge \overline{\delta^j_\tau(p_\tau)} \rightarrow \left(\chi_{\mu^-} \rightarrow \overline{\delta^j_{\mu^-}(p_{\mu^-})}\right)$

and the lemma follows by induction. ■

We will now restrict our attention to sets $s \in S_{\tau i}$, such that $\chi_\tau \in s$. We define $T_\lambda = S_\lambda$ and let $T_{\tau i} = \{s \in S_{\tau i} \mid \chi_\tau \in s\}$. By the construction of the set $S_{\tau i}$ and the rule, AUT, every consistent formula in $\Gamma^{\tau i}_\psi$ must be an element of some set in $T_{\tau i}$. Given any set, t, of formulas we let $t^i = \{\alpha \mid K_i\alpha \in t\}$. We require the following definition to allow us to compare maximal consistent subsets at different levels.

DEFINITION 8. For all $\tau \neq \lambda$, we define the relation $\prec_i \subset T_{\tau\backslash i} \times T_\tau$ and say $t \prec_i s$ (t *i-supports* s) if

- For some $q \in Q_{\tau^-}$, $\overline{q} \in s$ and there is some $a \in q$ such that $t \subseteq a$.

- $t^i \subseteq s^i \subseteq t$.

This definition is constructed such that for all $s \in T_\tau$, and for all $t \in T_{\tau\backslash i}$, $t \prec_i s$ if and only if $\widehat{s} \wedge L_i\widehat{t}$ is consistent.

LEMMA 9. *For all $s \in T_\tau$, for all $t \in T_{\tau\backslash i}$, if $t \not\prec_i s$ then $\vdash \hat{t} \rightarrow K_i\neg\hat{s}$.*

Proof. If $t \not\prec_i s$ then there are three cases we must consider:

1. for all $q \in Q_{\tau^-}$, if $\overline{q} \in s$ then for all $a \in q$, $t \not\subseteq a$. From Lemma 6 we know $\vdash \chi_{\tau^-} \wedge \overline{q} \rightarrow \bigvee_{a\in q} \hat{a}$, so it follows $\vdash \hat{t} \rightarrow \neg(\chi_{\tau^-} \wedge \overline{q})$, and thus by the SYNC rule, $\vdash \hat{t} \rightarrow K_i\neg(\chi_{\tau^-} \wedge \overline{q})$. As $\vdash \neg(\chi_{\tau^-} \wedge \overline{q}) \rightarrow \neg\hat{s}$ the proof follows from K2 and R1.

2. $s^i \not\subseteq t$. In this case there must be some $K_i\gamma \in s$ such that $\gamma \notin t$. Since $\gamma \in X_{\tau\backslash i}$ we must have $\vdash \hat{t} \rightarrow \neg\gamma$. Applying the epistemic axioms K3, K5 we can show $\vdash \hat{t} \rightarrow K_iL_i\neg\gamma$ and by the rule R2 we can show $\vdash K_i(L_i\neg\gamma \rightarrow \neg\hat{s})$. The result follows from K2.

3. $t^i \not\subseteq s^i$. In this case there must be some $K_i\gamma \in t$ such that $K_i\gamma \notin s$. In this case the result follows trivially from R1 and K2.

■

We note that the converse of this lemma is a consequence of Lemma 15. The following lemma allows us create a synchronous model from a set of runs corresponding to different knowledge sequences.

LEMMA 10. *For all τ, for all j, for all $s \in \delta^j_\tau(p_\tau) \cap T_\tau$, if $L_i\gamma \in s$, then there is some $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i}) \cap T_{\tau\backslash i}$ such that $\gamma \in t$ and $t \prec_i s$.*

Proof. We use the Lemma 6 and Lemma 7 to prove this lemma as follows. Suppose for contradiction that there exists some knowledge sequence, τ, some $j \in \omega$, some $s \in \delta^j_\tau(p_\tau) \cap T_\tau$ and some $L_i\gamma \in s$ such that for all $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i}) \cap T_{\tau\backslash i}$, if $t \prec_i s$, then $\gamma \notin t$. We can convert this statement into a formula and use the proof theory to derive a contradiction.

For all $t \in \delta^j_{\tau\backslash i}(p_{\tau\backslash i})$, either $t \not\prec_i s$, or $\gamma \notin t$, or $t \notin T_{\tau\backslash i}$. Thus, by Lemma 9, the following would be a tautology:

$$(23)\ \vdash \bigwedge_{t\in\delta^j_{\tau\backslash i}(p_{\tau\backslash i})} \left(\widehat{t} \rightarrow (K_i\neg\widehat{s} \vee \neg\gamma \vee \neg\chi_{\tau\backslash i^-})\right).$$

Note in the case that $\tau\backslash i = \lambda$ we can consider $\chi_{\tau\backslash i^-}$ to be the formula, *true.*

Since $s \in \delta^j_\tau(p_\tau) \cap T_\tau$ it follows that $\chi_{\tau^-}, \overline{\delta^j_{\tau^-}(p_{\tau^-})} \in s$. By the Lemma 6, Lemma 7 and the AUT rule we can deduce

(24) $\vdash \widehat{s} \to K_i \bigvee_{t \in \delta^j_{\tau\setminus i}(p_{\tau\setminus i})} \widehat{t}$

Putting this together with (23) we can show

(25) $\vdash \widehat{s} \to K_i(K_i\neg\widehat{s} \vee \neg\gamma \vee \neg\chi_{\tau\setminus i^-})$.

We can then apply basic epistemic reasoning and the AUT rule to show $\vdash \neg\widehat{s}$, contradicting the fact that s is consistent. ■

Lemma 10 gives us the sufficient machinery to complete the proof. If ψ is consistent, then for some knowledge sequence, σ, ψ must belong to Γ^σ_ψ and ψ must be consistent with χ_τ, for all τ. Therefore we can find some $s \in T_\sigma$ such that $\psi \in s$. It is clear that the relation $\leadsto$ can be restricted to T_σ for all σ, so we can use this to create a σ-history (an infinite $\leadsto$-sequence in T_σ) where all eventualities are satisfied. For every set in this history we can then satisfy any knowledge formulas using Lemma 10.

The construction we will use here is given as follows: A *ranked set of height* σ is a disjoint union $R = \bigcup_{\tau\leq\sigma} R_\tau$, where for each $\tau \leq \sigma$, R_τ is a set.

For each r in R_τ we associate a τ-history via a *labeling*, described in the following definition:

DEFINITION 11. A *labeling*, ℓ, of a ranked set R of height σ is a collection of functions $\ell_\tau : R_\tau \times \omega \longrightarrow T_\tau$ for $\tau \leq \sigma$ where:

1. for all $r \in R_\tau$, $\ell_\tau(r, 0) \in p_\tau$;
2. for all $r \in R_\tau$, for all $j \in \omega$, $\ell_\tau(r, j) \leadsto \ell_\tau(r, j + 1)$;
3. for all $r \in R_\tau$, for all $j \in \omega$ for all $\alpha\,\mathcal{U}\,\beta \in \ell_\tau(r, j)$ there is some $i \geq j$ such that $\beta \in \ell_\tau(r, i)$.

Hence, for any labeling ℓ, for any $r \in R_\tau$, $\ell_\tau(r, 0)\ell_\tau(r, 1)\ell_\tau(r, 2)\ldots$ will be a τ-history. The construction must also satisfy all the knowledge formulas. To do this we use the observation that if $L_i\gamma$ appears at some level (say, $L_i\gamma \in \ell_\tau(r, j)$ where $r \in R_\tau$), then a history labeled by an element of $R_{\tau\setminus i}$ is all that is required to satisfy this formula. In this case we need to find a *witness* for $L_i\gamma$ in $R_{\tau\setminus i}$ that is consistent with both $L_i\ell\tau(\hat{r}, j)$ and the time j. By Lemma 10 we know this is always possible. To allocate these witnesses we use the following definition:

DEFINITION 12. A *system of support*, ρ, for a ranked set R of height σ equipped with a labeling ℓ consists of, for all $\tau < \sigma$, for all agents i, a partial function $\rho^i_\tau : R_{\tau\setminus i} \hookrightarrow R_\tau \times \omega$, such that

1. for all $r \in R_\tau$, for all $j \in \omega$, if $\rho^i_\tau(t) = (r, j)$, then $\ell_{\tau\backslash i}(t, j) \prec_i \ell_\tau(r, j)$.
2. for all $r \in R_\tau$ for all $j \in N$, if $L_i\gamma \in \ell_\tau(r, j)$ then exactly one of the following holds:
   - there is some $t \in R_{\tau\backslash i}$ such that $\rho^i_\tau(t) = (r, j)$ and $\gamma \in \ell_\tau(t, j)$.
   - there is some $t \in R_\mu$ such that $\mu\backslash i = \tau$, and $\rho^i_\mu(r) = (t, j)$.

We note that in some cases the system of support does not directly allocate a witness for some formula $L_i\gamma$. This occurs if $L_i\gamma \in \ell_\tau(r, j)$, and for some μ, $\rho^i_\mu(r) = (t, j)$. In this case $L_i\gamma$ appears in a set that is itself a witness for a set, $\ell_\mu(t, j)$, at a higher level. In this case we must have $L_i\gamma \in \ell_\mu(t, j)$, and the witness for $L_i\gamma$ in $\ell_\mu(t, j)$ is sufficient to also witness $L_i\gamma$ in $\ell_\tau(r, j)$. This gives us enough to define the basic structure.

DEFINITION 13. Let $\psi \in \Gamma_n$ be a formula, and σ an index such that $\psi \in \mathcal{L}_\sigma$. A *$\psi$-frame* is a triple (R, ℓ, ρ) where R is a ranked set of height σ, ℓ is a labeling of R and ρ is a system of support for R and ℓ, such that for some $r \in R_\sigma$, and some $j \in \omega$, we have $\psi \in \ell_\sigma(r, j)$.

LEMMA 14. *Given any consistent formula, ψ, there exists a ψ-frame.*

This is left to the reader. The existence of i-τ-supports follows from lemma 10, and the existence of the τ-labellings follows from the usual reachability arguments.

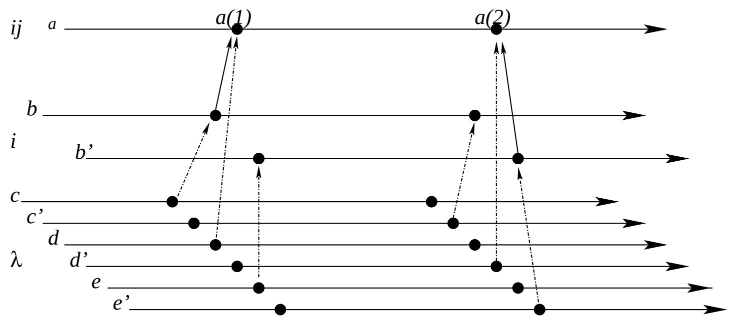

Figure 1. A basic φ-frame

A basic φ-frame is depicted in Figure 1. In this example $\sigma = ij$ and the levels of knowledge (λ, i, or ij) increase from bottom to top. Formulas containing K_j will only appear in the highest level (ij), whilst no formulas containing only epistemic operators will appear in the lowest level (λ). Time flows from left to right, a labels a run R_{ij}, b, b' label runs in R_i and $c, \ldots e'$

label runs in R_λ. The solid arrows between levels correspond to the partial function ρ^j_{ij} (so $\rho^j_{ij}(b) = (a, 1)$), and the dashed arrows correspond to ρ^i_i (so $\rho^i_i(d') = (a, 2)$).

Given a ψ-frame F, we can now construct a model, $M_F \subseteq \mathcal{R}$ (see (1)) as follows.

- We let the set of local states $\mathcal{L}_i$ be $R \times \omega$.
- For all $\tau \leq \sigma$, for each $r \in R_\tau$ we define a function $\pi_r : \omega \to \wp(\mathcal{V}) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_d$ by $\pi_r(j) = (a, l_1, \ldots, l_d)$ where:
  - $a = \ell_\tau(r, j) \cap \mathcal{V}$;
  - for each $i = 1, \ldots d$ if $\rho^i_\tau(r) = (t, j)$, then $l_i = \pi_t(j)_i$, and otherwise $l_i = (r, j)$.

It is clear that the model M_F is synchronous.

LEMMA 15. *For all $\tau \leq \sigma$, $r \in R_\tau$, for all $j \in \omega$, and for all $\varphi \in \Gamma^\tau_\psi$*

$$M_F, \pi_r, j \models \varphi \iff \varphi \in \ell_\tau(r, j).$$

Proof. This is shown in the usual way, by induction over the complexity of formulas. We note that the definition ensures that for all propositional atoms x, $M_F, \pi_r, j \models x$ if and only if $x \in \ell_\tau(r, j)$. Given a formula α, the inductive hypothesis is $M_F, \pi_r, j \models \alpha \iff \alpha \in \ell_\tau(r, j)$. Assuming α and α' satisfy the inductive hypothesis it can be shown $\alpha \wedge \alpha'$, $\neg\alpha$, $\bigcirc\alpha$, Ⓦα, $\alpha \mathcal{U} \alpha'$ and $\alpha \mathcal{S} \alpha'$ also satisfy the inductive hypothesis. This is relatively simple and is left to the reader (note that the $\mathcal{U}$ case relies on the last part of Definition 11).

The only interesting case is the inductive step for the knowledge operator, where $\varphi = K_i\alpha$. We assume by the inductive hypothesis that for all $\tau \leq \sigma$, for all $r \in R_\tau$, and for all $j \in \omega$, $M_F, \pi_r, j \models \alpha \iff \alpha \in \ell_\tau(r, j)$.
Suppose that $M_F, \pi_r, j \models \varphi$. In this case $M_F, \pi_r, j \models \alpha$ and for all t such that $\pi_r(j)_i = \pi_t(j)_i$ we have $M_F, \pi_t, j \models \alpha$. By the construction of M_F we have two possibilities:

1. For all $t \neq r$, such that $\pi_r(j)_i = \pi_t(j)_i$, $t \in R_{\tau\setminus i}$ and $\rho^i_\tau(t) = (r, j)$. By the induction hypothesis, for all such t where $\rho^i_\tau(t) = (r, j)$, $\alpha \in \ell_{\tau\setminus i}(t, j)$. Suppose for contradiction $\varphi \notin \ell_\tau(r, j)$. Then $L_i\neg\alpha \in \ell_\tau(r, j)$ and by the definition of ρ^i_τ there would be some t such that $\rho^i_\tau(t) = (r, j)$ and $\alpha \notin \ell_{\tau\setminus i}(t, j)$, giving us the required contradiction. Thus $\varphi \in \ell_\tau(r, j)$.

2. There is some μ and some $s \in T_\mu$ such that $\mu\backslash i = \tau$ and $\rho_\tau(j) = s$. For all $t \neq s$, such that

   $\pi_r(j)_i = \pi_t(j)_i$, we have $\rho^i_\mu(t) = (s, j)$ and hence $\alpha \in \ell_\tau(t, j)$. Since $\varphi \in \Gamma^{\mu\backslash i}_\psi$ we must have $K_i\varphi \in \Gamma^\mu_\psi$. If $L_i\neg\varphi \in \ell_\mu(s, j)$, by K4 we have $L_i\neg\alpha \in \ell_\mu(s, j)$ and by the definition of ρ^i_μ there must be some $t \in R_\tau$ such that $\rho^i_\mu(t) = (s, j)$ and $\alpha \notin \ell_\tau(t, j)$, contradicting the induction hypothesis. Therefore we must have $K_i\varphi \in \ell_\mu(s, j)$ and hence $\varphi \in \ell_\mu(s, j)^i$. By the definition of $\prec_i$ it follows that $\varphi \in \ell_\tau(r, j)$.

For the converse, suppose $\varphi \in \ell_\tau(r, j)$. Again we consider two possibilities:

1. For all $t \neq r$ such that $\pi_r(j)_i = \pi_t(j)_i$, $t \in R_{\tau\backslash i}$ and $\rho^i_\tau(t) = (r, j)$. In this case by the definition of ρ^i_τ and $\prec_i$, we have $\ell_\tau(r, j)^i \subseteq \ell_{\tau\backslash i}(t, j)$. Consequently $\alpha \in \ell_{\tau\backslash i}(t, j)^i$, and by K3, $\alpha \in \ell_\tau(r, j)$. By the inductive hypothesis, for all t such that $\pi_r(j)_i = \pi_t(j)_i$, we have $M_F, \pi_t, j \models \alpha$, so $M_F, \pi_r, j \models K_i\alpha$.

2. There is some μ, and $j \in \omega$ and some $s \in T_\mu$ such that $\mu\backslash i = \tau$ and $\rho(r) = (s, j)$. For all $t \neq s$ such that $\pi_r(j)_i = \pi_t(j)_i$, we have $\rho^i_\mu(t) = (s, j)$. Since $\ell_\tau(r, j) \prec_i \ell_\mu(s, j)$, $K_i\alpha \in \ell_\tau(r, j)$ implies $K_i\alpha \in \ell_\mu(s, j)$. For all $t \neq s$ such that $\pi_r(j)_i = \pi_t(j)_i$, we have $\rho^i_\mu(t) = (s, j)$ and thus $\ell_\tau(t, j) \prec_i \ell_\mu(s, j)$. By the definition of $\prec^i$, $K_i\alpha \in \ell_\mu(s, j)$ implies $\alpha \in \ell_\tau(t, j)$ for all such t. Since we must also have $\alpha \in \ell_\mu(s, j)$ (by K3) the results follows from the induction hypothesis.

■

8 Conclusion

In this paper we have presented sound and complete axiomatizations for logics of knowledge and past time with the synchrony and unique initial states constraints. While the proof of completeness for the unique initial states restriction is relatively straightforward, the proof of completeness for the synchrony restriction is surprisingly complicated. The axiomatization for synchrony relies on complex automata based rules, and finding a simpler axiomatization (and proof) would be of some interest.

For future work we will be investigating combinations of knowledge and past time given the semantic restrictions of *perfect recall* (where an agent retains the knowledge of previous times), and *no learning* (where an agent's knowledge can not increase over time) [1]. Along with synchrony and unique initial state restriction, this gives us sixteen different combinations of restrictions to consider. We will investigate axiomatizations for the resulting languages and extending these axiomatizations to include axioms for common knowledge.

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