Global constraints

• Capture common patterns
  – Alldifferent(X₁,…,Xₙ)
  – Nvalues(N,X₁,…,Xₙ)
  – Lex([X₁,…,Xₙ],[Y₁,…,Yₙ])

• Efficient & effective specialized propagators
  – Prune parts of search tree
Our hammer

- Use basic tools of computational complexity to study limits of
  - reasoning with global constraints
  - global constraints for breaking symmetry
Limits of Global Constraints

- Enforce lesser consistency
- Constraints cannot be combined
- Constraints cannot be generalized
- Decomposition will hurt pruning
In general

• Consider (generalized) arc-consistency
  – Every value for every variable can be extended to satisfy the constraint
  – That is, every value has support

• Similar results for other local consistencies
  – Bounds consistency for integer variables
  – Bounds consistency for set variables
  …
In general

- Global constraints are intractable
  - \text{ACSupport?} \text{ is NP-complete}
  - Does this value have support?
- Consider \( C(X_1, \ldots, X_n) \)
  - Where \( X_i=j \) implies \( X_j=\text{true} \), \( X_i=-j \) implies \( X_j=\text{false} \)
  - \text{SAT in} \( k \) \text{ vars,} \( j \) clauses \( \rightarrow \) \( C(X_1, \ldots, X_{j+k}) \)
  - \( X_1 \) to \( X_k \in \{\text{true, false}\} \)
  - \text{ith clause is} \( x_1 \lor -x_3 \lor x_5 \rightarrow X_{k+i} \in \{1, -3, 5\} \)
  - Consider reduction of: \( \{x_1, -x_1 \lor x_2\} \)
In general

• Global constraints are intractable
  – ACSupport? is NP-complete
    • Does this value have support?
  – MaxAC? is DP-complete
    • Are these domains the maximal arc-consistent domains?
• DP is NP \cup \text{coNP}
  • Answers NP question: are these domains AC? Yes!
  • Answers coNP questions: is any smaller domain AC? No!
In general

- Global constraints are intractable
  - ACSupport? is NP-complete
  - MaxAC? is DP-complete

- Even some specific constraints proposed in the past are intractable
  - NValues(N,X1,..Xn)
  - AtMost1(S1,..,Sn)
  - ...
NV

Values

- NValues\(N,X_1,\ldots,X_n\)
  - N values used in \(X_1,\ldots,X_n\)
  - Useful for resource allocation
- Simple reduction of SAT to NValues
  - SAT problem in \(k\) vars, \(j\) clauses
  - \(X_i = \{i,-i\}\) for \(1 \leq i \leq k\)
  - \(X_{k+i} = \{1,-3,5\}\) if \(i\)-th clause is: \((x_1 \lor -x_3 \lor x_5)\)
  - \(N = \{n\}\)
  - Consider reduction of: \(\{x_1, -x_1 \lor x_2\}\)
NValues

• NValues(N,X1,...,Xn)
  – N values used in X1,...,Xn
  – Useful for resource allocation
• Simple reduction of SAT to NValues
  – Finding support (and hence enforcing arc-consistency) is NP-hard
  – Look to enforce lesser level of local consistency like bound consistency
Composing constraints

- Take two tractable constraints
  - E.g. Disjoint(S₁,…,Sₙ) and FixedCard(S₁,…,Sₙ)
- Could we combine them into one bigger global constraint?
  - E.g. FixedCardDisjoint(S₁,…,Sₙ)
  - No, NP-hard to propagate!
GCC

- Take a tractable constraint
  - E.g. GCC([X₁,..,Xₙ],[l₁,..,lₘ],[u₁,..,uₘ])
  - Value j occurs between lᵢ and oᵢ times in X₁,..,Xₙ
- Generalize some constants to variables
  - E.g. GCC([X₁,..,Xₙ],[O₁,..,Oₘ])
  - Value j occurs Oᵢ times in X₁,..,Xₙ
- NP-hard to make generalized arc-consistent
  - [Claude-Guy Quimper 2003]
• Reduction of 1 in 3 SAT on +ve clauses to GCC
• If ith clause is \( x_1 \lor x_3 \lor x_5 \) then \( X_i \in \{1, 3, 5\} \)
• \( O_j \in \{0, k\} \) where \( k \) is number of occurrences of \( x_j \) in clauses
• Consider \( \{x_1 \lor x_2 \lor x_4, x_2 \lor x_3 \lor x_4, x_1 \lor x_3 \lor x_4\} \)
Decomposing constraints

• Consider a global constraint that is NP-hard to propagate
  – E.g. AtMost1 (S1,…,Sn)
• Consider a decomposition into smaller constraints
  – |Si ∩ Sj| ≤ 1 for all i<j
• If it is polynomial to propagate decomposition
  – decomposition must hinder propagation (assuming P≠NP)
Symmetry breaking

• Add (global) constraints to eliminate symmetries
  – E.g. lex order rows, lex order cols
• Can we break all row & col symmetry with a single global constraint?
  – Enforcing GAC on such a global constraint is NP-hard
Conclusions

• Computational complexity is a useful hammer to study global constraints
• Uncovers fundamental limits of reasoning with global constraints
  – Lesser consistency needs to be enforced
  – Decomposition hurts pruning
  – Composition or generalization intractable
  – Symmetry breaking is inherently limited

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