ON DIRECTED BALANCED INCOMPLETE BLOCK DESIGNS WITH BLOCK SIZE FIVE

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ABSTRACT. A directed balanced incomplete block design (DBIBD), with parameters (v,b,r,k,λ^*), is a balanced incomplete block design (BIBD), with parameters $(v,b,r,k,2\lambda^*)$, in which the blocks are regarded as ordered k-tuples and in which each ordered pair of elements occurs in λ^* blocks. By generalizing results of Hanani, we show that, when $k = 5$, the necessary conditions for the existence of these designs are sufficient, with one exception: namely the DBIBD(15,21,7,5,1*) does not exist.

1. *Introduction.*

A *directed balanced incomplete block design* (DBIBD), with parameters (v,b,r,k,λ^*), is a balanced incomplete block design (BIBD), with parameters $(v,b,r,k,2\lambda^*)$, in which blocks are regarded as k-tuples and in which each *ordered* pair of elements occurs in λ^* blocks. Thus, given the block (x,y,z), we say the three ordered pairs (x,y), (y,z), and (x,z) occur in it.

These designs are, in fact, designs on a directed graph; see, for example, Harary, Wallis, and Heinrich [5] and the references therein. They may also be regarded as a generalization of crossover designs.

It has been shown [6,7] that when $k = 3$ or 4 the necessary conditions for the existence of a DBIBD are sufficient. By generalizing results of Hanani [4], we show that for $k = 5$, with the exception of the non-existent DBIBD(15,21,7,5,1*), the necessary conditions are again sufficient.

2. *Preliminary Definitions and Results.*

We define a group divisible design as in Hanani [4]:

let X be a v-set such that $X = \bigcup_{i=1}^{t} G_i$, $G_i \cap G_j = \phi$, $i \neq j$, $|G_i| = m$ for all i. The G_i are called *groups*. A *group divisible design,* $GD[k,\lambda,m;v]$, is a collection of k-subsets of the v-set X (called *blocks*) such that each block intersects each group in at most one element and a pair of elements of X from different groups occurs in exactly λ

blocks. In a similar way we can define a $GD[K,\lambda,M;v]$, where the size of each block is an element of K and the size of each group is an element of M. In any case, $GD(K,\lambda,M)$ denotes the set of all v such that a $GD[K,\lambda,M;v]$ exists.

A *directed* group divisible design, $DGD[K,\lambda^*,M;v]$, is a $GD[K,2\lambda^*,M;v]$ in which each *ordered* pair of elements from different groups occurs in exactly λ^* blocks. Again $DGD(K,\lambda^*,M)$ is the set of all v such that a $DGD[K,\lambda^*,M;v]$ exists.

Similarly $B(k,\lambda)$ is the set of all v such that a $BIBD(v,b,r,k,\lambda)$ exists and $DB(k,\lambda^*)$ is the set of all v such that a $DBIBD(v,b,r,k,\lambda^*)$ exists.

Often, designs will be specified by giving one, or more, initial blocks, and instructions on how they should be developed. Thus "mod p" means: "to each element of the initial block, add in turn each of the non-zero elements of $GF(p)$, using the addition in $GF(p)$". Similarly, "mod(p,q)" means: "to each ordered pair in the initial block, add in turn each nonzero element of $GF(p) \times GF(q)$". Where appropriate, the primitive polynomial of the field will also be given.

We will now prove a number of basic results, the first of which is a generalization of a lemma in [7].

LEMMA 1. *If* $n \in GD(S,1,R)$, $mR + 1 \subset DB(k,\lambda^*)$ *and* $mS \subset DGD(k,\lambda^*,m)$ *then* $mn + 1 \in DB(k,\lambda^*)$.

Let q denote a prime power and let $GF(q)^*$ denote the multiplicative group of the finite field $GF(q)$. The proof of the following result is straight-forward.

LEMMA 2. *Let* $S = \{1,x,x^2,x^3,x^4\}$, *where* x *is a generator of* $GF(q)^*$ *and* $q \equiv 3 \pmod 4$, $q > 3$. *Then the initial blocks* $S, x^2S, x^4S, \ldots, x^{q-3}S$ *give a* $\left(q, \binom{q}{2}, \frac{5(q-1)}{2}, 5, 10\right)$ *BIBD. If* S *can be arranged so that the* 10 *ordered differences of* S *contain* 5 *squares and* 5 *nonsquares then there exists a* $\left(q, \binom{q}{2}, \frac{5(q-1)}{2}, 5, 5^*\right)$ *DBIBD.*

COROLLARY 2.1. $N_5 \subset DB(5,5^*)$, *where* $N_5 = \{7,19,23,43,47,67,27\}$.

Proof. Suitable orderings for S, in order, are: $\{1,3,2,6,4\}$, $\{1,2,4,16,8\}$, $\{1,5,10,2,4\}$, $\{1,3,27,9,38\}$, $\{1,5,25,31,14\}$, $\{1,2,4,8,16\}$, and $\{1,x,x^2,x^3,x^4\}$, where $x^3 = x + 2$, and x is a generator of $GF(27)^*$.

The next result is a generalization of [4, Lemma 4.12].

LEMMA 3. *If* q *is an odd prime power then* $5q \in DB(5,1^*)$.

Proof. If $q \equiv 1 \pmod 4$ the result follows from [4, Lemma 4.13]. We therefore assume $q \equiv 3 \pmod 4$, $q > 3$, $q = 2d + 1$, and construct a $DGD[5,1^*,5;5q]$. Consider the blocks $((0;0),(1;x^{2y}),(4;x^{2y+1}),(4;x^{2y+d+1}),(1;x^{2y+d}))$, $y = 0,1,\ldots,d-1$, to be developed $\mathrm{mod}(5,q)$. As $-1 = x^d$ is a non-square element, it is clear that the differences resulting in $(1,x^k)$ or $(4,x^k)$ account for all elements of this type; those of the type $(0,x^k)$ also all arise. Now consider those differences with 3 in the first position. The second position is filled with $x^{2y}(x-1)$, or $x^{2y}(x^{d+1}-1)$, $y = 0,1,\ldots,d-1$ and these are all elements of $GF(q)^*$ if $(x - 1)$ and $(x^{d+1} - 1)$ are a square and a non-square. In this case the differences with 2 first have $x^{2y+1}(x^{d-1}-1)$ or $x^{2y}(x-1)$, $y = 0,1,\ldots,d-1$ in the second position. Now

$$\begin{aligned} x^{d+1} - 1 &= x^d(x-x^d) \\ &= x^{d+1}(1 - x^{d-1}) \\ &= x(x^{d-1} - 1) , \end{aligned}$$

so that we again have all elements of $GF(q)^*$ in the second position. If $(x - 1)$ and $(x^{d+1} -1)$ are both non-squares or both squares then a similar argument shows that

$$((0;0),(4;x^{2y+1}),(1;x^{2y}),(1;x^{2y+d}),(4;x^{2y+d+1})), \quad y = 0,1,\ldots,d-1,$$

is a suitable ordering. By writing each group in some order and the reverse and adjoining these as blocks of the design, we see that $5q \in DB(5,1^*)$ as required.

LEMMA 4. *For all* $r \geq 1$, $5r \in DGD(5,2^*,r)$.

Proof. By [4, Theorem 3.11, Lemma 3.1], for all $r \in N$ there exists a GD[5,2,r;5r]. Thus by writing each block twice, once in some order and once in the reverse order, we have the result.

COROLLARY 4.1. *If* $r \in DB(5,2^*)$ *then* $5r \in DB(5,2^*)$; *if* $r + 1 \in DB(5,2^*)$ *then* $5r + 1 \in DB(5,2^*)$.

Proof. In the first case adjoin to the blocks of a DGD[5,2*,r;5r] the blocks of a DBIBD(r,b,r_1,5,2*), one constructed on each group of the DGD. In the second case, construct the additional blocks using each of the groups with a new point adjoined.

LEMMA 5. $\{10,12\} \subset DGD(5,1^*,2)$.

Proof. The blocks of the DGD[5,1*,2;10] are as follows: (4,0,8,7,6),(6,2,8,0,9),(3,7,9,1,0),(6,7,4,5,3),(8,5,4,2,1),(9,3,2,5,6), (1,5,9,7,8),(0,1,2,3,4), where the groups are $\{x,x+5\}$, $x = 0,1,2,3,4$. To obtain a DGD[5,1*,2;12] develop the initial block (1,2,0,4,9) mod 12. This is the design R144 in Clatworthy [1].

LEMMA 6. $14 \in DGD(5,5^*,2)$.

Proof. One such design is that with initial blocks:

((0;0),(0;1),(0;6),(1;3),(1;4)),
((0;0),(0;4),(0;3),(1;2),(1;5)),
((0;0),(0;5),(0;2),(1;6),(1;1)),
((1;0),(0;1),(0;6),(0;3),(0;4)),
((1;0),(0;4),(0;3),(0;2),(0;5)),
((1;0),(0;5),(0;2),(0;6),(0;1)),

to be developed mod(2;7).

3. *Main Result.*

THEOREM. *Let* λ^* *and* $v \geq 5$ *be given positive integers. A necessary and sufficient condition for the existence of a* DBIBD(v,5,λ^*) *is that* $v \neq 15$, $\lambda^* \neq 1$, *and that* $\lambda^*(v-1) \equiv 0 \pmod 2$ *and* $\lambda^* v(v-1) \equiv 0 \pmod{10}$.

Method of Proof. That these conditions are necessary follows from the usual counting arguments for BIBDs. We need only consider values of λ^* which are factors of 10, since if $\lambda_1^* | \lambda_2^*$ then $DB(k,\lambda_1^*) \subset DB(k,\lambda_2^*)$. Thus we have the following cases:

$$\begin{array}{ll} \lambda^* = 1 & v \equiv 1 \text{ or } 5 \pmod{10}, \\ \lambda^* = 2 & v \equiv 0 \text{ or } 1 \pmod{5}, \\ \lambda^* = 5 & v \equiv 1 \pmod{2}, \\ \lambda^* = 10 & \text{all } v. \end{array}$$

The following lemmas show that designs do indeed exist in all these cases, with the exception of the DBIBD(15,5,1*), the non-existence of which is established by Hall and Connor [2,3].

LEMMA 7. *If* $v \equiv 1$ *or* $5 \pmod{10}$ *and* $v \neq 15$ *then* $v \in DB(5,1^*)$.

Proof. Let $v = 2u + 1$, where $u \equiv 0$ or $2 \pmod{5}$ and $u \neq 7$. By [4, Lemma 5.16], $u \in GD(\{5,6\},1,M_5')$, where $M_5' = \{2,5,10,12,15,17,20,$ $22,32,35,37,40,42,45,47,50,52,55,57,67,75,77,80,82,92,105,107,110,112,$ $115,117,120,122,132,167\}$. By Lemmas 1 and 5 it suffices to show $v = 2t + 1 \in DB(5,1^*)$ for all $t \in M_5'$. For $t \equiv 0$ or $2 \pmod{10}$ this follows from [4, Lemma 5.19]; for $t \in \{17,47,57,67,77,107,117,167\}$, $v \in \{35,95,115,135,155,215,235,335\}$, and so $v \in DB(5,1^*)$ by Lemma 3. For the remaining values of v a solution is given in Table I.

LEMMA 8. *If* $v \equiv 0$ *or* $1 \pmod{5}$ *then* $v \in DB(5,2^*)$.

Proof. As in Hanani [4, Lemma 5.23], it suffices to show that $v \in DB(5,2^*)$ for all $v \in M_5 = \{1,5,6,10,11,15,16,20,21,31,35,36,40,41,$ $45,46,50,51,70,71,75,76,100,101,105,106,151\}$. For $v \equiv 1$ or $5 \pmod{10}$ and $v \neq 15$ this follows from Lemma 7. For the remaining values of v a solution is given in Table II.

LEMMA 9. *If* $\lambda^* > 1$ *then* $15 \in DB(5,\lambda^*)$.

Proof. For $\lambda^* = 2$ see Table II. For $\lambda^* = 3$ a solution is ((0,0),(0,1),(0,4),(1,0),(2,0)),((0,0),(0,2),(0,3),(2,0),(1,0)), ((0,0),(1,4),(1,1),(2,3),(2,2)), all mod(3,5), ((0,0),(0,1),

$(0,2),(0,3),(0,4))$, $((0,0),(2,4),(2,1),(1,3),(1,2))$, $((0,1),$ $(2,2),(2,0),(1,4),(1,3))$, $((0,2),(2,3),(2,1),(1,4),(1,0))$, $((0,3),(2,4),(2,2),(1,1),(1,0))$, $((0,4),(2,3),(2,0),(1,2),(1,1))$, all mod$(3,-)$. A design for any other value of λ^* may be constructed by taking an appropriate number of copies of these designs.

LEMMA 10. *If* $v \equiv 1 \pmod 2$ *and* $v \geq 5$ *then* $v \in DB(5,5^*)$.

Proof. Let $v = 2t + 1$, $t \geq 2$. By [4, Lemma 5.17], $t \in GD(\{5,6,7\},1,M_5'')$, where $M_5'' = \{2,3,\ldots,24,26,31,32,33,34,36\}$. By Lemmas 1, 5, and 6 it suffices to show $v = 2t + 1 \in DB(5,5^*)$ for all $t \in M_5''$. If $t \equiv 0 \pmod 2$ this follows from [4, Lemma 5.21]; for $t \equiv 0$ or $2 \pmod 5$, Lemmas 7 and 9 give the result; for $t \in \{3,9,11,13,21,23,33\}$, $v \in N_5$ and $v \in DB(5,5^*)$ by Corollary 2.1. For the remaining values of v a solution is given in Table III.

LEMMA 11. *For every integer* $v \geq 5$, $v \in DB(5,10^*)$.

Proof. By [4, Lemma 5.18] it is sufficient to show that $v \in DB(5,10^*)$ for all $v \in K_5 = \{5,6,\ldots,20,22,23,24,27,28,29,32,33,34,39\}$. For $v \equiv 1 \pmod 2$, Lemma 10 gives the result; for $v \equiv 0$ or $1 \pmod 5$, Lemma 8 gives the result. For the remaining values of v a solution is given in Table IV.

TABLE I

v	DBIBD(v,b,r,5,1^*)
11	(3,5,1,4,9), mod 11.
31	(1,16,8,2,4),(6,12,24,3,17),(20,10,5,9,18), all mod 31.
71	(1,54,5,25,57),(11,62,59,26,55),(10,43,50,37,2),(18,24,49,19,32),(51,56,68,67,42),(35,44,23,7,33),(40,30,58,6,8), all mod 71.
75	((0;0,0),(1;1,0),(1;2,2),(4;1,4),(4;2,3)),((4;1,1),(1;1,0),(0;0,0),(4;2,2),(1;2,3)),((1;2,0),(4;2,3),(0;0,0),(1;1,4),(4;1,1)),((4;2,2),(4;1,4),(0;0,0),(1;2,0),(1;1,1)),((3;1,3),(3;1,2),(2;2,0),(2;1,0),(0;0,0)),((3;2,1),(0;0,0),(3;2,4),(2;1,0),(2;2,0)),((2;1,2),(0;0,0),(2;1,3),(3;1,4),(3;1,1)), all mod (5;3,5), showing 75 ϵ DGD(5,1^*,5).
91	Use Lemma 1 with S = {5}, R = {5}, m = 2 and 45 ϵ GD(5,1,5) by [4, Lemma 4.13].
111	((0,0),(1,1),(1,36),(2,29),(2,8)),((0,0),(2,27),(2,10),(1,8),(1,29)),((0,0),(2,36),(2,1),(1,23),(1,14)),((0,20),(0,17),(1,0),(0,30),(0,7)),((1,0),(0,16),(0,9),(0,12),(2,0)),((0,0),(1,10),(1,27),(2,31),(2,6)),

	((0,35),(0,2),(1,0),(0,3),(0,34)), ((2,0),(0,32),(0,18),(0,24),(1,0)), ((0,0),(1,26),(1,11),(2,14),(2,23)), ((0,15),(0,22),(1,0),(0,33),(0,4)), ((0,0),(2,11),(2,26),(1,6),(1,31)), all mod (3,37).
151	(1,59,8,64,19),(83,27,65,67,60),(72,78,20,9,123), (127,125,148,94,110),(53,70,107,122,101), and these blocks multiplied by 6 and 36, all mod 151.
211	(1,107,71,188,55),(84,126,56,178,189), (32,48,72,162,108),(160,29,149,177,118), (59,180,194,120,80),(5,113,144,96,64), (200,63,89,42,28),(165,110,142,3,2), (70,117,105,78,52),(145,112,167,41,168), (24,36,16,54,81),(143,25,109,58,87), (135,97,90,40,60),(77,10,192,128,15), (100,150,137,14,21),(9,6,119,73,4), (35,158,164,26,39),(123,79,13,82,125), (18,12,27,8,146),(75,50,174,7,116), (154,173,20,45,30), all mod 211.
231	By [4, Lemma 2·16], with $n = 21$ and $m = 11$, $231 \in GD(5,1,11)$. Thus $231 \in DGD(5,1^*,11)$ (use the blocks of GD(5,1,11) and their reverses); as $11 \in DB(5,1^*)$ the result follows using the argument of Corollary 4·1.

TABLE II

v	DBIBD($v,b,r,5,2^*$)
6	Blocks: $(2,3,1,0,4),(\infty,1,2,3,4),(4,3,2,\infty,0),$ $(4,1,0,\infty,3),(\infty,0,4,2,1),(0,3,1,2,\infty)$.
10	$(1,2,\infty,2x+1,x+2),(0,2,1,x+2,2x+1)$, both mod 9 (where $x^2 = 2x + 1$).
15	$((\infty,1),(\infty,4),(0,0),(1,2),(1,3)),$ $((\infty,2),(\infty,3),(0,0),(1,1),(1,4)),$ $((\infty,0),(0,0),(0,1),(0,4),(1,0)),$ $((\infty,0),(0,0),(0,2),(0,3),(1,0)),$ $((1,0),(0,3),(0,2),(\infty,4),(\infty,1)),$ $((1,0),(0,4),(0,1),(\infty,3),(\infty,2)),$ $((1,0),(1,4),(1,1),(0,0),(\infty,0)),$ $((1,0),(1,3),(1,2),(0,0),(\infty,0)),$ all mod $(-,5)$, and the two blocks $((\infty,0),(\infty,1),(\infty,2),(\infty,4),(\infty,3)),$ $((\infty,3),(\infty,4),(\infty,2),(\infty,1),(\infty,0))$.
16	$(x^{\alpha},x^{\alpha+3},x^{\alpha+6},x^{\alpha+9},x^{\alpha+12})$, $\alpha = 0,1,2$, all mod 16 (where $x^4 = x + 1$).
20	$(0,7,\infty,11,1),(1,4,16,7,9),(9,7,11,6,17),$ $(6,4,11,5,1)$, all mod 19.
36	By [4, Lemma 4·12], $35 \in DGD(5,2^*,5)$. From above $6 \in DB(5,2^*)$ and the construction of

Corollary 4·1 gives the result.

40 We show $40 \in DGD(5,2^*,5)$.
$((0;0),(1;x^{\alpha}),(4;x^{\alpha+3}),(4;x^{\alpha+2}),(1;x^{\alpha+1}))$,
$\alpha = 0,1,\ldots,6$, all mod (5;8) (where $x^3 = x+1$).

46 Use Corollary 4·1 with $r = 9$.

50 Use Corollary 4·1 with $r = 10$.

70 We show $70 \in DGD(5,2^*,5)$.
$((0;0),(1;2^{\alpha}),(1;2^{\alpha+6}),(4;2^{\alpha+3}),(4;2^{\alpha+9}))$ in
this order and reverse, $\alpha = 0,1,2$,
$((3;3),(0;\infty),(2;1),(1;0),(4;9))$,
$((0;\infty),(2;8),(1;0),(4;7),(3;11))$,
$((1;0),(3;10),(2;12),(4;4),(0;\infty))$,
$((2;5),(1;0),(4;6),(0;\infty),(3;2))$,
$((0;3),(0;1),(0;\infty),(0;9),(0;0))$,
$((4;5),(4;8),(1;12),(0;0),(1;1))$,
$((0;0),(1;2),(1;11),(4;3),(4;10))$,
$((0;0),(1;4),(4;6),(1;9),(4;7))$, all mod (5;13).

76 Use Corollary 4·1 with $r = 15$.

100 Use Corollary 4·1 with $r = 20$.

106 By [4], $105 \in DGD(5,2^*,5)$. As $6 \in DB(5,2^*)$, the result follows.

TABLE III

v	DBIBD(v,b,r,5,5*)
39	$((0,1),(0,3),(0,9),(2,2^{\alpha+6}),(1,2^{\alpha+2}))$, $((0,2),(0,6),(0,5),(2,2^{\alpha+7}),(1,2^{\alpha+3}))$, $((0,12),(0,10),(0,4),(2,2^{\alpha+8}),(1,2^{\alpha+4}))$, $((0,11),(0,7),(0,8),(2,2^{\alpha+9}),(1,2^{\alpha+5}))$, $\alpha = 0,4,8$
	$((0,1),(0,3),(0,9),(1,0),(2,0))$,
	$((0,2),(0,6),(0,5),(1,0),(2,0))$,
	$((0,12),(0,10),(0,4),(2,0),(1,0))$,
	$((0,11),(0,7),(0,8),(2,0),(1,0))$,
	$((0,0),(0,2^{2\beta}),(1,2^{2\beta+6}),(0,2^{2\beta+6}),(1,2^{2\beta}))$, $\beta = 0,1,2$,
	all mod (3,13).
63	$63 \in B(\{7,9\},1)$ (see [4]), $\{7,9\} \subset DB(5,5^*)$.

TABLE IV

v	DBIBD(v,b,r,5,10*)
8	$(x^{\alpha}, x^{\alpha+1}, x^{\alpha+2}, x^{\alpha+3}, x^{\alpha+4})$, $\alpha = 0,1,\ldots,6$, all mod 8 (where $x^3 = x+1$).
12	(1,10,∞,9,2),(2,9,∞,7,4),(4,7,∞,3,8),(8,3,∞,6,5), (5,6,∞,10,1),(3,5,1,4,9) (7 times), all mod 11.
14	(0,1,∞,3,9),(0,1,12,8,5),(0,3,10,11,2), (0,4,9,7,6), each occurring 3 times, (0,1,∞,3,9),(9,3,∞,1,0), all mod 13.
18	$(0, 3^{\beta}, 3^{\beta+4}, 3^{\beta+8}, 3^{\beta+12})$, in this order and reverse, $\beta = 0,1,2,3$, (0,16,4,1,13),(4,16,∞,7,6), (0,6,7,11,10),(13,1,∞,10,11),(0,14,12,3,5), (5,14,∞,15,9),(0,2,9,8,15),(14,12,∞,15,8), (0,7,13,11,16),(16,11,∞,13,7), all mod 17.
22*	(0,2,∞,3,7),(5,9,∞,3,0),(11,1,∞,0,7), (0,2,∞,8,11),(0,9,∞,14,1),(11,9,4,3,0)(9 times), (0,3,4,9,11)(8 times), all mod 21.
24	(1,5,0,2,10),(10,0,5,2,4),(4,0,10,2,20), (20,0,4,10,8),(20,4,0,17,8),(8,20,0,16,17), (17,8,11,16,0),(16,17,0,11,9),(11,9,0,16,22), (11,9,22,0,18),(22,9,18,21,0),(0,21,22,13,18), (18,13,21,19,0),(21,0,3,19,13),(13,3,19,15,0), (6,15,19,0,3),(7,6,0,3,15),(12,7,0,15,6),

* The first five blocks are given in Takeuchi [8].

$(0,14,6,7,12),(7,14,\infty,1,0),(0,1,\infty,5,19),$
$(5,16,\infty,0,2),(\infty,0,1,12,14),(14,12,1,0,\infty),$
all mod 23.

28 $((0,1),(0,2),(0,5),(x^{\alpha},0),(x^{\alpha+1},0))$ forward and reverse twice each, $\alpha = 0,1,2,$
$((0,3^{\beta}),(0,3^{\beta+3}),(1,0),(x,0),(x^2,0))$ forward and reverse, $\beta = 0,1,2,$
$((0,0),(1,1),(1,6),(x,3),(x,4)),$
$((0,0),(x,4),(x,3),(x+1,2),(x+1,5)),$
$((0,0),(x+1,5),(x+1,2),(1,6),(1,1)),$
$((0,0),(x,1),(x,6),(x+1,3),(x+1,4)),$
$((0,0),(x+1,4),(x+1,3),(1,2),(1,5)),$
$((0,0),(1,5),(1,2),(x,6),(x,1)),$
$((0,0),(x+1,1),(x+1,6),(1,3),(1,4)),$
$((0,0),(1,4),(1,3),(x,2),(x,5)),$
$((0,0),(x,5),(x,2),(x+1,6),(x+1,1))$, all mod $(4,7)$ (where $x^2 = x + 1$).

32 $(x^{\alpha},x^{\alpha+1},x^{\alpha+2},x^{\alpha+3},x^{\alpha+4})$, $\alpha = 0,1,\ldots,30$, all mod 32 (where $x^5 = x^2 + 1$).

34 $((0,0),(1,2^{\alpha}),(1,2^{\alpha+5}),(2,2^{\alpha+1}),(2,2^{\alpha+6}))$, forward and reverse twice each, $\alpha = 0,1,2,3,4,$
$((0,1),(0,4),(0,5),(0,9),(0,3))$, forward and reverse twice each, $((0,2^{\alpha}),(0,2^{\alpha+5}),\infty,(1,0),(2,0)),$
$\alpha = 0,1,\ldots,4$, $((2,0),(1,0),(0,2^{\alpha+5}),(0,2^{\alpha}),(0,0)),$
$\alpha = 0,1,\ldots,4$, all mod $(3,11)$.

REFERENCES

[1] W. H. Clatworthy, *Tables of Two-Associate Class Partially Balanced Designs*, National Bureau of Standards, Washington, 1973.

[2] W. S. Connor, Jr., *On the structure of balanced incomplete block designs*, Ann. Math. Statist. *23* (1952), 57-71.

[3] M. Hall, Jr. and W. S. Connor, Jr., *An embedding theorem for balanced incomplete block designs*, Can. J. Math. *6* (1954), 35-41.

[4] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. *11* (1975), 255-369.

[5] F. Harary, W. D. Wallis and K. Heinrich, "Decompositions of complete symmetric digraphs into the four oriented quadrilaterals" in *Combinatorial Mathematics*, edited by D. A. Holton and J. Seberry, Lecture Notes in Mathematics, Vol. 686, Springer-Verlag, Berlin, Heidelberg, New York, 1974, 165-173.

[6] J. Seberry and D. Skillicorn, *All directed BIBDs with* $k = 3$ *exist*, J. Combin. Theory (to appear).

[7] D. J. Street and J. Seberry, *All DBIBDs with block size four exist*, Utilitas Math. *18* (1980), 27-34.

[8] K. Takeuchi, *A table of difference sets generating balanced incomplete block designs*, Rev. Internl. Statist. Inst. *30* (1962), 361-366.

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