BALANCED DESIGNS FOR TWO-VARIETY COMPETITION EXPERIMENTS

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ABSTRACT. For the square grid we find the inequivalent building designs of sizes 1×5, 2×5, 3×5, 4×5, 1×10 and 2×10 and the inequivalent bordered 10×10 and 11×11 squares with nested balance which are constructed by rotation of suitable subarrays.

1. Introduction

In this note we consider the construction of balanced designs for two-variety competition experiments on the square grid. Several authors have investigated the construction of such designs on the triangular, square and hexagonal grids. The survey paper of A. P. Street [2] gives the relevant references. Gates [1] discusses some of the applications of such designs.

We will represent the two varieties by 0 and 1. There are 16 neighbour configurations for 0 and 16 for 1. The 16 configurations for 0 are given in Figure 1, subdivided into 5 equivalence classes depending on whether the central plant is surrounded by 0, 1, 2, 3 or 4 plants of the other variety.

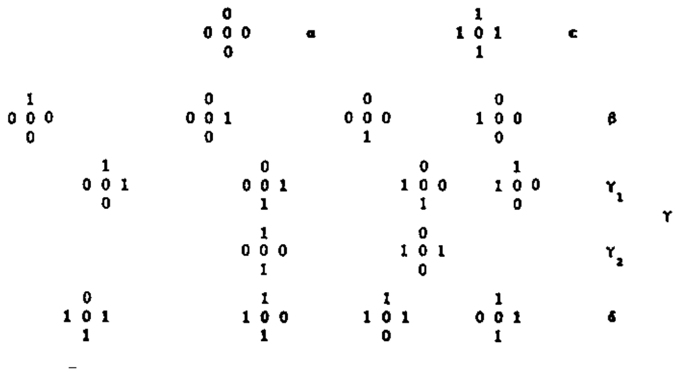


Figure 1.

A design is said to be *balanced* if, within each equivalence class, there are equal numbers of 0s and 1s and if each equivalence class is represented equally often.

Utilitas Mathematica, **28** (1985), pp.113-120

A design is said to have *directional balance* if the design is balanced and if, for each variety within each equivalence class, each configuration appears equally often.

In Section 2 we discuss the use of building designs in the construction of balanced designs and give the number of inequivalent building designs of small size. In Section 3, we consider a similar construction method for designs with nested balance.

2. Balanced Designs From Building Designs

For the square grid an $r \times c$ *building design* is a pair of bordered $r \times c$ arrays such that

(i) each array is the (0,1) complement of the other;
(ii) the complementary pair forms a balanced design;
(iii) the two border rows are equal;
(iv) each array of the pair interlocks with itself in such a way that the shared border row can be augmented at either end and the resulting pair of bordered $(2r+1) \times c$ arrays form a balanced design.

Building designs were introduced by Veevers and Boffey [**3**] who showed that for the hexagonal grid there are 12 building designs, symmetric about the major axis, with 25 points of which seven are observable (that is, are surrounded by six neighbours). Veevers, Boffey and Zafar-Yab [**4**] give one example of a 3×5 building design on the square grid and remark that it is the smallest size for which a building design exists for the square grid.

Example 1

a) The following is one member of the complementary pair. The original bordered $r \times c$ array is shown in the box. The augmenting varieties are in italics. The equivalence class of each of the internal varieties is given in the array to the right.

	0	0	0	1	0	
0	0	0	0	0	0	0
0	0	0	1	1	0	1
0	1	0	1	0	1	0
0	0	0	0	1	0	*1*
0	0	0	0	0	0	0
0	0	0	1	1	0	1
0	1	0	1	0	1	0
	0	0	0	1	0	

α	α	β	γ	α
β	β	γ	δ	δ
ε	γ	δ	ε	ε
β	α	γ	ε	δ

b) The following array, together with its complement, forms a balanced design but it is not a building design as when two are interlocked the three new internal varieties are from the equivalence classes δ, β and ε and possibilities from augmentation are β or γ and δ or ε so it is impossible to obtain an α.

	0	1	1	0	1	
0	0	0	1	1	0	1
0	0	0	1	0	1	1
0	0	0	1	1	0	1
	0	1	1	0	1	

■

A building design is said to have ***horizontal symmetry*** if it is symmetric about the 'row' axis and to have ***vertical symmetry*** if it is symmetric about the column axis.

LEMMA 2. *There are no vertically symmetric* 3×5 *building designs.*

PROOF: Suppose that such an array exists. The neighbour equivalence classes of each of the 15 internal varieties is given below.

$$\begin{matrix} N_1 & N_2 & N_3 & N_2 & N_1 \\ N_4 & N_5 & N_6 & N_5 & N_4 \\ N_7 & N_8 & N_9 & N_8 & N_7 \end{matrix}$$

Thus the six neighbour classes N_1, N_2, N_4, N_5, N_7, N_8 appear twice each and the three neighbour classes N_3, N_6 and N_9 appear once each. For balance each neighbour class must appear three times. As there are five neighbour classes at least two of N_1, N_2, N_4, N_5, N_7 and N_8 are the same and so at least one neighbour class appears at least four times, a contradiction. ∎

A similar argument shows that if $rc/5$ is odd there is no vertically symmetric $r \times c$ building design unless c is odd and $r \geq 5$.

We say two building designs are *equivalent* if one can be obtained from the other by reflection through the horizontal axis, the vertical axis or through both.

Using a back-track algorithm written in Pascal we found all inequivalent building designs of sizes 3×5, 4×5, 1×10 and 2×10. The results are summarised in Table 1. (A complete list is available from the authors.)

Array Size	Balanced Arrays	Number of Building Designs	Horizontally Symmetric Building Designs	Vertically Symmetric Building Designs
1 x 5	20	0	-	-
2 x 5	43	0	-	-
3 x 5	1146	172	2	0
4 x 5	40242	3259	9	0
1 x 10	4880	460	460	0
2 x 10	55580	276	0	0

Table 1. Number of Inequivalent Arrays

In Table 2 we list one member of each pair of the two horizontally symmetric 3×5 building designs. In Table 3 we list the top half of one member of each pair of the nine horizontally symmetric 4×5 building designs.

	0	0	0	0	1	
0	0	0	1	1	0	1
0	0	0	1	0	1	1
0	0	0	1	1	0	1
	0	0	0	0	1	

	0	0	1	0	1	
0	0	0	1	1	0	1
1	0	0	0	0	1	0
0	0	0	1	1	0	1
	0	0	1	0	1	

Horizontally Symmetric 3 x 5 Building Designs

Table 2.

	0	1	0	0	0	
0	0	0	1	0	1	0
0	0	1	0	0	0	0

	0	0	0	0	1	
0	0	1	0	1	0	0
0	0	0	0	0	1	0

	0	0	0	0	1	
0	0	1	0	1	0	1
0	0	0	1	1	1	1

	0	0	0	1	0	
1	0	1	0	0	1	0
0	1	0	0	0	0	0

	0	0	1	0	0	
1	0	1	0	0	1	0
0	1	0	0	0	0	0

	0	1	1	1	0	
1	0	1	1	0	1	0
0	1	1	1	1	0	1

	1	0	0	0	0	
0	0	0	1	0	1	0
0	0	0	0	1	0	0

	1	1	1	0	1	
1	0	1	1	1	0	1
0	1	0	1	1	1	1

	1	1	1	0	1	
1	0	1	1	1	0	1
0	1	1	1	0	1	1

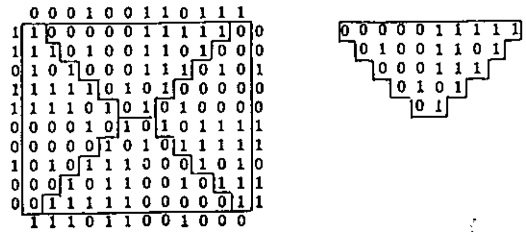
Horizontally Symmetric 4 x 5 Building Designs

Table 3.

3. Designs With Directional Balance

Williams and Bailey [5] gave two construction methods for these designs.

One was to construct a bordered 10×12 array, given in Figure 2, obtained by rotating the array to the right, as indicated by the bold lines in the figure. A back-track algorithm written in Pascal found this to be the unique array with this property.



Bordered 10 x 12 Array With Nested Balance.

Figure 2.

The other method was to construct a bordered 11×11 array which, when the centre square was discarded, had nested balance. The array was obtained by rotating a bordered 6×5 array. If we insist that, for a particular array, all the border rows have the same sequence then there are 244 inequivalent bordered 6×5 arrays.

None of these designs have any additional symmetry properties — this is not surprising as we require an odd number of α's and ε's in the original bordered 6×5 array. In Figure 3 we illustrate the method of construction and give four of the generating arrays we found. (A complete list is available on request.)

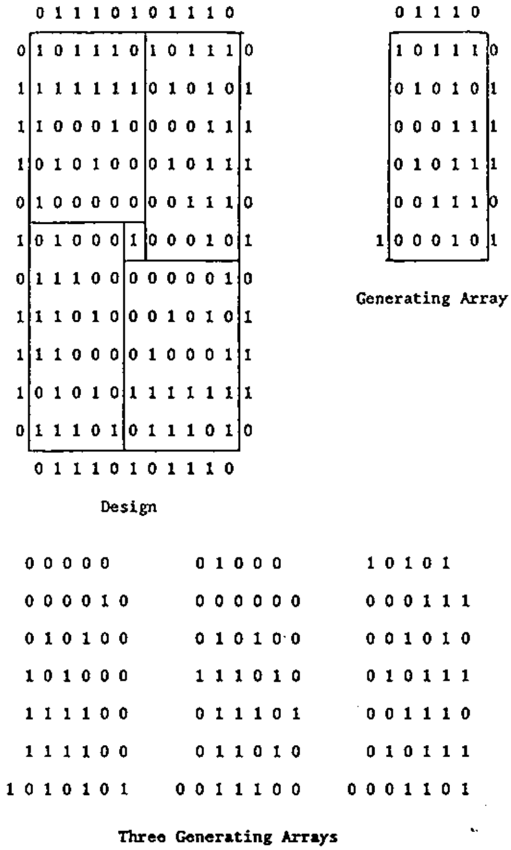


Figure 3.

Acknowledgement

We thank the referee for carefully reading the paper and in particular for pointing out the extension of Lemma 2.

References

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Received October 1, 1984; Revised December 14, 1984