

A functorial version of a construction of Hochschild and Mostow for representations of Lie algebras

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Let \underline{g} be a Lie algebra, \underline{h} a complemented ideal of \underline{g} , and W an \underline{h} -module. Hochschild and Mostow have described the construction of a \underline{g} -module "induced" from W , which is finite-dimensional provided W is finite-dimensional and satisfies a nilpotent action condition. This note describes a modification of their construction which is functorial and a weak adjoint to the restriction functor from \underline{g} -modules to \underline{h} -modules.

Throughout this paper we shall suppose that \underline{g} is a Lie algebra over a field k , that \underline{h} is an ideal of \underline{g} , and that there is a subalgebra \underline{s} of \underline{g} such that $\underline{g} = \underline{h} \oplus \underline{s}$. $U\underline{h}$, $U\underline{g}$ will denote the universal enveloping algebras of \underline{h} and \underline{g} . Clearly, every $g \in \underline{g}$ can be written uniquely as $g = h + s$ with $h \in \underline{h}$ and $s \in \underline{s}$. This allows us to define, (with Hochschild and Mostow [1]), a composition $*$ by

$$(1) \quad g * u = hu + (su - us) \quad \text{for } g \in \underline{g}, \quad u \in U\underline{h}.$$

It can be shown that $su - us \in U\underline{h}$, hence (1) determines a \underline{g} -module structure on $U\underline{h}$.

We shall use $\text{mod-}\underline{h}$, $\text{mod-}\underline{g}$ to denote the categories of right \underline{h} - and \underline{g} -modules, and $F : \text{mod-}\underline{g} \rightarrow \text{mod-}\underline{h}$ to denote the restriction functor. Now let $W \in \text{mod-}\underline{h}$. Then $\text{hom}_k(U\underline{h}, W)$ has a \underline{g} -module structure given by

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$f^{\mathcal{G}}(u) = f(g* u)$ for $f \in \text{hom}_K(U_{\underline{h}}, W)$, $g \in \underline{g}$, and $u \in U_{\underline{h}}$.

Construction of the functor I

Define a map $\hat{j}_W : W \rightarrow \text{hom}_K(U_{\underline{h}}, W)$ by setting $\hat{j}_W(w)(u) = w.u$ for $w \in W$ and $u \in U_{\underline{h}}$. It is easy to check that \hat{j}_W is a $U_{\underline{h}}$ -monomorphism. We define a \underline{g} -submodule IW of $\text{hom}_K(U_{\underline{h}}, W)$ by setting $IW = (\text{im } \hat{j}_W) \cdot U_{\underline{g}}$. Now let j_W be \hat{j}_W with codomain restricted to be IW . Let $W, W' \in \text{mod-}\underline{h}$ and $\psi \in \text{hom}_{U_{\underline{h}}}(W, W')$. We define $I\psi : IW \rightarrow IW'$ by $[(I\psi)(f)](u) = (\psi \circ f)(u)$ for $u \in U_{\underline{h}}$ and $f \in IW$. We must show $(I\psi)(f) \in IW'$. Since $f \in IW$, f may be written as

$$f = \sum_{i=1}^n j_W(w_i) \cdot x_i$$

for suitable $w_i \in W$, $x_i \in U_{\underline{g}}$. Then, for $u \in U_{\underline{h}}$,

$$\begin{aligned} [(I\psi)(f)](u) &= \left[\psi \circ \sum_{i=1}^n j_W(w_i) \cdot x_i \right](u) \\ &= \psi \left\{ \sum_{i=1}^n x_i (x_i * u) \right\} \\ &= \sum_{i=1}^n \psi(w_i) \cdot (x_i * u) \\ &= \left[\sum_{i=1}^n j_{W'}(\psi(w_i)) \cdot x_i \right](u), \end{aligned}$$

so $(I\psi)(f) = \sum_{i=1}^n j_{W'}(\psi(w_i)) \cdot x_i \in IW'$. If W'' is another \underline{h} -module and $\psi' \in \text{hom}_{U_{\underline{h}}}(W', W'')$, then $(\psi' \circ \psi) \circ f = \psi' \circ (\psi \circ f)$; it follows that I has the multiplicative property of a functor.

LEMMA 1. j_W is natural in W .

Proof. We must show that if $\psi \in \text{hom}_{U_{\underline{h}}}(W, W')$ then $FI\psi \circ j_W = j_{W'} \circ \psi$. Suppose $w \in W$ and $u \in U_{\underline{h}}$. Then

$$\begin{aligned} (FI\psi \circ j_W)(w)(u) &= (\psi \circ j_W(w))(u) = \psi(w.u) = \psi(w).u = j_W(\psi(w))(u) \\ &= (j_W \circ \psi)(w)(u) \end{aligned}$$

as required. \square

LEMMA 2. I is a faithful functor.

Proof. If $I\psi = 0$, then for all $w \in W$, $0 = (I\psi)(j_W(w))$; so $0 = [(I\psi)(j_W(w))][1_{U\mathfrak{h}}] = \psi(w)$. That is, $\psi = 0$. \square

THEOREM 3. The functor $I : \text{mod-}\mathfrak{h} \rightarrow \text{mod-}\mathfrak{g}$, described above, is an injective weak left adjoint to F . That is, for $W \in \text{mod-}\mathfrak{h}$ and $V \in \text{mod-}\mathfrak{g}$, there is an injection

$$\theta_{WV} : \text{hom}_{U\mathfrak{g}}(IW, V) \rightarrow \text{hom}_{U\mathfrak{h}}(W, FV)$$

which is natural in W and V .

Proof. For $\phi \in \text{hom}_{U\mathfrak{g}}(IW, V)$, we define $\theta_{WV}(\phi) = F\phi \circ j_W$. The naturality of θ_{WV} follows from that of j_W and the definition of θ_{WV} . We must show that θ_{WV} is injective. Suppose that $\phi_1, \phi_2 \in \text{hom}_{U\mathfrak{g}}(IW, V)$, and that $F\phi_1 \circ j_W = F\phi_2 \circ j_W$. Then ϕ_1 and ϕ_2 coincide on $\text{im } j_W$. Since ϕ_1, ϕ_2 are $U\mathfrak{g}$ -homomorphisms, it follows that they must coincide on $(\text{im } j_W).U\mathfrak{g} = IW$. \square

THEOREM 4 (compare Hochschild and Mostow [1] and Zassenhaus [2]).

Let \mathfrak{g} be a finite-dimensional Lie algebra over a field k of characteristic zero, and let \mathfrak{h} be an ideal of \mathfrak{g} with complementary subalgebra \mathfrak{s} . Let W be a finite-dimensional \mathfrak{h} -module on which $[\mathfrak{h}, \mathfrak{s}]$ acts nilpotently. Then IW , as defined above, is a finite-dimensional \mathfrak{g} -module.

Proof. If $\{0\} = W_0 < W_1 < \dots < W_n = W$ is a composition series for W , then set $S(W) = (W_n/W_{n-1}) \oplus \dots \oplus (W_2/W_1) \oplus (W_1/W_0)$. By the Jordan-Hölder theorem, $S(W)$ is determined up to isomorphism. Clearly, a subalgebra of \mathfrak{h} acts nilpotently on W if and only if it annihilates $S(W)$. Let us write $d = \dim_k W$, and let $\text{ann}_{U\mathfrak{h}}(M)$ denote the annihilator in $U\mathfrak{h}$ of an \mathfrak{h} -module M . Obviously,

$$(2) \quad (\text{ann}_{U_{\underline{h}}}(S(W)))^d \subseteq \text{ann}_{U_{\underline{h}}}(W) \subseteq \text{ann}_{U_{\underline{h}}}(S(W)) .$$

Since, by hypothesis, $[\underline{h}, \underline{s}] \subseteq \text{ann}_{U_{\underline{h}}}(S(W))$, it follows that $\text{ann}_{U_{\underline{h}}}(S(W))$ is a \underline{g} -submodule of W . Hence $(\text{ann}_{U_{\underline{h}}}(S(W)))^d$ is a \underline{g} -submodule of $U_{\underline{h}}$.

If $f \in \text{hom}_k(U_{\underline{h}}, W)$ and $f(\text{ann}_{U_{\underline{h}}}(W)) = \{0\}$, then for all $x \in U_{\underline{g}}$,

$$\begin{aligned} f^x \left\{ (\text{ann}_{U_{\underline{h}}}(S(W)))^d \right\} &\subseteq f \left\{ x \cdot (\text{ann}_{U_{\underline{h}}}(S(W)))^d \right\} \\ &\subseteq f(\text{ann}_{U_{\underline{h}}}(W)) \text{ by (2),} \\ &= \{0\} . \end{aligned}$$

Now $\text{im } j_W$ annihilates $\text{ann}_{U_{\underline{h}}}(W)$; so $IW = (\text{im } j_W) \cdot U_{\underline{g}}$ annihilates $(\text{ann}_{U_{\underline{h}}}(S(W)))^d$. Let us write $J = (\text{ann}_{U_{\underline{h}}}(S(W)))^d$. Then it is easy to see that IW is embedded in $\text{hom}_k(U_{\underline{h}}/J, W)$. Since W is finite-dimensional, $\text{ann}_{U_{\underline{h}}}(W)$ is of finite codimension in $U_{\underline{h}}$. Hence, by (2), $\text{ann}_{U_{\underline{h}}}(S(W))$ is of finite codimension in $U_{\underline{h}}$. Now we appeal to a result of Zassenhaus [2, page 263], which states that if X, Y are ideals of $U_{\underline{h}}$ of finite codimension, then so is XY . We deduce from this that J is of finite codimension in $U_{\underline{h}}$, so that $\dim_k \text{hom}_k(U_{\underline{h}}/J, W) < \infty$, and so $\dim_k IW < \infty$. \square

References

- [1] G. Hochschild and G.D. Mostow, "Extensions of representations of Lie groups and Lie algebras, I", *Amer. J. Math.* 79 (1957), 924-942.
- [2] Hans Zassenhaus, "Über die Darstellungen der Lie-Algebren bei Charakteristik 0", *Comment. Math. Helv.* 26 (1952), 252-274.

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