

On coinduced corepresentations

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We prove a Frobenius reciprocity type theorem for coinduced comodules.

We shall assume familiarity with the notation and results of the first two chapters of Sweedler [1]. Throughout this paper, C and D will denote coalgebras over a field k , and $f : C \rightarrow D$ will denote a coalgebra surjection. W will be a right D -comodule with structure map ψ_W and V will be a right C -comodule with structure map ψ_V . $\varepsilon_C, \varepsilon_D, \Delta_C, \Delta_D$ will denote the counits and comultiplications of C and D .

Trushin [2, page 176], has defined the coinduced comodule $W \otimes^D C$ to be the kernel of the map

$$\xi = \left\{ (1 \otimes f \otimes 1)(1 \otimes \Delta_C) - (\psi_W \otimes 1) \right\} : \cancel{W} \otimes_k C \rightarrow \cancel{W} \otimes_k D \otimes_k C,$$

with structure map $\psi_{W \otimes^D C} = (1 \otimes \Delta_C)|_{\ker \xi}$.

We shall use $\text{comod-}C$ and $\text{comod-}D$ to denote the categories of right D - and C -comodules respectively, and F to denote the forgetful functor $F : \text{comod-}C \rightarrow \text{comod-}D$.

THEOREM. *For every $W \in \text{comod-}D$ and every $V \in \text{comod-}C$, there is an isomorphism*

$$\rho_{VW} : \text{comod-}C(V, W \otimes^D C) \rightarrow \text{comod-}D(FV, W)$$

which is natural in V and W .

Proof. If $\phi : V \rightarrow W \otimes^D C$ is a morphism of C -comodules, then we set

$\rho_{VW}(\phi) = \mu \circ (1 \otimes \varepsilon_C) \circ F\phi$, where $\mu : W \otimes k \rightarrow W$ is the scalar multiplication map. We also define

$$\lambda_{VW} : \text{comod-}D(FV, W) \rightarrow \text{comod-}C(V, W \otimes^D C)$$

by

$$\lambda_{VW}(\chi) = (\chi \otimes 1) \circ \psi_V \text{ for } \chi \in \text{comod-}D(FV, W) .$$

We have to check that

- (1) $\rho_{VW}(\phi)$ is a D -comodule morphism,
 - (2) $\text{im } \lambda_{VW}(\chi) \subseteq \ker \xi = W \otimes^D C$,
 - (3) $\lambda_{VW}(\chi)$ is a D -comodule morphism,
 - (4) ρ_{VW} and λ_{VW} are inverse to each other, and
 - (5) ρ_{VW} is natural in V and W .
- (1) We must show that

$$\psi_W \circ (\mu \circ (1 \otimes \varepsilon_C) \circ \phi) = [(\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(\phi \otimes 1)] \circ [(1 \otimes f) \circ \psi_V] ;$$

that is, that the outside rectangle of the following diagram commutes. This will follow if we show that subdiagrams (A) to (G) commute.

$$\begin{aligned}(\psi_W \otimes 1)(\chi \otimes 1)\psi_V &= (\chi \otimes 1 \otimes 1)(1 \otimes f \otimes 1)(\psi_V \otimes 1)\psi_V \\ &= (1 \otimes f \otimes 1)(\chi \otimes 1 \otimes 1)(1 \otimes \Delta_C)\psi_V,\end{aligned}$$

since ψ_V is a C -comodule structure map; so

$$(\psi_W \otimes 1)(\chi \otimes 1)\psi_V = (1 \otimes f \otimes 1)(1 \otimes \Delta_C)(\chi \otimes 1)\psi_V,$$

as required.

(3) Obviously the diagram below commutes, so $\lambda_{VW}(\chi)$ is a C -comodule morphism:

$$\begin{array}{ccccc}V & \xrightarrow{\psi_V} & W \otimes C & \xrightarrow{\chi \otimes 1} & W \otimes^D C \\ \psi_V \downarrow & & \downarrow 1 \otimes \Delta_C & & \downarrow 1 \otimes \Delta_C \\ W \otimes C & \xrightarrow{\psi_V \otimes 1} & W \otimes C \otimes C & \xrightarrow{\chi \otimes 1 \otimes 1} & W \otimes C \otimes C\end{array}$$

(4) If $\phi \in \text{comod-}C(V, W \otimes^D C)$, then

$$\begin{aligned}\lambda_{VW}(\rho_{VW}(\phi)) &= (\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(\phi \otimes 1)\psi_V \\ &= (\mu \otimes 1)(1 \otimes \varepsilon_C \otimes 1)(1 \otimes \Delta_C)\phi,\end{aligned}$$

since ϕ is a C -comodule morphism,

$$\begin{aligned}&= (\mu \otimes 1)(1 \otimes \varepsilon_D \otimes 1)(1 \otimes f \otimes 1)(1 \otimes \Delta_C)\phi, \text{ since } \varepsilon_D^f = \varepsilon_C, \\ &= (\mu \otimes 1)(1 \otimes \varepsilon_D \otimes 1)(\psi_W \otimes 1)\phi, \text{ since } \text{im } \phi \subseteq \ker \xi, \\ &= \phi, \text{ by a comodule axiom.}\end{aligned}$$

If $\chi \in \text{comod-}D(FV, W)$, then

$$\begin{aligned}\rho_{VW}(\lambda_{VW}(\chi)) &= \mu(1 \otimes \varepsilon_C)(\chi \otimes 1)\psi_V \\ &= \mu(1 \otimes \varepsilon_D)(1 \otimes f)(\chi \otimes 1)\psi_V \text{ since } \varepsilon_D^f = \varepsilon_C, \\ &= \mu(1 \otimes \varepsilon_D)(\chi \otimes 1)(1 \otimes f)\psi_V \\ &= \mu(1 \otimes \varepsilon_D)\psi_W \chi \text{ since } \chi \text{ is a } D\text{-morphism,} \\ &= \chi \text{ by a comodule axiom.}\end{aligned}$$

(5) Naturality. We must show that the following diagram commutes:

$$\begin{array}{ccc}
 \text{comod-}C(V_2, W_2 \otimes^D C) & \xrightarrow{\rho_{V_2 W_2}} & \text{comod-}D(FV_2, W_2) \\
 \downarrow \text{comod-}C(\eta, \theta \otimes 1) & & \downarrow \text{comod-}D(F\eta, \theta) \\
 \text{comod-}C(V_1, W_1 \otimes^D C) & \xrightarrow{\rho_{V_1 W_1}} & \text{comod-}D(FV_1, W_1)
 \end{array}$$

where $\eta \in \text{comod-}C(V_1, V_2)$ and $\theta \in \text{comod-}D(W_2, W_1)$.

If $\phi \in \text{comod-}C(V_2, W_2 \otimes^D C)$, then

$$\begin{aligned}
 \rho_{V_1 W_1}(\text{comod-}C(\eta, \theta \otimes 1)(\phi)) &= \mu(1 \otimes \varepsilon_C) \circ F[(\theta \otimes 1) \circ \theta \circ \eta] \\
 &= \theta \circ \mu(1 \times \varepsilon_C) \circ F\phi \circ F\eta \\
 &= \theta \circ (\mu(1 \times \varepsilon_C) \circ F\phi) \circ F\eta \\
 &= \text{comod-}D(F\eta, \theta)(\rho_{V_2 W_2}(\phi)) . \quad \square
 \end{aligned}$$

References

- [1] Moss E. Sweedler, *Hopf algebras* (Benjamin, New York, 1969).
- [2] David Trushin, "A theorem on induced corepresentations and applications to finite group theory", *J. Algebra* 42 (1976), 173-183.

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