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Abstraction, Refinement and Proof for Probabilistic Systems

With 62 Figures

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Preface

Probabilistic techniques in computer programs and systems are becoming more and more widely used, for increased efficiency (as in random algorithms), for symmetry breaking (distributed systems) or as an unavoidable artefact of applications (modelling fault-tolerance). Because interest in them has been growing so strongly, stimulated by their many potential uses, there has been a corresponding increase in the study of their correctness — for the more widespread they become, the more we will depend on understanding their behaviour, and their limits, exactly.

In this volume we address that last concern, of understanding: we present a method for rigorous reasoning about probabilistic programs and systems. It provides an operational model — “how they work” — and an associated program logic — “how we should reason about them” — that are designed to fit together. The technique is simple in principle, and we hope that with it we will be able to increase dramatically the effectiveness of our analysis and use of probabilistic techniques in practice.

Our contribution is a probabilistic calculus that operates at the level of the program text, and it is light-weight in the sense that the amount of reasoning is similar in size and style to what standard assertional techniques require. In the fragment at right, for example, each potential loop entry occurs with probability 1/2; the resulting iteration establishes $x \geq 1/2$ with probability exactly $p$ for any $0 \leq p \leq 1$. It is thus an implementation of the general operation choose with probability $p$, but it uses only simple tests of unbiased random bits (to implement the loop guard). It should take only a little quantitative logic to confirm that claim, and indeed we will show that just four lines of reasoning suffice.

Economy and precision of reasoning are what we have come to expect for standard programs; there is no reason we should accept less when they are probabilistic.

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The cover illustration comes from page 59.
The program fragment is adapted from Fig. 7.7.10 on page 210.
Scope and applicability

Methods for the analysis of probabilistic systems include automata, labelled transition systems, model checking and logic (e.g. dynamic or temporal). Our work falls into the last category: we overlay the Hoare-logic paradigm with probabilistic features imported from Markov processes, taking from each the essential characteristics required for a sound mathematical theory of refinement and proof. The aim is to accommodate modelling and analysis of both sequential and distributed probabilistic systems, and to allow — even encourage — movement between different levels of abstraction.

Our decision to focus on logic — and a proof system for it — was motivated by our experience with logical techniques more generally: they impose a discipline and order which promotes clarity in specifications and design; the resulting proofs can often be carried out, and checked, with astonishing conciseness and accuracy; and the calculation rules of the logic lead to an algebra that captures useful equalities and inequalities at the level of the programs themselves.

Although we rely ultimately on an operational model, we use it principally to validate the logic (and that, in turn, justifies the algebra) — direct reliance on the model’s details for individual programs is avoided if possible. (However we do not hesitate to use such details to support our intuition.) We feel that operational reasoning is more suited to the algorithmic methods of verification used by model checkers and simulation tools which can, for specific programs, answer questions that are impractical for the general approach that a logic provides.

Thus the impact of our approach is most compelling when applied to programs which are intricate either in their implementation or their design, or have generic features such as undetermined size or other parameters. They might appear as probabilistic source-level portions of large sequential programs, or as abstractions from the probabilistic modules of a comprehensive system-level design; we provide specific examples of both situations. In the latter case the ability to abstract modules’ properties has a significant effect on the overall verification enterprise.

Technical features

Because we generalise the well-established assertional techniques of specifications, pre- and postconditions, there is a natural continuity of reasoning style evident in the simultaneous use of the new and the familiar approaches: the probabilistic analysis can be deployed more, or less, as the situation warrants.

A major feature is that we place probabilistic choice and abstraction together, in the same framework, without having to factor either of them out for separate treatment unless we wish to (as in fact we do in Chap. 11). This justifies the abstraction and refinement of our title, and is what gives
us access to the stepwise-development paradigm of standard programming
where systems are “refined” from high levels of abstraction towards the low
levels that include implementation detail.

As a side-effect of including abstraction, we retain its operational
counterpart demonic choice as an explicit operator \( \boxplus \) in the cut-down
probabilistic programming language \( pGCL \) which we use to describe our
algorithms — that is, the new probabilistic choice operator \( p \boxplus \) refines de-
monic choice rather than replacing it. In Chap. 8 we consider angelic choice
\( \uplus \) as well, which is thus a further refinement.

Probabilistic and demonic choice together allow an elementary treatment
of the hybrid that selects “with probability at least \( p \)” (or similarly “at most
\( p \)”), an abstraction which accurately models our unavoidable ignorance of
exact probabilities in real applications. Thus in our mathematical model
we are able to side-step the issue of “approximate refinement.”

That is, rather than saying “this coin refines a fair coin with probability
95\%,” we would say “this coin refines one which is within 5\% of being
fair.” This continues the simple view that either an implementation refines
a specification or it does not, which simplicity is possible because we have
retained the original treatment in terms of sets of behaviours: abstraction
is inclusion; refinement is reverse inclusion; and demonic choice is union.
In that way we maintain the important relationship between the three
concepts. (Section 6.5 on pp. 169ff illustrates this geometrically.)

Organisation and intended readership

The material is divided into three major parts of increasing specialisation,
each of which can to a large extent be studied on its own; a fourth part
contains appendices. We include a comprehensive index and extensive cross-
referencing.

Definitions of notation and explanations of standard mathematical tech-
niques are carefully given, rather than simply assumed; they appear as
footnotes at their first point of use and are made visually conspicuous by
using SMALL CAPITALS for the defined terms (where grammar allows). Thus
in many cases a glance should be sufficient to determine whether any foot-
note contains a definition. In any case all definitions, whether or not in
footnotes, may be retrieved by name through the index; and those with
numbers are listed in order at page xvii.

Because much of the background material is separated from the main
text, the need for more advanced readers to break out of the narrative
should be reduced. We suggest that on first reading it is better to consult
the footnotes only when there is a term that appears to require definition
— otherwise the many cross-references they contain may prove distracting,
as they are designed for “non-linear” browsing once the main ideas have
already been assimilated.
Part I, *Probabilistic guarded commands*, gives enough introduction to the probabilistic logic to prove properties of small programs such as the one earlier, for example at the level of an undergraduate course for Formal-Methods-inclined students that explains “what to do” but not necessarily “why it is correct to do that.” These would be people who need to understand how to reason about programs (and why), but would see the techniques as intellectual tools rather than as objects of study in their own right.

We have included many small examples to serve as models for the approach (they are indexed under *Programs*), and there are several larger case studies (for example in Chap. 3).

Part II, *Semantic structures*, develops in detail the mathematics on which the probabilistic logic is built and with which is it justified. That is, whereas the earlier sections present and illustrate the new reasoning techniques, this part shows where they have come from, why they have the form they do and — crucially — why they are correct.

That last point is especially important for students intending to do research in logic and semantics, as it provides a detailed and extended worked example of the fundamental issue of proving reasoning techniques *themselves* to be correct (more accurately, “valid”), a higher-order concept than the more familiar theme of the previous part in which we presented the techniques *ex cathedra* and used them to verify particular programs.

This part would thus be suitable for an advanced final-year undergraduate or first-year graduate course, and would fit in well with other material on programming semantics. It defines and illustrates the use of many of the standard tools of the subject: lattices, approximation orders, fixed points, semantic injections and retractions *etc.*

Part III, *Advanced topics*, concentrates on more exotic methods of specification and design, in this case probabilistic temporal/modal logics. Its final chapter, for example, contains material only recently discovered and leads directly into an up-to-date research area. It would be suitable for graduate students as an introduction to this specialised research community.

Part IV includes appendices collecting material that either leads away from the main exposition — *e.g.* alternative approaches and why we have not taken them — or supports the text at a deeper level, such as some of the more detailed proofs.

It also contains a short list of algebraic laws that demonic/probabilistic program fragments satisfy, generated mainly by our needs in the examples and proofs of earlier sections. An interesting research topic would be a more systematic elaboration of that list with a view to incorporating it into probabilistic Kleene- or omega algebras for distributed computations.
Overall, readers seeking an introduction to probabilistic formal methods could follow the material in order from the beginning. Those with more experience might instead sample the first chapter from each part, which would give an indication of the scope and flavour of the approach generally.

Original sources

Much of the material is based on published research, done with our colleagues, in conference proceedings and journal articles; but here it has been substantially updated and rationalised — and we have done our best to bring the almost ten years’ worth of developing notation into a uniform state.

For self-contained presentations of the separate topics, and extra background, readers could consult our earlier publications as shown overleaf.

At the end of each chapter we survey the way in which our ideas have been influenced by — and in some cases adopted from — the work of other researchers, and we indicate some up-to-date developments.

Acknowledgements

Our work on probabilistic models and logic was carried out initially at the University of Oxford, together with Jeff Sanders and Karen Seidel and with the support of the UK’s Engineering and Physical Sciences Research Council (the EPSRC) during two projects led by Sanders and Morgan over the years 1994–2001.

Morgan spent sabbatical semesters in 1995–6 at the University of Utrecht, as the guest of S. Doaitse Swierstra, and at the University of Queensland and the Software Verification and Research Centre (SVRC), as the guest of David Carrington and Ian Hayes. The foundational work the EPSRC projects produced during that period — sometimes across great distances — benefited from the financial support of those institutions but especially from the academic environment provided by the hosts and by the other researchers who were receptive to our initial ideas [MMS96].

Ralph Back at Åbo Akademi hosted our group’s visit to Turku for a week in 1996 during which we were able to explore our common interests in refinement and abstraction as it applied to the new domain; that led later to a three-month visit by Elena Troubitsyna from the Turku Center for Computer Science (TUCS), to our group in Oxford in 1997, and contributed to what has become Chap. 4 [MMT98].

David Harel was our host for a two-week visit to Israel in 1996, during which we presented our ideas and benefited from the interaction with researchers there.
Chapters’ dependence on original sources

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The sources listed opposite are in chronological order of writing, thus giving roughly the logical evolution of the ideas.

Subsequently we have continued to work with Sanders and with Ken Robinson, Thai Son Hoang and Zhendong Jin, supported by the Australian Research Council (ARC) over the (coming) years 2001–8 in their Large Grant and Discovery programmes, at the Universities of Macquarie and of New South Wales.

Joe Hurd from the Computer Laboratory at Cambridge University visited us in 2002, with financial assistance from Macquarie University; and Orieta Celiku was supported by TUCS when she visited in 2003. Both worked under McIver’s direction on the formalisation of pGCL, and its logic, in the mechanised logic HOL.

Hoang, Jin and especially Eric Martin have helped us considerably with their detailed comments on the typescript; also Ralph Back, Ian Hayes, Michael Huth, Quentin Miller and Wayne Wheeler have given us good advice. Section B.1 on the algebraic laws satisfied by probabilistic programs has been stimulated by the work (and the critical eyes) of Steve Schneider and his colleagues at Royal Holloway College in the U.K.

We thank the members of IFIP Working Groups 2.1 and 2.3 for their many comments and suggestions.

Annabelle McIver                         LRI Paris,
Carroll Morgan                            May 2004

_In memoriam AJMvG_
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Part I

Probabilistic guarded commands

and their refinement logic

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1.1 Sequential program logic

Since the mid-1970’s, any serious student of rigorous program development will have encountered “assertions about programs” — they are predicates which, when inserted into program code, are supposed to be “true at that point of the program.” Formalised — *i.e.* made into a logic — they look like either

\[
\{ \text{pre} \} \text{prog} \{ \text{post} \} \quad \text{Hoare-style} \]
\[
\text{pre} \Rightarrow \wp.\text{prog}.\text{post} , \quad \text{Dijkstra-style}
\]

in each case meaning “from any state satisfying precondition pre, the sequential program prog is guaranteed to terminate in a state satisfying postcondition post.” \(^1\) Formulae pre and post are written in first-order predicate logic over the program variables, and prog is written in a sequential programming language. Often Dijkstra’s *Guarded Command Language* [Dij76], called GCL, is used in simple expositions like this one, since it contains just the essential features, and no clutter.

A conspicuous feature of Dijkstra’s original presentation of guarded commands was the novel “demonic” choice. He explained that it arose naturally if one developed programs hand-in-hand with their proofs of correctness: if a single specification admitted say two implementations, then a third possibility was program code that seemed to choose unpredictably between the two. Yet in its pure form, where for example

\[
\text{prog} \sqcap \text{prog}’
\]

is a program that can unpredictably behave either as prog or as prog’, this “demonic” nondeterminism seemed at first — to some — to be an unnecessary and in fact gratuitously confusing complication. Why would anyone ever want to introduce unpredictability *deliberately*? Programs are unpredictable enough already.

If one really wanted programs to behave in some kind of “random” way, then more useful surely would be a construction like the

\[
\text{prog} \quad \sqcup \quad \text{prog}’
\]

that behaves as prog on half of its runs, and as prog’ on the other half. Of course on any particular run the behaviour is unpredictable, and even over many runs the proportions will not necessarily be exactly “50/50” — but over a long enough period one will find approximately equal evidence of each behaviour.

A logic and a model for programs like (1.3) was in fact provided in the early 1980’s [Koz81, Koz85], where in the “Kozen style” the pre- and post-formulae became real- rather than Boolean functions of the state, and \(\sqcap\) was replaced by \(\sqcup\) in the programming language. Those logical statements

\(^1\)We will use the Dijkstra-style.
(1.1) now took on a more general meaning, that “if program \(\text{prog}\) is run many times from the same initial state, the average value of \(\text{post}\) in the resulting final states is at least the actual value that \(\text{pre}\) had in the initial state.” Naturally we are relying on the expressions’ \(\text{pre}\) and \(\text{post}\) having real- rather than Boolean type when we speak of their average, or expected value.

The original — standard, we call it — Boolean logic was still available of course via the embedding \(\text{false, true} \mapsto 0, 1\).

Dijkstra’s demonic \(\sqcap\) was not so easily discarded, however. Far from being “an unnecessary and confusing complication,” it is the very basis of what is now known as refinement and abstraction of programs. (The terms are complementary: an implementation refines its specification; a specification abstracts from its implementation.) To specify “set \(r\) to a square-root of \(s\)” one could write directly in the programming language GCL

\[
\begin{align*}
r &:= -\sqrt{s} \\
r &:= \sqrt{s},
\end{align*}
\]

something that had never been possible before. This explicit, if accidental, “programming feature” caught the tide that had begun to flow in that decade and the following: the idea that specifications and code were merely different ways of describing the same thing (as advocated by Abrial, Hoare and others; making an early appearance in Back’s work [Bac78] on what became the Refinement Calculus [Mor88b, Bac88, Mor87, Mor94b, BvW98]; and as found at the heart of specification and development methods such as Z [Sp88] and VDM [Jon86]).

Unfortunately, probabilistic formalisms were left behind, and did not embrace the new idea: replacing \(\sqcap\) by \(\not\sqcap\), they lost demonic choice; without demonic choice, they lost abstraction and refinement; and without those, they had no nontrivial path from specification to implementation, and no development calculus or method.

---

\(^2\)Admittedly this is a rather clumsy notation when compared with those designed especially for specification, e.g.

\[
\begin{align*}
r &:= [r^2 = s] & \text{a specification statement (Back, Morgan, Morris)} \\
(r')^2 &:= s & \text{(the body of) a Z schema (Abrial, Oxford)} \\
&\vdash r = s & \text{VDM (Björner, Jones)} \\
\text{any } r' \text{ with } (r')^2 = s &\text{ then } r := r' \text{ end} & \text{a generalised substitution (Abrial)}
\end{align*}
\]

But the point is that the specification could be written in a “programming language” at all; it was beginning to be realised that there was no reason to distinguish the meanings of specifications and of programs (a point finally crystallised in the subtitle Assigning Programs to Meanings of Abrial’s book [Abr96a], itself a reference 30 years further back to Floyd’s paper [Flo67] where it all began).
To have a probabilistic development method, we need both $\sqcap$ and $\rho \oplus$ — we cannot abandon one for the other. Using them together, we can for example describe “flip a nearly fair coin” as

$$c := \text{heads}_{0.49} \oplus \text{tails} \quad \sqcap \quad c := \text{heads}_{0.51} \oplus \text{tails}.$$  

What we are doing here is specifying a coin which is within 1% of being fair — just as well, since perfect $0.5 \oplus$ coins do not exist in nature, and so we could never implement a specification that required one.\textsuperscript{3} This program abstracts, slightly, from the precise probability of heads or tails.

In this introduction we will see how the seminal ideas of Floyd, Hoare, Dijkstra, Abrial and others can be brought together and rephrased in the probabilistic context suggested by Kozen, and how the milestones of sequential program development and refinement — the concepts of

- program assertions;
- loop invariants;
- loop variants;
- program algebra (e.g. monotonicity and conjunctivity)

— can be generalised to include probability. Our simple programming language will be Dijkstra’s, but with $\rho \oplus$ added and — crucially — demonic choice $\sqcap$ retained: we call it $pGCL$.

Section 1.2 gives a brief overview of $pGCL$ and its use of so-called expectations rather than predicates in its accompanying logic; Section 1.3 then supplies operational intuition by relating $pGCL$ operationally to a form of gambling game. (The rigorous operational semantics is given in Chap. 5, and a deeper connection with games is given in Chap. 11.) Section 1.4 completes the background by reviewing elementary probability theory.

Section 1.5 gives the precise syntax and expectation-transformer semantics of $pGCL$, using the infamous “Monty Hall” game as an example. Finally, in Sec. 1.6 we make our first acquaintance with the algebraic properties of $pGCL$ programs.

Throughout we write $f.x$ instead of $f(x)$ for function application of $f$ to argument $x$, with left association so that $f.g.x$ is $(f(g))(x)$; and we use “:=” for is defined to be. For syntactic substitution we write $\text{expr} \ (\text{var} \mapsto \text{term})$.

\textsuperscript{3}That means that probabilistic formalisms without abstraction in their specifications must introduce probability into their refinement operator if they are to be of any practical use: writing for example $\text{prog} \square_{0.99} \text{prog’}$ can be given a sensible meaning even if the probability in $\text{prog}$ is exact [DGJP02, vBMOW03, Yin03]. But we do not follow that path here.
to indicate replacing \textit{var} by \textit{term} in \textit{expr}. We use “overbar” to indicate \textit{complement} both for Booleans and probabilities: thus \texttt{true} is \texttt{false}, and \texttt{\overline{P}} is \(1 - p\).

### 1.2 The programming language \textit{pGCL}

We’ll use \textit{square brackets} \([\cdot]\) to convert Boolean-valued predicates to arithmetic formulae which, for reasons explained below, we call \textit{expectations}. Stipulating that \texttt{[false]} is zero and \texttt{[true]} is one makes \(|P|\) in a trivial sense the probability that a given predicate \(P\) holds: if false, it holds with probability zero; if true, it holds with probability one.\footnote{Note that this nicely complements our “overbar” convention, because for any predicate \(P\) the two expressions \(\overline{|P|}\) and \(|\overline{P}|\) are therefore the same.}

For our first example, consider the simple program

\[
x := -y \quad \text{\texttt{\textcircled{↓}}} \quad x := +y
\]

over integer variables \(x, y; \mathbb{Z}\), using the new construct \(\texttt{\textcircled{↓}}\) which we interpret as “choose the left branch \(x := -y\) with probability 1/3, and choose the right branch with probability \(1 - 1/3\).”

Recall [DiJ76] that for any predicate \texttt{post} over \textit{final} states, and a standard command \texttt{prog},\footnote{Throughout we use \textsc{standard} to mean “non-probabilistic.”} the “weakest precondition” predicate \texttt{wp.prog.post} acts over \textit{initial} states: it holds just in those initial states from which \texttt{prog} is guaranteed to reach \texttt{post}. Now suppose \texttt{prog} is probabilistic, as Program (1.5) is: what can we say about the \textit{probability} that \texttt{wp.prog.post} holds in some initial state?

It turns out that the answer is just \texttt{wp.prog.[post]}, once we generalise \texttt{wp.prog} to expectations instead of predicates. For that, we begin with the two definitions \footnote{Here we are defining the language as we go along; but all the definitions are collected together in Fig. 1.5.3 (p. 26).}

\[
\begin{align*}
\texttt{wp.(x := E).postE} & := \quad \text{“postE with \textit{x} replaced everywhere by \textit{E}”} \quad (1.6) \\
\texttt{wp.(\texttt{prog} \cup \texttt{prog'}).postE} & := \quad p \cdot \texttt{wp.prog.postE} \quad (1.7) \\
& \quad + \overline{p} \texttt{wp.prog'.postE},
\end{align*}
\]

in which \texttt{postE} is an expectation, and for our example program we ask \textit{what is the probability that the predicate “the final state will satisfy \textit{x} ≥ 0” holds in some \textit{given} initial state of the program (1.5)}.\footnote{In the usual way, we take account of free and bound variables, and if necessary rename to avoid variable capture.}

To find out, we calculate \texttt{wp.prog.[post]} using the definitions above; that is
1. Introduction to $pGCL$

\[
wp.(x: = -y \oplus x: = +y); [x \geq 0] \\
\equiv^8 (1/3) * wp.(x: = -y); [x \geq 0] \\
+ (2/3) * wp.(x: = +y); [x \geq 0] \\
\equiv (1/3) [-y \geq 0] + (2/3) [+y \geq 0] \\
\equiv [y < 0]/3 + [y = 0] + 2[y > 0]/3 .
\]

using (1.7)

Thus our answer is the last arithmetic formula above, which we call a “pre-expectation” — and the probability we seek is found by reading off the formula’s value for various initial values of $y$, getting

- when $y < 0$, $1/3 + 0 + 2(0)/3 = 1/3$
- when $y = 0$, $0/3 + 1 + 2(0)/3 = 1$
- when $y > 0$, $0/3 + 0 + 2(1)/3 = 2/3$ .

Those results indeed correspond with our operational intuition about the effect of $\frac{1}{3}\oplus$.

For our second example we illustrate abstraction from probabilities: a demonic version of Program (1.5) is much more realistic in that we set its probabilistic parameters only within some tolerance. We say informally (but still precisely) that

\[
\begin{align*}
\bullet & \quad \text{$x: = -y$ is to be executed with probability at least $1/3$,} \\
\bullet & \quad \text{$x: = +y$ is to be executed with probability at least $1/4$} \\
\bullet & \quad \text{it is certain that one or the other will be executed.}
\end{align*}
\]

(1.8)

Equivalently we could say that alternative $x: = -y$ is executed with probability between $1/3$ and $3/4$, and that otherwise $x: = +y$ is executed (therefore with probability between $1/4$ and $2/3$).

With demonic choice we can write Specification (1.8) as

\[
x: = -y \frac{1}{3} \oplus x: = +y \quad \text{and} \\
x: = -y \frac{1}{4} \oplus x: = +y ,
\]

(1.9)

because we do not know or care whether the left or right alternative of $\sqcap$ is taken — and it may even vary from run to run of the program, resulting in an “effective” $\frac{1}{2} \oplus$ with $p$ somewhere between the two extremes.\(^9\)

\(^8\)Later we explain the use of “$\equiv$” rather than “$=$”.

\(^9\)We will see later that a convenient notation for (1.9) uses the abbreviation

\[
\text{prog } p \oplus q \quad \text{prog’} \quad := \quad \text{prog } p \oplus \text{prog’} \quad \sqcap \quad \text{prog’} \oplus \text{prog} ;
\]

we would then write it $x: = -y \frac{1}{3} \oplus x: = +y$, or even $x: = -y \frac{1}{4} \oplus +y$.\(^9\)
To treat Program (1.9) we need a third definition,
\[ wp.(prog \sqcap prog').postE := wp.prog.postE \min wp.prog',postE, \]
using \( \min \) because we regard demonic behaviour as attempting to make the achieving of \( post \) as improbable as it can. Repeating our earlier calculation (but more briefly) gives this time
\[ wp.(\text{Program (1.9)}).[x \geq 0] \]
\[ \equiv \min \left[ \frac{[y \leq 0]}{3} + 2 \frac{[y \geq 0]}{3} \right] \text{ using (1.6), (1.7), (1.10)} \]
\[ \equiv \frac{[y < 0]}{3} + \frac{[y = 0]}{4} + \frac{[y > 0]}{4} \text{ using arithmetic} \]

Our interpretation has become

- When \( y \) is initially negative, a demon chooses the left branch of \( \sqcap \) because that branch is more likely (2/3 vs. 1/4) to execute \( x := +y \) — the best we can say then is that \( x \geq 0 \) will hold with probability at least 1/3.

- When \( y \) is initially zero, a demon cannot avoid \( x \geq 0 \) — either way the probability of \( x \geq 0 \) finally is one.

- When \( y \) is initially positive, a demon chooses the right branch because that branch is more likely to execute \( x := -y \) — the best we can say then is that \( x \geq 0 \) finally with probability at least 1/4.

The same interpretation holds if we regard \( \sqcap \) as abstraction instead of as run-time demonic choice. Suppose Program (1.9) represents some mass-produced physical device and, by examining the production method, we have determined the tolerance (1.8) we can expect from a particular factory. If we were to buy one from the warehouse, all we could conclude about its probability of establishing \( x \geq 0 \) is just as calculated above.

Refinement is the converse of abstraction: we have

**Definition 1.2.1 Probabilistic Refinement** For two programs \( prog, prog' \) we say that \( prog' \) is a refinement of \( prog \), written \( prog \sqsubseteq prog' \), whenever for all post-expectations \( postE \) we have
\[ wp.prog.postE \Rightarrow wp.prog'.postE \quad (1.11) \]

We use the symbol \( \Rightarrow \) for \( \leq \) (extended pointwise) between expectations, which emphasises the similarity between probabilistic- and standard refinement.\(^{10}\)

\(^{10}\)We are aware that “\( \Rightarrow \)” looks more like “\( \geq \)” than it does “\( \leq \)” ; but for us its resemblance to “\( \Rightarrow \)” is the important thing.
1. Introduction to \( pGCL \)

From (1.11) we see that in the special case when expectation \( postE \) is an embedded predicate \([post]\), the meaning of \( \Rightarrow \) ensures that a refinement \( \text{prog}' \) of \( \text{prog} \) is at least as likely to establish \( post \) as \( \text{prog} \) is.\(^{11}\) That accords with the usual definition of refinement for standard programs — for then we know \( \text{wp.} \text{prog}.[post] \) is either zero or one, and whenever \( \text{prog} \) is certain to establish \( post \) (whenever \( \text{wp.} \text{prog}.[post] \equiv 1 \)) we know that \( \text{prog}' \) also is certain to do so (because then \( 1 \Rightarrow \text{wp.} \text{prog}'.[post] \)).

For our third example we prove a refinement: consider the program

\[
x := -y \quad \downarrow \uplus \quad x := +y ,
\]

which clearly satisfies Specification (1.8); thus it should refine Program (1.9), which is just that specification written in \( pGCL \). With Definition (1.11), we find for any \( postE \) that

\[
\text{wp.} (\text{Program (1.12)}) \cdot postE \equiv \text{wp.}(x := -y).postE/2 + \text{wp.}(x := +y).postE/2 \quad \text{definition } \downarrow \uplus , \text{ at (1.7)}
\]

\[
\equiv postE^-/2 + postE^+/2 \quad \text{introduce abbreviations}
\]

\[
\equiv (3/5)(postE^-/3 + 2postE^+/3) + (2/5)(3postE^-/4 + postE^+/4) \quad \text{arithmetic}
\]

\[
\leq \min \quad 3postE^-/4 + postE^+/4 \quad \text{any linear combination exceeds } \text{min}
\]

\[
\equiv \text{wp.} (\text{Program (1.9)} \cdot postE .
\]

The refinement relation (1.11) is indeed established for the two programs.

The introduction of 3/5 and 2/5 in the third step can be understood by noting that demonic choice \( \boxdot \) can be implemented by any probabilistic choice whatever: in this case we used \( \downarrow \uplus \). Thus a proof of refinement using program algebra might read

Program (1.12)

\[
x := -y \quad \downarrow \uplus \quad x := +y
\]

\( ^{10} \)Similar conflicts of interest arise when logicians use “\( \supset \)” for implies although, interpreted set-theoretically, implies is in fact “\( \subseteq \)”. And then there is “\( \subseteq \)" for refinement, which corresponds to “\( \supset \)" of behaviours.

\( ^{11} \)We see later in this chapter, however, and in Sec. A.1, that it is not sound to consider only post-expectations \( postE \) of the form \([post]\) in Def. 1.2.1: it is necessary for refinement, but not sufficient, that \( \text{prog}' \) be at least as likely to establish any postcondition \( post \) as \( \text{prog} \) is.
1.3 An informal computational model for pGCL

\[
\begin{align*}
= & \quad (x := -y \quad \frac{4}{7} \oplus \quad x := +y) \quad \text{Sec. B.1 Law 4} \\
\frac{4}{7} \oplus & \quad (x := -y \quad \frac{4}{7} \oplus \quad x := +y) \\
\sqsubseteq & \quad x := -y \quad \frac{4}{7} \oplus \quad x := +y \\
\sqcap & \quad x := -y \quad \frac{4}{7} \oplus \quad x := +y \\
= & \quad \text{Program (1.9).}
\end{align*}
\]

1.3 An informal computational model: pGCL describes gambling

We now use a simple card-and-dice game as an informal introduction to the computational model for pGCL, to support the intuition for probabilistic choice, demonic choice and their interaction. To start with, we consider the simplest case: non-looping programs without \( \sqcap \) or \( p \oplus \).

1.3.1 The standard game

Imagine we have a board of numbered squares, and a selection of numbered cards laid on it with at most one card per square; winning squares are indicated by coloured markers. The squares are the program states; the program is the pattern of numbered cards; the coloured markers indicate the postcondition.

To play the game:

An initial square is chosen (according to certain rules which do not concern us); subsequently

- if the square contains a card the card is removed, and play continues from the square whose number appeared on the card, and
- if the square does not contain a card, the game is over.

When the game is over the player has won if his final square contains a marker — otherwise he has lost.

This simple game is deterministic: any initial state always leads to the same final state. And because the cards are removed after use it is also guaranteed to terminate, if the board is finite. It is easily generalised however to include other features of standard programs:

\[\text{By } (\sqcap) \sqsubseteq (p \oplus) \text{ we mean that for all } \text{prog, prog}' \text{ we have } \text{prog} \sqcap \text{prog}' \sqsubseteq \text{prog} \oplus \text{prog}', \]

which is an instance of our Law 7 given on p. 323, in Sec. B.1 on program algebra.
looping  If the cards are not removed after use, the game can “loop.” A looping-forever player loses.

aborting  If a card reads go to jail, the program is said to “abort” and the player can be sent to any square whatever, including a special supplementary “jail” square from which there is no escape. A jailed player loses.

demonic nondeterminism  If each square can contain several cards, face-down, and the rules are modified so that the next state is determined by choosing just one of them “blind,” then play is nondeterministic. Taking the demonic (pessimistic) view, the player should expect to lose unless he is guaranteed to reach a winning position no matter which blind choices he makes.

In the standard game, for each (initial) square one can examine the cards before playing to determine whether a win is guaranteed from there. But once the game has started, the cards are turned face-down.

The set of squares from which a win is guaranteed is the weakest precondition.\textsuperscript{13}

1.3.2  The probabilistic game

Suppose now that each card contains not just one but, rather, a list of successor squares, and the choice from the list is made by rolling a die. In this deterministic game,\textsuperscript{14} play becomes a succession of die rolls, taking the player from square to square; termination (no card) and winning (marker) are defined as before.

When squares can contain several cards face down, each with a separate list of successors to be resolved by die roll, we are dealing with probability and demonic nondeterminism together: first the card is chosen “blind” (\textit{i.e.} demonically); the card is turned over and a die roll (probability) determines which of its listed alternatives to take.

In the probabilistic game one can ask for the greatest guaranteed probability of winning; as in the standard case, the prediction will vary depending on the initial square. (It’s because of demonic nondeterminism, as illustrated below, that the probability might be only a lower bound.)

\textsuperscript{13}A glance at Fig. 6.7.1 (p. 173) will show where we are headed in the visualisation of probabilistic preconditions!

\textsuperscript{14}Note that we still call this game “deterministic,” in spite of the probabilistic choices, and there are good mathematical reasons for doing so. (In Chap. 5, for example, we see that such programs are maximal in the refinement order.) An informal justification is that deterministic programs are those with repeatable behaviours and, even for probabilistic programs, the output \textit{distribution} is repeatable (to within statistical confidence measures) provided the program contains no demonic choice; see \textit{e.g.} p. 135.
In Fig. 1.3.1 is an example game illustrating some of the above points. The greatest guaranteed probability of winning from initial state 0 is only 1/2, in spite of the fact that the player can win every time if he is lucky enough to choose the first card in the pile; but he might be unlucky enough never to choose the first card, and we must assume the worst.

1.3.3 Expected winnings in the probabilistic game

For standard programs, the computational model of execution supports a complementary, “logical” view — given a set of final states (the postcondition) we can examine the program to determine the largest set of initial states (the weakest precondition) from which execution of the program is guaranteed to reach the designated final states. The sets of states are predicates, and the program is being regarded as a predicate transformer.

Regarding sets of states as characteristic functions (from the state space into \{0, 1\}), we generalise to “probabilistic predicates” by extending the range of those functions to all of \( \mathbb{R}_{\geq} \), the non-negative reals.\(^{15}\)

Probabilistic programs become functions from probabilistic postconditions to probabilistic weakest preconditions — we call them post-expectations and greatest pre-expectations. The corresponding generalisation in the game is as follows.

Rather than placing winning markers on the board, we place money — rather than strictly winning or losing, the player simply keeps whatever money he finds in his final square. In Fig. 1.3.2 we show the effect of translating our original game. In fact, not much changes: the probability of winning (in Fig. 1.3.1) translates into the equivalent expected payoff (Fig. 1.3.2) as the corresponding fraction of £1, illustrating this important fact:

The expected value of a characteristic function over a distribution is the same as the probability assigned to the set that function describes.

Thus using expectations is at least as general as using probabilities explicitly, since we can always restrict ourselves to \{0, 1\}-valued functions from which probabilities are then recovered.

For probabilistic programs, the operational interpretation of execution thus supports a “logical” view also — given a function from final states to \( \mathbb{R}_{\geq} \) (the post-expectation) one can examine the program beforehand to determine for each initial state the minimum expected (or “average”) win when the game is played repeatedly from there (the greatest pre-expectation) — also therefore a function from states to \( \mathbb{R}_{\geq} \).

\(^{15}\)In later chapters we will be more precise about the range of expectations, requiring them in particular to be bounded above.
To play from a square, you first pick one of the face-down cards. (In the diagram, we are seeing what’s on the cards with our x-ray vision.) Then you roll a die to choose one of the alternatives on the card. (In this case the die is two-sided, i.e. it is a coin.)

As special cases, a standard step (non-probabilistic) has only one alternative per card, but possibly many cards; and a deterministic step has only one card, but possibly many alternatives on it. A standard and deterministic step has one card, and only one alternative.

The winning final positions — the postcondition — are the states \{4, 5\}, marked with a £1 coin. From initial state 2 a win is guaranteed; from state 0 or 1 the minimum guaranteed probability of winning is 1/2; from state 3 the minimum probability is zero, since the second card might be chosen every time.

The probabilities are summarised in Fig. 1.3.2.

Figure 1.3.1. CARD-AND-DICE GAME OPERATIONAL SEMANTICS FOR pGCL
The post-expectation:

<table>
<thead>
<tr>
<th>Final state</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff awarded if this state reached</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>£1</td>
<td>£1</td>
<td>0</td>
</tr>
</tbody>
</table>

The probability of winning (ending on a £1) (from Fig. 1.3.1):

<table>
<thead>
<tr>
<th>Initial state</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greatest guaranteed probability of winning</td>
<td>1/2</td>
<td>1/2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The greatest pre-expectation:

<table>
<thead>
<tr>
<th>Initial state</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greatest guaranteed expected payoff</td>
<td>50p</td>
<td>50p</td>
<td>£1</td>
<td>£1</td>
<td>£1</td>
<td>£1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 1.3.2. A probabilistic and nondeterministic gambling game

Since the functions are expectations, the program is being regarded as an expectation transformer.\(^\text{16}\)

We are not limited to £1 coins for indicating postconditions — that is only an artefact of embedding standard postconditions into the probabilistic world. In general any amount of money can be placed in a square, and that is the key to allowing a smooth sequential composition of programs at the logical level — for if the program game of Fig. 1.3.2 were executed after some other program prog, the precondition of the two together with respect to the postcondition \{4, 5\} would be calculated by applying wp.prog to the greatest pre-expectation table for game. That is because sequential composition of programs becomes, as usual, functional composition of the corresponding transformers: we have

\[
\text{wp.(prog; game).}\{4, 5\} := \text{wp.prog.( wp.game.}\{4, 5\}\text{),}
\]

and that table contains non-integer values (for example 50p).

Another reason for allowing arbitrary values in \(\mathbb{R}_\geq\) is that using only standard postconditions (\{0, 1\}-valued) — equivalently, using explicit probabilities (recall the important fact above) — is not discriminating enough when nondeterminism is present: certain programs are identified that should be distinguished, and the semantics becomes non-compositional. (See Sec. A.1 for why this happens.)

\(^\text{16}\)For deterministic (yet probabilistic) programs, the card-game model and the associated transformers are essentially Kozen’s original construction [Koz81, Koz85]. We have added demonic (and later angelic) nondeterminism.
1.4 Behind the scenes: elementary probability theory

In probability theory, an event is a subset of some given sample space $S$, so that the event is said to have occurred if the sampled value is in that set; a probability distribution $\Pr$ over the sample space is a function from its events into the closed interval $[0, 1]$, giving for each event the probability of its occurrence. In the general case, for technical reasons, not necessarily all subsets of the sample space are events.\footnote{This may occur if the sample space is uncountable, for example; the general technique for such cases involves $\sigma$-algebras [GS92]. See Footnote 7 on p. 297 for an example.}

In our case we consider countable sample spaces, and take every (sub-)set of $S$ to be an event — and so we can regard a probability distribution more simply as a function from $S$ directly to probabilities (rather than from its subsets). Thus $\Pr: S \rightarrow [0, 1]$, and the probability of a more general event is now just the sum of the probabilities of its elements: we are using discrete distributions.\footnote{The price paid for using discrete distributions is that there are some “everyday” situations we cannot describe, such as the uniform “continuous” distribution over the real interval $[0, 1]$ that might be the result of the program “choose a real number $x$ randomly so that $0 \leq x \leq 1.$” We get away with it because no such program can be written in pGCL — at least, not at this stage.}

A random variable $X$ is a function from the sample space to the non-negative reals;\footnote{Footnote 12 on p. 134 gives a more generous definition.} and the expected value $\text{Exp}X$ of that random variable is defined in terms of the (discrete) probability distribution $\Pr$; we have the summation

$$\text{Exp}X := \left( \sum_{s \in S} \Pr.s \ast X.s \right). \quad \text{(1.13)}$$

It represents the “average” value of $X.s$ over many repeated samplings of $s$ according to the distribution $\Pr$.\footnote{Although the parentheses may look odd around $\sum$ — we write $(\sum \cdots)$ rather than $\sum(\cdots)$ — we always indicate the scope of bound variables (like $s$) with explicit delimiters, since it helps to avoid errors when doing calculations.}

In fact expected values can also be characterised without referring directly to an underlying probability distribution:

If a function $\text{Exp}$ is of type $(S \rightarrow \mathbb{R}_\geq) \rightarrow \mathbb{R}_\geq$, and it is

- non-negative so that $\text{Exp}X \geq 0$ for all $X: S \rightarrow \mathbb{R}_\geq$,
- linear so that for $X, Y: S \rightarrow \mathbb{R}_\geq$ and $c, d: \mathbb{R}_\geq$ we have

$$\text{Exp}(c \ast X + d \ast Y) = c \ast \text{Exp}X + d \ast \text{Exp}Y$$
and normalised so that it satisfies \( \text{Exp.1} = 1 \), where \( 1 \) is the constant function returning 1 for all arguments in \( S \),
then it is an expectation over some probability distribution: it can be shown that it is expressible uniquely in the form (1.13) for some \( \text{Pr} \).

The relevance of the above is that our real-valued expressions over the state — what we are calling “expectations” — are random variables, and that the expression

\[ \text{wp.prog}.\text{postE} \text{,} \tag{1.14} \]

as a function of initial values for the state variables, is a random variable as well. As a function of state variables, it is the expected value of the random variable \( \text{postE} \) (also a function of state variables, but those taken after execution) over the distribution of final states produced by executions of \( \text{prog} \), and so

\[ \text{preE} \Rightarrow \text{wp.prog}.\text{postE} \tag{1.15} \]

says that \( \text{preE} \) gives in any initial state a lower bound for the expected value of \( \text{postE} \) in the final distribution reached via execution of \( \text{prog} \) begun in that initial state.

In general, we call random variables post-expectations when they are to be evaluated in a final state, and we call them pre-expectations when they are calculated as at (1.14). And, like pre- and postconditions in standard programs, if placed “between” two programs a single random variable is a post-expectation for the first and a pre-expectation for the second.

But how do \( \text{prog} \) and an initial state determine a distribution? In fact the underlying distributions are found on the cards of the game from Sec. 1.3 — the sample space is the set of squares, and each card gives an explicit distribution over that space. If we consider the deterministic game, and regard “make one move in the game” as a program in its own right, then we have a function from initial state to final distribution — the function taking a square to the card that square contains.\(^{23}\) For any postcondition \( \text{postE} \) written, say, as an expression over names \( N \) of squares, and initial square \( N_0 \), the expression \( \text{wp.move}.\text{postE} \ (N \mapsto N_0) \) is the expectation of \( \text{postE} \) over the distribution of square names given on the card found at \( N_0 \).

\(^{22}\)It is a special case of the Riesz Representation Theorem which states, loosely speaking, that knowledge of the expectation (assumed to be given directly) of every random variable uniquely determines an underlying probability distribution. See for instance Feller [Fel71, p. 135].

\(^{23}\)For nondeterministic programs we are thus considering a function from state to sets of distributions, from a square to the set of cards there; again we see the general computational model underlying the expectation-transformer semantics.
For example, in Figs. 1.3.1 and 1.3.2 we see the above features: program 
move is given by the layout of the cards (Fig. 1.3.1); and the resulting pre-
and post-expectations are tabulated in Fig. 1.3.2. All three tables there are 
random variables over the state space \{0, \cdots, 6\}.

When we move to more general programs, we must relax the conditions 
that characterise expectations. If \textit{prog} is possibly nonterminating — if it is 
recursive or contains \texttt{abort} — then \textit{wp.prog.postE} may violate the normal-
isation condition \(\text{Exp}_1 = 1\). However as a function which satisfies the first 
two conditions it can still be regarded as an expectation in a weak sense.
That was shown by Kozen [Koz81] and later Jones [Jon90], who defined 
expectations with regard to “probability distributions” which may sum to 
less than one. Those are in fact a special case of Jones’s \textit{evaluations},\(^{24}\) and 
she gave conditions similar to the above for their existence [Jon90, p. 117].

Finally, if program \textit{prog} is not deterministic then we move further away 
from elementary theory, because \textit{wp.prog.postE} is no longer an expectation 
even in the weak sense: it not linear. It is still however the minimum of a 
set of expectations: if \textit{prog} and \textit{prog'} are deterministic programs then 
\(\text{wp.(prog \cap prog')}\text{.postE}\) is the pointwise minimum of the two expecta-
tions \textit{wp.prog.postE} and \textit{wp.prog'.postE}. This definition is one of the main 
features of this approach.

Thus although linearity is lost, it is not gone altogether: we retain so-
called sub-linearity,\(^{25}\) which implies that for any \(c_1, c_2 : \mathbb{R}_\geq\) and any program 
\textit{prog} we still have

\[
\text{wp.prog.(} c_1 * \text{post}E_1 + c_2 * \text{post}E_2 \text{)} \leq c_1 * \text{wp.prog.post}E_1 + c_2 * \text{wp.prog.post}E_2.
\]

And clearly non-negativity continues to hold.

The characterisations of expectations given above for the simpler cases 
might suggest that non-negative and sublinear functionals uniquely deter-
mine a set of probability distributions — and, in Chap. 5, that is indeed 
shown to be the case: sublinearity is the key “healthiness condition” for 
expectation transformers.\(^{26}\)

1.5 Basic syntax and semantics of \textit{pGCL}

1.5.1 Syntax

Let \textit{prog} range over programs and \textit{p} over real number expressions taking 
values between zero and one inclusive; assume that \textit{x} stands for a list of 
distinct variables, and \textit{expr} for a list of expressions (of the same length as \textit{x}

\(^{24}\)She was working in a much more general context.

\(^{25}\)The actual property is slightly more general than we give here; see Sec. 1.6.

\(^{26}\)Halpern and Pucella [HP02] have recently studied similar properties.
where appropriate); and let the program scheme \( C \) be a program in which program names like \( xxx \) can appear. The syntax of \( pGCL \) is as follows:

\[
prog \ := \ \text{abort} \mid \text{skip} \mid x := E \mid \text{prog} ; \text{prog} \\
\text{prog} \oplus \text{prog} \mid \text{prog} \oslash \text{prog} \\
\text{(mu } xxx \bullet C) \tag{1.16}
\]

The first four constructs, namely \text{abort}, \text{skip}, assignment and sequential composition, are just the conventional ones [Dij76].

The remaining constructs are for probabilistic choice, nondeterministic choice and recursion: given \( p \) in the closed interval \( [0, 1] \) we write \( \text{prog} \oplus \text{prog}' \) for the probabilistic choice between programs \( \text{prog} \) and \( \text{prog}' \); they have probability \( p \) and \( 1-p \) respectively of being selected. In many cases \( p \) will be a constant, but in general it can be an expression over the state variables.

1.5.2 Shortcuts and “syntactic sugar”

For convenience we extend our logic and language with the following notations.

\textbf{Boolean embedding} — For predicate \texttt{pred} we write \([\texttt{pred}]\) for the expectation “1 if \texttt{pred} else 0”.\(^{27}\)

\textbf{Conditional} — The conditional

\[
\text{prog if } \texttt{pred} \text{ else } \text{prog}' \\
\text{or } \text{if } \texttt{pred} \text{ then } \text{prog} \text{ else } \text{prog}' \text{ fi },
\]

chooses program \( \text{prog} \) (resp. \( \text{prog}' \)) if Boolean \texttt{pred} is \texttt{true} (resp. \texttt{false}).

It is defined \( \text{prog} \ [\texttt{pred}] \oplus \text{prog}' \).

If \texttt{else} is omitted then \texttt{else skip} is assumed. (See also the “hybrid” conditional of Sec. 3.1.2.)

\textbf{Implication-like relations} — For expectations \texttt{exp}, \texttt{exp}' we write

\[
\texttt{exp} \Rightarrow \texttt{exp}' \text{ for } \texttt{exp} \text{ is everywhere less than or equal to } \texttt{exp}' \\
\texttt{exp} \equiv \texttt{exp}' \text{ for } \texttt{exp} \text{ and } \texttt{exp}' \text{ are everywhere equal} \\
\texttt{exp} \subseteq \texttt{exp}' \text{ for } \texttt{exp} \text{ is everywhere greater than or equal to } \texttt{exp}'
\]

We distinguish \( \texttt{exp} \Rightarrow \texttt{exp}' \) from \( \texttt{exp} \leq \texttt{exp}' \) — the former is a statement about \texttt{exp} and \texttt{exp}', thus true or false as a whole; the latter is itself a Boolean-valued expression over the state, possibly true in some states and false in others.\(^{28}\) Similarly we regard \( \texttt{exp} = \texttt{exp}' \) as

\(^{27}\)We will not distinguish predicates from Boolean-valued expressions.

\(^{28}\)Note that \( \texttt{exp} \Rightarrow \texttt{exp}' \) is different again, in fact badly typed if \texttt{exp} and \texttt{exp}' are expectations: one real-valued function cannot “imply” another.
true in just those states where \( \text{exp} \) and \( \text{exp}' \) are equal, and false in the rest.

The closest standard equivalent of \( \Rightarrow \) is the entailment relation \( \models \) between predicates\(^{29}\) — and in fact \( \text{post} \models \text{post}' \) exactly when \( [\text{post}] \Rightarrow [\text{post}'] \), meaning that the “embedding” of \( \models \) is \( \Rightarrow \).

**Multi-way probabilistic choices** — A probabilistic choice over \( N \) alternatives can be written horizontally
\[
(\text{prog}_1 \oplus p_1 | \cdots | \text{prog}_N \oplus p_N)
\]

or vertically
\[
\begin{align*}
\text{prog}_1 & \oplus p_1 \\
\text{prog}_2 & \oplus p_2 \\
\vdots \\
\text{prog}_N & \oplus p_N
\end{align*}
\]

in which the probabilities are enumerated and sum to no more than one.\(^{30}\) We can also write a “probabilistic comprehension” (\( \{ i : \text{prog}_i \oplus p_i \} \)) over some countable index set \( I \). In general, we have
\[
\text{wp}(\text{prog}_1 \oplus p_1 | \cdots | \text{prog}_N \oplus p_N) \cdot \text{postE} \\
: = \quad p_1 \cdot \text{wp} \cdot \text{prog}_1 \cdot \text{postE} + \cdots + p_N \cdot \text{wp} \cdot \text{prog}_N \cdot \text{postE}.
\]

It means “execute \( \text{prog}_1 \) with probability at least \( p_1 \), and \( \text{prog}_2 \) with probability at least \( p_2 \)”\(^31\)

If the probabilities sum to 1 exactly, then it is a simple \( N \)-way probabilistic branch; if there is a deficit \( 1 - \Sigma_i p_i \), it gives the probability of aborting.

When all the programs \( \text{prog}_i \) are assignments with the same left-hand side, say \( x = \text{expr}_i \), we write even more briefly
\[
x = (\text{expr}_1 \oplus p_1 | \cdots | \text{expr}_N \oplus p_N).
\]

**Variations on \( \oplus \)** — By \( \text{prog} \oplus_{p} \text{prog}' \) we mean \( \text{prog}' \oplus p \text{prog} \), and in general we write \( \text{prog} \oplus_{p, p'} \text{prog}' \) for
\[
\begin{align*}
\text{prog} & \oplus p \\
\text{prog}' & \oplus p' \\
\text{prog} \land \text{prog}' & \oplus 1 - (p+p')
\end{align*}
\]

the program that executes \( \text{prog} \) with probability at least \( p \) and \( \text{prog}' \)

---

\(^{29}\)One predicate entails another, written \( \models \), just when it implies the other in all states.

\(^{30}\)See Sec. 4.3 for an example of the vertical notation.

\(^{31}\)It is “at least \( p_i \)” because if the probabilities sum to less than one there will be an “aborting” component, which might behave like \( \text{prog}_i \).
1.5. Basic syntax and semantics of pGCL

with probability at least \( p' \); we assume \( p + p' \leq 1 \).

By \( \geq p \oplus \) we mean \( p \oplus 0 \), and so on. (See also (B.3) on p. 328.)

**Demonic choice** — We write demonic choice between assignments to
the same variable \( x \) as

\[
x: \in \{ \text{expr}_1, \text{expr}_2, \cdots \} , \quad \text{or} \quad x := \text{expr}_1 \cap \text{expr}_2 \cap \cdots ,
\]

in each case abbreviating \( x := \text{expr}_1 \cap x := \text{expr}_2 \cap \cdots \). More generally
we can write \( x \in \text{expr} \) or \( x \notin \text{expr} \) if \( \text{expr} \) is set-valued, provided the
implied choice is finite.\(^{32}\)

**Iteration** — The construct \( (\mu \text{xxx} \cdot C) \) behaves as prescribed by the
program context \( C \) except that it invokes itself recursively whenever
it reaches a point where the program name \( \text{xxx} \) appears in \( C \). Then,
in the usual way, iteration is a special case of recursion:

\[
\text{do } \text{pred} \rightarrow \text{body od} \quad := \quad (\mu \text{xxx} \cdot (\text{body}; \text{xxx}) \text{ if } \text{pred else skip}) . \quad 33
\]

1.5.3 Example of syntax: the “Monty Hall” game

We illustrate the syntax of our language with the example program of
Fig. 1.5.1. There are three curtains, labelled \( A \), \( B \) and \( C \), and a prize is
hidden nondeterministically behind one of them, say \( p_c \). A contestant hopes
to win the prize by guessing where it is hidden: he chooses randomly to

\(^{32}\)None of our examples requires a choice from the empty set. We see later that the
finiteness requirement is so that our programs will be continuous (Footnote 60 on p. 71); and in some cases — for example, the third and fourth statements of the program shown in Fig. 1.5.1 — we rely on type information for that finiteness.

\(^{33}\)An equivalent but simpler formulation is given by the least fixed-point definition

\[
\text{wp.(do pred } \rightarrow \text{body od).R} \quad := \quad (\mu Q \cdot \text{wp.body.Q if } \text{pred else R}) , \quad (1.19)
\]

which matches Dijkstra’s original formulation more closely [Dij76]. But there is some
technical work required to get between the two, as we explain later at (7.12). The
expression on the right can be read

the least pre-expectation \( Q \) such that

\[
Q \equiv \text{wp.body.Q if } \text{pred else R} ,
\]

and is called a fixed point because placing \( Q \) in the expression does not alter its value
— this is the mathematical equivalent of “and the same again” when the loop returns
to its starting point for potentially more iterations.

The “least,” for us, means the lowest expectation — that reflects the view, appropriate
for elementary sequential programming, that unending iteration should have little worth
(in fact, zero). For standard programming, the order is \( \text{false } \leq \text{true} \) so that taking the
least fixed-point means adopting the view that an infinite loop does not establish any
postcondition (i.e., has precondition \( \text{false} \)).

A more discriminating treatment of unending computations is given in Part III.
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\[ pc \in \{A, B, C\}; \]  
\[ cc = (A @ \frac{1}{3} | B @ \frac{1}{3} | C @ \frac{1}{3}); \]  
\[ ac \notin \{pc, cc\}; \]  
\[ (cc \notin \{cc, ac\}) \text{ if clever else skip} \]

The three “curtain” variables \( ac, cc, pc \) are of type \( \{A, B, C\} \).

Written in full, the first three statements would be

\[
\begin{align*}
pc & = A \sqcap pc; = B \sqcap pc; = C; \\
cc & = A \sqcup (cc; = B \sqcup cc; = C); \\
ac & \in \{A, B, C\} - \{pc, cc\}.
\end{align*}
\]

The fourth statement is written using \( \notin \) just for convenience — in fact it executes deterministically, since \( cc \) and \( ac \) are guaranteed to be different at that point.

Figure 1.5.1. The “Monty Hall” program

point to curtain \( cc \). The host then tries to get the contestant to change his choice, showing that the prize is not behind some other curtain \( ac \) — which means that either the contestant has chosen it already or it is behind the other closed curtain.

Should the contestant change his mind?

1.5.4 Intuitive interpretation of pGCL expectations

In its full generality, an expectation is a function describing how much each program state is “worth.”

The special case of an embedded predicate \([pred] \) assigns to each state a worth of zero or of one: states satisfying \( pred \) are worth one, and states not satisfying \( pred \) are worth zero. The more general expectations arise when one estimates, in the initial state of a probabilistic program, what the worth of its final state will be. That estimate, the “expected worth” of the final state, is obtained by summing over all final states

the worth of the final state multiplied by the probability the program “will go there” from the initial state.

Naturally the “will go there” probabilities depend on “from where,” and so that expected worth is a function of the initial state.

When the worth of final states is given by \([post] \), the expected worth of the initial state turns out to be just the probability that the program will reach \( post \). That is because
expected worth of initial state

\[ \equiv (\text{probability } \text{prog} \text{ reaches } \text{post}) \]
\[ \times (\text{worth of states satisfying } \text{post}) \]
\[ + (\text{probability } \text{prog} \text{ does not reach } \text{post}) \]
\[ \times (\text{worth of states not satisfying } \text{post}) \]
\[ \equiv (\text{probability } \text{prog} \text{ reaches } \text{post}) \times 1 \]
\[ + (\text{probability } \text{prog} \text{ does not reach } \text{post}) \times 0 \]
\[ \equiv \text{probability } \text{prog} \text{ reaches } \text{post} \]

note we have relied on the fact that all states satisfying \text{post} have worth one.

More general analyses of programs \text{prog} in practice lead to conclusions of the form

\[ p \equiv \text{wp}.\text{prog}.[\text{post}] \]

for some \( p \) and \( \text{post} \) which, given the above, we can interpret in two equivalent ways:

- the expected worth \([\text{post}]\) of the final state is at least the value of \( p \) in the initial state; or
- the probability that \text{prog} will establish \text{post} is at least \( p \).\(^{34}\)

Each interpretation is useful, and in the following example we can see them acting together: we ask for the probability that two fair coins when flipped will show the same face, and calculate

\[ \text{wp}. (x := H \frac{1}{\text{H}} \oplus x := T; y := H \frac{1}{\text{H}} \oplus y := T) \]
\[ \equiv \frac{1}{\text{H}} \oplus \text{ and sequential composition}^{35} \]
\[ \text{wp}.(x := H \frac{1}{\text{H}} \oplus x := T).([x = \text{H}] / 2 + [x = \text{T}] / 2) \]
\[ \equiv \frac{1}{\text{H}}((\text{H} = \text{H}) / 2 + (\text{H} = \text{T}) / 2) \]
\[ + \frac{1}{\text{H}}((\text{T} = \text{H}) / 2 + (\text{T} = \text{T}) / 2) \]
\[ \frac{1}{\text{H}} \oplus \text{ and :=} \]

\(^{34}\)We must say “at least” in general, because possible demonic choice in \text{prog} means that the pre-expectation is only a lower bound for the actual expected value the program could deliver; and some analyses give only the weaker \( p \Rightarrow \text{wp}.\text{prog}[\text{post}] \) in any case. See also Footnote 14 on p. 89.

\(^{35}\)See Fig. 1.5.3 for this definition.
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\[ (1/2)(1/2 + 0/2) + (1/2)(0/2 + 1/2) \text{ \quad definition } \]
\[ = 1/2 \text{ \quad arithmetic } \]

We can then use the second interpretation above to conclude that the faces are the same with probability 1/2.\(^{36}\)

But part of the above calculation involves the more general expression

\[ \text{wp}(x; = H \frac{1}{2} + x; = T).([x = H]/2 + [x = T]/2), \quad (1.20) \]

and what does that mean on its own? It must be given the first interpretation, that is as an expected worth, since “will establish \([x = H]/2 + [x = T]/2\)” makes no sense. Thus it means

the expected value of the expression \([x = H]/2 + [x = T]/2\)
after executing the program \(x; = H \frac{1}{2} + x; = T\),

which the calculation goes on to show is in fact 1/2. But for our overall conclusions we do not need to think about the intermediate expressions — they are only the “glue” that holds the overall reasoning together.\(^{37}\)

1.5.5 Semantics

The probabilistic semantics is derived from generalising the standard semantics in the way suggested in Sec. 1.3. Let the state space be \(S\).

**Definition 1.5.2 Expectation space** The space of expectations over \(S\) is defined

\[ \mathbb{E}S := (S \rightarrow \mathbb{R}_\geq, \Rightarrow), \]

where the entailment relation \(\Rightarrow\), as we have seen, is inherited pointwise from the normal \(\leq\) ordering in \(\mathbb{R}_\geq\). The expectation-transformer model for programs is

\[ TS := (\mathbb{E}S \leftarrow \mathbb{E}S, \sqsubseteq), \]

where we write the functional arrow backward just to emphasise that such transformers map final post-expectations to initial pre-expectations, and where the refinement order \(\sqsubseteq\) is derived pointwise from entailment \(\Rightarrow\) on \(\mathbb{E}S\). \(\Box\)

\(^{36}\)Recall Footnote 34.) If we do know, by other means say, that the program is deterministic (though still probabilistic), then we can say the pre-expectation is exact.

\(^{37}\)See p. 271 for an example of this same analogy, but in the context of temporal logic.
Although both $\mathcal{E}S$ and $\mathcal{T}S$ are lattices, neither is a complete partial order,\footnote{A partial order differs from the familiar “total” orders like “$\leq$” in that two elements can be “incomparable”; the most common example is subset $\subseteq$ between sets, which satisfies Reflexivity (a set is a subset of itself), Anti-Symmetry (two sets cannot be subsets of each other without being the same set) and Transitivity (one set within a second within a third is a subset of the third directly as well). But it is not true that for any two sets one is necessarily a subset of the other. A lattice is a non-empty partially ordered set where for all $x, y$ in the set there is a greatest lower bound $x \cap y$ and and a least upper bound $x \cup y$. This holds e.g. for the lattice of sets, as above; but the collection of non-empty sets is not a lattice, because $x \cap y$ (which is how $x \cap y$ is written for sets) is not necessarily non-empty even if $x$ and $y$ are.

A partial order $\subseteq$ is chain- or directed complete — then called a cpo — when it contains all limits of chains or directed sets respectively, where a chain is a set totally ordered by $\subseteq$ and a set is $\subseteq$-directed if for any $x, y$ in the set there is a $z$ also in the set such that $x, y \subseteq z$. (Since a chain is directed, directed completeness implies chain completeness; in fact with the Axiom of Choice, chain- and directed completeness are equivalent.)

All of these details can be found in standard texts [DP90].} because $\mathbb{R}_\geq$ itself is not. (It lacks an adjoined $\infty$ element.) In addition, when $S$ is infinite (see e.g. Sec. 8.2 of Part II) we must impose the condition on elements of $\mathcal{E}S$ that each of them be bounded above by some non-negative real.\footnote{There is a difference between requiring that there be an upper bound for all expectations (we do not) and requiring that each expectation separately have an upper bound (we do).

In the first case, we would be saying that there is some $M$ such that every expectation $\alpha$ in $\mathcal{E}S$ satisfies $\alpha \Rightarrow M$. That would be convenient because it would make both $\mathcal{E}S$ and $\mathcal{T}S$ complete partial orders, trivially; and that would e.g. allow us to use a standard treatment of fixed points.

But we adopt the second case where, for each expectation $\alpha$ separately, there is some $M_\alpha$ such that $\alpha \Rightarrow M_\alpha$; and, as $\alpha$ varies, these $M_\alpha$’s can increase without bound. That is why $\mathcal{E}S$ is not complete and is, therefore, why we will need a slightly special argument when dealing with fixed points.} In Fig. 1.5.3 we give a probabilistic semantics to the constructs of our language. It has the important feature that the standard programming constructs behave as usual, and are described just as concisely.

Note that our semantics states how $\text{wp} \cdot \text{prog}$ in each case transforms an expression in the program variables; that is, we give a procedure for calculating the greatest pre-expectation by purely syntactic manipulation. An alternative view is to see the post-expectations as mathematical functions of type $\mathcal{E}S$, and the expressions $\text{wp} \cdot \text{prog}$ are then of type $\mathcal{T}S$.

The expression-based view is more convenient in an introduction, and for the treatment of specific programs; the function-based view is more convenient (and, for recursion, necessary) for general properties of expectation transformers. In this chapter and the rest of Part I we retain the
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\[
\begin{align*}
wp.\text{abort}.postE & := 0 \\
wp.\text{skip}.postE & := postE \\
wp.(x:= expr).postE & := postE \langle x \mapsto expr \rangle \\
wp.(\text{prog} ; \text{prog}').postE & := wp.\text{prog}.(wp.\text{prog}'.postE) \\
wp.\text{prog} /\text{prog}'.postE & := wp.\text{prog}.postE \min wp.\text{prog}'.postE \\
wp.\text{prog} \parallel \text{prog}'.postE & := p \ast wp.\text{prog}.postE + \overline{p} \ast wp.\text{prog}'.postE
\end{align*}
\]

Recall that \(\overline{p}\) is the complement of \(p\).

The expression on the right gives the greatest pre-expectation of \(postE\) with respect to each \(pGCL\) construct, where \(postE\) is an expression of type \(\text{ES}\) over the variables in state space \(S\). (For historical reasons we continue to write \(wp\) instead of \(gp\).)

In the case of recursion, however, we cannot give a purely syntactic definition. Instead we say that

\[
(\text{mu } \text{x} \times E \times \text{C}) := \text{least fixed-point of the function } cntx.\langle TS \rightarrow TS \rangle \\
\text{defined so that } cntx.(wp.\text{x} \times E) = wp.C. \tag{40}
\]

Figure 1.5.3. Probabilistic \(wp\)-semantics of \(pGCL\)

expression-based view as far as possible; but in Part II we use the more mathematical notation. (See for example Sec. 5.3.)

The worst program \text{abort} cannot be guaranteed to terminate in any proper state and therefore maps every post-expectation to 0. The immediately terminating program \text{skip} does not change anything, therefore the expected value of post-expectation \(postE\) after execution of \text{skip} is just its actual value before. The pre-expectation of the assignment \(x:= expr\) is the postcondition with the expression \(expr\) substituted for \(x\). Sequential composition is functional composition. The semantics of demonic choice \(\parallel\) reflects the dual metaphors for it: as abstraction, we must take the minimum because we are giving a guarantee over all possible implementations; as a demon’s behaviour, we assume he acts to make our expected winnings as small as possible.

The pre-expectation of probabilistic choice is the weighted average of the pre-expectations of its branches. Since any such average is no less than the minimum it follows immediately that probabilistic choice refines demonic

\[40\text{Because } TS \text{ is not complete, to ensure existence of the fixed point we insist that the transformer-to-transformer function } cntx \text{ be "feasibility-preserving," i.e. that if applied to a feasible transformer it returns a feasible transformer again. "Feasibility" of transformers is one of the "healthiness conditions" we will encounter in Sec. 1.6. For convenience, we usually assume that } cntx \text{ is continuous as well.}

See Lem. 5.6.8 on p. 148.
choice, which corresponds to our intuition. In fact we consider probabilistic choice to be a deterministic programming construct; that is we say that a program is deterministic if it is free of demonic nondeterminism unless it aborts.\footnote{Some writers call that \textit{pre-determinism}: “deterministic if terminating.”}

Finally, recursive programs have least-fixed-point semantics as usual.

1.5.6 Example of semantics: Monty Hall again

We illustrate the semantics by returning to the program of Fig. 1.5.1. Consider the post-expectation $[pc = cc]$, which takes value one just in those final states in which the candidate has correctly chosen the prize. Working backwards through the program’s four statements, we have first (by standard \textit{wp} calculations) that

\[
\text{wp. } ((cc \notin \{cc, ac\}) \text{ if clever else skip}) . [pc = cc] \\
\equiv \ [\text{clever}] * \{(ac, cc, pc) = \{A, B, C\}\} + \ [\neg\text{clever}] * \ [pc = cc] ,
\]

because (in case \text{clever}) the nondeterministic choice is guaranteed to pick \textit{pc} only when it cannot avoid doing so.\footnote{In Fig. 1.5.1 we said that this fourth statement “executes deterministically”; yet here we have called it nondeterministic. On its own, it is nondeterministic; but in the context of the program its nondeterminism is limited to making a choice from a singleton set, as our subsequent calculations will show.}

Standard reasoning suffices for our next step also:

\[
\text{wp. } (ac \notin \{pc, cc\}). \\
\text{([clever] * ([ac, cc, pc] = \{A, B, C\}] + \ [\neg\text{clever}] * \ [pc = cc])} \\
\equiv \ [\text{clever}] * \ [pc \neq cc] + \ [\neg\text{clever}] * \ [pc = cc] .
\]

For the \text{clever} case note that $\{ac, cc, pc\} = \{A, B, C\}$ holds (in the post-expectation) iff all three elements differ, and that the statement itself establishes only two of the required three inequalities — that $ac \neq pc$ and $ac \neq cc$. The weakest precondition supplies the third.

For the $\neg\text{clever}$ case note that neither $pc$ nor $cc$ is assigned to by $ac \notin \{pc, cc\}$, so that $pc = cc$ holds afterwards iff it held before.

The next statement is probabilistic, and so produces a probabilistic pre-expectation involving the factors $1/3$ given explicitly in the program; we have

\[
\text{wp. } (cc = (A @ \frac{1}{3} | B @ \frac{1}{3} | C @ \frac{1}{3})). \\
\text{([clever] * [pc \neq cc] + [\neg\text{clever}] * [pc = cc])} \\
\equiv \ [\text{clever}] /3 * \ [(pc \neq A] + [pc \neq B]) + [pc \neq C]) + \ [\neg\text{clever}] /3 * \ ([pc = A] + [pc = B] + [pc = C])
\]
\[ \equiv (\lceil \text{clever} \rceil /3) \times 2 + (\lceil \neg \text{clever} \rceil /3) \times 1 \quad \text{type of pc is } \{A, B, C\} \]
\[ \equiv 2 \lceil \text{clever} \rceil /3 + [\neg \text{clever}] /3. \]

Then for the first statement \( pc : \in \{A, B, C\} \) we only note that \( pc \) does not appear in the final condition above, thus leaving it unchanged under \( wp \); with simplification it becomes
\[ (1 + [\text{clever}]) / 3, \]
which is thus the pre-expectation for the whole program.

Since the post-expectation \( \lceil pc = cc \rceil \) is standard (it is the characteristic function of the set of states in which \( pc = cc \)), we are able to interpret the pre-expectation directly as the probability that \( pc = cc \) will be satisfied on termination: we conclude that the contestant has \( 2 / 3 \) probability of finding the prize if he is clever, and only \( 1 / 3 \) if he is not.

### 1.6 Healthiness and algebra for pGCL

Recall that all standard GCL constructs satisfy the important property of conjunctivity\(^{44}\) — that is, for any GCL command \( \text{prog} \) and post-conditions \( \text{post, post}' \) we have
\[
wp.\text{prog}.(\text{post} \land \text{post}') = wp.\text{prog}.'\text{post} \land wp.\text{prog}.\text{post}'.
\]

That “healthiness condition” [Dij76] is used to prove many general properties of programs.

In pGCL the healthiness condition becomes “sublinearity,” a generalisation of conjunctivity: \(^{45}\)

**Definition 1.6.1 Sublinearity of pGCL** Let \( c_0, c_1, c_2 \) be non-negative reals, and \( \text{post}E_1, \text{post}E_2 \) expectations; then all pGCL constructs \( \text{prog} \) satisfy
\[
wp.\text{prog}.(c_1 \ast \text{post}E_1 + c_2 \ast \text{post}E_2 \ominus c_0) \equiv c_1 \ast wp.\text{prog}.\text{post}E_1 + c_2 \ast wp.\text{prog}.\text{post}E_2 \ominus c_0,
\]
which property of \( \text{prog} \) is called sublinearity. Truncated subtraction \( \ominus \) is defined
\[
x \ominus y := (x - y) \max 0,
\]

\(^{43}\)Footnote 50 on p.33 explains how typing might be propagated this way.

\(^{44}\)They satisfy monotonicity too, which is implied by conjunctivity.

\(^{45}\)Having discovered a probabilistic analogue of conjunctivity, we naturally ask for an analogue of disjunctivity. That turns out to be “super-linearity” — which when combined with sublinearity gives (just) linearity, and is characteristic of deterministic probabilistic programs, just as disjunctivity (with conjunctivity) characterises deterministic standard programs. See Sec. 8.3.
the maximum of the normal difference and zero. It has syntactic precedence lower than $+$. □

Although it has a strange appearance, from sublinearity we can extract a number of very useful consequences, as we now show. We begin with monotonicity, feasibility and scaling.\footnote{These properties are collected together in Sec. 5.6, and restated in Part II as Defs. 5.6.3–5.6.5.}

**Definition 1.6.2 Healthiness conditions**

- **monotonicity:** increasing a post-expectation can only increase the pre-expectation. Suppose $\text{post}E \Rightarrow \text{post}E'$ for two expectations $\text{post}E, \text{post}E'$; then
  
  $\text{wp.prog.post}E' \equiv \text{wp.prog.}(\text{post}E + (\text{post}E' - \text{post}E))$

  $\Leftrightarrow \text{post}E' - \text{post}E \leq 0$, hence well defined;
  sublinearity with $c_0, c_1, c_2 : = 0, 1, 1$

  $\text{wp.prog.post}E + \text{wp.prog.}(\text{post}E' - \text{post}E)$

  $\Leftrightarrow \text{wp.prog.post}E . \ 0 \Rightarrow \text{wp.prog.}(\text{post}E' - \text{post}E)$

- **feasibility:** pre-expectations cannot be “too large.” First note that

  $\text{wp.prog.0} \equiv \text{wp.prog.}(2 * 0)$

  $\Leftrightarrow 2 * \text{wp.prog.0} , \ \text{sublinearity with} \ c_0, c_1, c_2 : = 0, 2, 0$

  so that $\text{wp.prog.0}$ must be zero.

  Now write $\max \ \text{post}E$ for the maximum of $\text{post}E$ over all its variables’ values; then

  $0 \equiv \text{wp.prog.0}$ \hspace{1cm} \text{feasibility above}

  $\equiv \text{wp.prog.}(\text{post}E \circ \max \ \text{post}E)$ \hspace{0.5cm} \text{post}E \circ \max \ \text{post}E \equiv 0$

  $\Leftrightarrow \text{wp.prog.post}E \circ \max \ \text{post}E . \ c_0, c_1, c_2 : = \max \ \text{post}E, 1, 0$

  But from $0 \Leftrightarrow \text{wp.prog.post}E \circ \max \ \text{post}E$ we have trivially that

  $\text{wp.prog.post}E \Rightarrow \max \ \text{post}E$, \hspace{1cm} (1.21)

  which we identify as the feasibility condition for $pGCL$.\footnote{Note how the general (1.21) implies the strictness condition $\text{wp.prog.0} \equiv 0$, a direct numeric embedding of Dijkstra’s Law of the Excluded Miracle.}

- **scaling:** multiplication by a non-negative constant distributes through commands. Note first that $\text{wp.prog.}(c * \text{post}E) \Leftrightarrow c * \text{wp.prog.post}E$ directly from sublinearity.
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For $\Rightarrow$ we have two cases: when $c$ is zero, trivially from feasibility

$$wp.prog.(0 \ast postE) \equiv wp.prog.0 \equiv 0 \equiv 0 \ast wp.prog.postE;$$

and for the other case $c \neq 0$ we reason

$$wp.prog.(c \ast postE) \equiv c(1/c) \ast wp.prog.(c \ast postE) \quad c \neq 0$$

$$\Rightarrow c \ast wp.prog.((1/c) \ast postE)) \quad \text{sublinearity using } 1/c$$

$$\equiv c \ast wp.prog.postE,$$

thus establishing $wp.prog.(c \ast postE) \equiv c \ast wp.prog.postE$ generally. (See p. 53 for an example of scaling’s use.)

The remaining property we examine is so-called “probabilistic conjunctivity.” Since standard conjuction “∧” is not defined over numbers, we have many choices for a probabilistic analogue “&” of it, requiring only that

$$
\begin{align*}
0 \& 0 &= 0 \\
0 \& 1 &= 0 \\
1 \& 0 &= 0 \\
1 \& 1 &= 1
\end{align*}
$$

(1.22)

for consistency with embedded Booleans.

Obvious possibilities for & are multiplication * and minimum min, and each of those has its uses; but neither satisfies anything like a generalisation of conjunctivity. Return for example to the program of Fig. 1.5.1, and consider its second statement

$$cc = \langle A \oplus \frac{1}{3} | B \oplus \frac{1}{3} | C \oplus \frac{1}{3} \rangle.$$ 

Writing $prog$ for the above, with postcondition $[cc \neq C] \min [cc \neq A]$ we find

$$wp.prog.([cc \neq C] \min [cc \neq A]) \equiv wp.prog.[cc \neq C \land cc \neq A] \equiv wp.prog.[cc = B] \equiv 1/3 \neq 2/3 \min 2/3 \equiv wp.prog.[cc \neq C] \min wp.prog.[cc \neq A].$$

Thus probabilistic programs do not distribute min in general, and we must find something else. Instead we define

$$exp \& exp' := exp + exp' \ominus 1,$$  

(1.23)
whose right-hand side is inspired by sublinearity when \( c_0, c_1, c_2 := 1, 1, 1 \). The operator is commutative; and if we restrict expectations to \([0, 1]\) it is associative as well. Note however that it is not idempotent.\(^{48}\)

We now state a (sub-)distribution property for \&, a direct consequence of sublinearity.

**sub-conjunctivity:** the operator \& sub-distributes through expectation transformers. From sublinearity with \( c_0, c_1, c_2 := 1, 1, 1 \) we have

\[
\text{wp.prog} (\text{postE} \& \text{postE}') \quad \subseteq \quad \text{wp.prog.postE} \& \text{wp.prog.postE}'
\]

for all \text{prog}.

(Unfortunately there does not seem to be a full (\( \equiv \)) conjunctivity property for expectation transformers.)

Beyond sub-conjunctivity, we say that \& generalises conjunction for several other reasons as well. The first is of course that it satisfies the standard properties (1.22).

The second reason is that sub-conjunctivity (a consequence of sublinearity) implies “full” conjunctivity for standard programs. Standard programs, containing no probabilistic choices, take standard \([\text{post}]\)-style post-expectations to standard pre-expectations: they are the embedding of GCL in \( pGCL \), and for standard \text{prog} we now show that

\[
\text{wp.prog}([\text{post}] \& [\text{post}']) \quad \equiv \quad \text{wp.prog.}[\text{post}] \& \text{wp.prog.}[\text{post}'] .
\]

(1.24)

First note that “\( \subseteq \)” comes directly from sub-conjunctivity above, taking \( \text{postE}, \text{postE}' \) to be \([\text{post}]\), \([\text{post}']\).

For “\( \supseteq \)” we appeal to monotonicity, because \([\text{post}] \& [\text{post}'] \supseteq [\text{post}]\) whence \( \text{wp.prog}([\text{post}] \& [\text{post}']) \supseteq \text{wp.prog.}[\text{post}]\), and similarly for \( \text{post}'\).

Putting those together gives

\[
\text{wp.prog}([\text{post}] \& [\text{post}']) \quad \supseteq \quad \text{wp.prog.}[\text{post}] \min \text{wp.prog.}[\text{post}'] ,
\]

by elementary arithmetic properties of \( \supseteq \). But on standard expectations — which \( \text{wp.prog.}[\text{post}]\) and \( \text{wp.prog.}[\text{post}']\) are, because \text{prog} is standard — the operators \( \min \) and \& agree.

A last attribute linking \& to \( \wedge \) comes straight from elementary probability theory. Let \( X \) and \( Y \) be two events, not necessarily independent: then

if the probability of \( X \) is at least \( p \), and the probability of \( Y \) is at least \( q \), the most that can be said in general about the joint event \( X \cap Y \) is that it has probability at least \( p \& q \).

---

\(^{48}\)A binary operator \( \odot \) is idempotent just when \( x \odot x = x \) for all \( x \).
To see this, we begin by recalling that for any events \( X, Y \) and any probability distribution \( \Pr \) we have\(^{49}\)

\[
\Pr(X \cap Y) = \Pr(X) + \Pr(Y) - \Pr(X \cup Y)
\]

\[
\geq (\Pr(X) + \Pr(Y) - 1) \uplus 0 .
\]

We are not dealing with exact probabilities however; when demonic nondeterminism is present we have only lower bounds. Thus we address the question

Given only \( \Pr.X \geq p \) and \( \Pr.Y \geq q \), what is the most precise lower bound for \( \Pr.(X \cap Y) \) in terms of \( p \) and \( q \)?

From the reasoning above we obtain

\[
(p + q - 1) \uplus 0 \quad (1.25)
\]

immediately as a lower bound. But to see that it is the greatest lower bound we must show that for any \( X, Y, p, q \) there is a probability distribution \( \Pr \) such that the bound is attained; and that is illustrated in Fig. 1.6.3, where an explicit distribution is given in which \( \Pr.X = p \), \( \Pr.Y = q \) and \( \Pr.(X \cap Y) \) is as low as possible, reaching \( (p + q - 1) \uplus 0 \) exactly.

Returning to our example, but using \( \& \), we now have equality:

\[
\text{wp.prog}([cc \neq C] \& [cc \neq A])
\equiv \text{wp.prog}([cc = B])
\equiv 1/3
\equiv 2/3 \& 2/3
\equiv \text{wp.prog}([cc \neq C] \& \text{wp.prog}([cc \neq A]) .
\]

The \( \& \) operator also plays a crucial role in the proof (Chap. 7) of our probabilistic loop rule, presented in Chap. 2 and used in the examples to come.

### 1.7 Healthiness example: modular reasoning

As an example of the use of healthiness conditions, we formulate and prove a simple but very powerful property of \( pGCL \) programs, important for “modular” reasoning about them.

By *modular* reasoning in this case we mean determining, first, that a program \( \text{prog} \) of interest has some standard property; then for subsequent (possibly probabilistic) reasoning we assume that property. This makes

\(^{49}\)The first step is the *modularity law* for probabilities.
1.7. Healthiness example: modular reasoning

![Diagram of sets X, Y with intersections and unions]

Pr. \( X \) = \( p \cap (1 - q) + (p + q - 1) \cup 0 \) = \( p \)
Pr. \( Y \) = \( q \cap (1 - p) + (p + q - 1) \cup 0 \) = \( q \)
Pr. \( (X \cap Y) \) = \( (p + q - 1) \cup 0 \) = \( p \& q \)

The lower bound \( p \& q \) is the best possible.

**Figure 1.6.3. Probabilistic conjunction & depicted**

the reasoning modular in the sense that we do not have to prove all the properties at once.\(^{50}\)

We formulate the principle as a lemma.

**Lemma 1.7.1 Modular reasoning** Suppose for some program prog and predicates pre and post we have

\[
\text{[pre]} \Rightarrow \text{wp.prog.[post]},
\]

which is just the embedded form of a standard Hoare-triple specification. Then in any state satisfying pre we have for any bounded post-expectations \( postE, postE' \) that

\[
\text{wp.prog.postE} = \text{wp.prog.postE'}, \quad 51
\]

provided \( post \) implies that \( postE \) and \( postE' \) are equal.

That is, with (1.26) we can assume the truth of \( post \) when reasoning about the post-expectation, provided \( pre \) holds in the initial state.

\(^{50}\) A typical use of this appeals to standard reasoning, in a “first pass,” to establish that some (Boolean) property — such as a variable’s typing — is invariant in a program; then, in the “second pass” during which probabilistic reasoning might be carried out, we can assume that invariant everywhere without comment. Recall Footnote 43 on p. 28; see also the treatment of Fig. 7.7.11 on p. 211 to come.

\(^{51}\) We write “\( \equiv \)” rather than “\( \equiv \)” because the equality holds only in some states (those satisfying pre), as indicated in the text above. Thus writing “\( \equiv, \Rightarrow, \Leftrightarrow \)” as we do elsewhere is just an alternative for the text “in all states”.


1. Introduction to pGCL

Proof: We use the healthiness conditions of the previous section, and we assume that the post-expectations \( \text{post}E, \text{post}E' \) are bounded above by some nonzero \( M \). Given that the current state satisfies \( \text{pre} \), we then have

\[
\begin{align*}
\text{wp}.\text{prog}.([\text{post}] \ast \text{post}E) \\
= M \ast \text{wp}.\text{prog}.([\text{post}] \ast \text{post}E/M) & \quad \text{scaling} \\
= M \ast \text{wp}.\text{prog}.([\text{post}] \& (\text{post}E/M)) & \quad [\text{post}] \text{ is standard;} \allowbreak \text{post}E/M \geq 1 \\
\end{align*}
\]

\[
\begin{align*}
\wedge \vee \\
M \ast ([\text{pre}] \& \text{wp}.\text{prog}.(\text{post}E/M)) & \quad \text{sub-conjunctivity} \\
M \ast ([\text{pre}] \& \text{wp}.\text{prog}.(\text{post}E/M)) & \quad \text{Assumption (1.26)} \\
\wedge \vee \\
M \ast (1 \& \text{wp}.\text{prog}.(\text{post}E/M)) & \quad \text{pre holds in current state} \\
M \ast \text{wp}.\text{prog}.(\text{post}E/M) & \quad \text{arithmetic} \\
\wedge \vee \\
\text{wp}.\text{prog}.\text{post}E & \quad \text{scaling}
\end{align*}
\]

The opposite inequality is immediate (in all states) from the monotonicity healthiness property, since \([\text{post}] \ast \text{post}E \Rightarrow \text{post}E\). Thus, still assuming \( \text{pre} \) in the current state, we conclude with

\[
\begin{align*}
\text{wp}.\text{prog}.\text{post}E \\
= \text{wp}.\text{prog}.([\text{post}] \ast \text{post}E) & \quad \text{above} \\
= \text{wp}.\text{prog}.([\text{post}] \ast \text{post}E') & \quad \text{assumption about } \text{post}E, \text{post}E' \\
= \text{wp}.\text{prog}.\text{post}E' & \quad \text{as above, but for } \text{post}E'
\end{align*}
\]

This kind of reasoning is nothing new for standard programs, and indeed is usually taken for granted (although its formal justification appeals to conjunctivity). It is important that it is available in pGCL as well.\textsuperscript{52}

1.8 Interaction between probabilistic- and demonic choice

We conclude with some illustrations of the interaction of demonic and probabilistic choice. Consider two variables \( x, y \), one chosen demonically and the other probabilistically. Suppose first that \( x \) is chosen demonically and \( y \) probabilistically, and take post-expectation \([x = y]\). Then

\textsuperscript{52}Lem. 1.7.1 holds even when \( \text{post}E, \text{post}E' \) are unbounded, provided of course that \( \text{wp}.\text{prog} \) is defined for them; the proof of that can be given by direct reference to the definition of \( \text{wp} \) over the model, as set out in Chap. 5.

We will need that extension for our occasional excursions beyond the “safe” bounded world we have formally dealt with in the logic (e.g. Sections 2.11 and 3.3).
\[
wp.(x = 1 \land x = 2); \ (y = 1 \oplus y = 2). [x = y]
\]
\[
\equiv \ wp.(x = 1 \land x = 2). ([x = 1]/2 + [x = 2]/2)
\]
\[
\equiv \ ([1 = 1]/2 + [1 = 2]/2) \ \min \ ([2 = 1]/2 + [2 = 2]/2)
\]
\[
\equiv \ (1/2 + 0/2) \ \min \ (0/2 + 1/2)
\]
\[
\equiv \ 1/2,
\]
from which we see that program establishes \(x = y\) with probability at least \(1/2\); no matter which value is assigned to \(x\), with probability \(1/2\) the second command will assign the same to \(y\).

Now suppose instead that it is the second choice that is demonic. Then we have
\[
wp.(x = 1 \oplus x = 2); \ (y = 1 \land y = 2). [x = y]
\]
\[
\equiv \ wp.(x = 1 \oplus x = 2). ([x = 1] \ \min \ [x = 2])
\]
\[
\equiv \ ([1 = 1] \ \min \ [1 = 2])/2 + ([2 = 1] \ \min \ [2 = 2])/2
\]
\[
\equiv \ (1 \ \min \ 0)/2 + (0 \ \min \ 1)/2
\]
\[
\equiv \ 0,
\]
reflecting that no matter what value is assigned probabilistically to \(x\), the demon could choose subsequently to assign a different value to \(y\).

Thus it is clear that the execution order of occurrence of the two choices plays a critical role in their interaction, and in particular that the demon in the first case cannot make the assignment “clairvoyantly” to \(x\) in order to avoid the value that later will be assigned to \(y\).

1.9 Summary

Being able to reason formally about probabilistic programs does not of course remove \emph{per se} the complexity of the mathematics on which they rely: we do not now expect to find astonishingly simple correctness proofs for all the large collection of randomised algorithms that have been developed over the decades [MR95]. However it should be possible in principle to locate and determine reliably what are the probabilistic/mathematical facts the construction of a randomised algorithm needs to exploit... which is of course just what standard predicate transformers do for conventional algorithms.

In the remainder of Part I we concentrate on proof rules that can be derived for \(pGCL\) — principally for loops — and on examples.

The theory of expectation transformers with nondeterminism is given in Part II, where in particular the role of sublinearity is identified and proved: it characterises a subspace of the predicate transformers that has an equivalent operational semantics of relations between initial and final probabilistic distributions over the state space — a formalisation of the
gambling game of Sec. 1.3. All the programming constructs of the probabilistic language of guarded commands belong to that subspace, which means that the programmer who uses the language can elect to reason about it either axiomatically or operationally.

Chapter notes

In the mid-1970’s, Rabin demonstrated how randomisation could be used to solve a variety of programming problems [Rab76]; since then, the range of applications has increased considerably [MR95], and indeed we analyse several of them as case studies in later chapters. In the meantime — fuelled by randomisation’s impressive applicability — the search for an effective logic of probabilistic programs became an important research topic around the beginning of the 1980’s, and remained so until the mid-1990’s. Ironically, the major technical difficulty was due, in the main, to one of standard programming’s major successes: demonic nondeterminism, the basis for abstraction. It was a challenging problem to decide what to do about it, and how it should interact with the new probabilistic nondeterminism.

The first probabilistic logics did not treat demonic nondeterminism at all — Feldman and Harel [FH84] for instance proved soundness and completeness for a probabilistic PDL which was (in our terms) purely deterministic. The logical language allowed statements about programs to be made at the level of probability distributions and, as we discuss in Sec. A.2, that proves to be an impediment to the natural introduction of a demon. A Hoare-style logic based on similar principles has also been explored by den Hartog and de Vink [dHdV02].

The crucial step of a quantitative logic of expectations was taken by Kozen [Koz85]. Subsequently Jones [Jon90], with Plotkin and using the evaluations from earlier work of Saheh-Djahromi [SD80] that were based directly on topologies rather than on σ- or Borel algebras, worked on more general probabilistic powerdomains; as an example of her technique she specialised it to the Kozen-style logic for deterministic programs, resulting in the sub-probability measures that provide a neat way to quantify nontermination.53

In 1997 He et al. [HSM97] finally proposed the operational model containing all the ingredients for a full treatment of abstraction and program refinement in the context of probability — and that model paved the way for the “demonic/probabilistic” program logic based on expectation transformers. Subsequently Ying [Yin03] has worked towards a probabilistic refinement calculus in the style of Back [BvW98].

53 The notion of sub-probability measures to characterise termination was present much earlier, for example in the work of Feldman and Harel [FH84].