Instantaneously Decodable Codes

A coding scheme is said to be *instantaneously decodable* if encoded text using it can be decoded into the original text by scanning the encoded text from left to right (no back-tracking), picking off the decoded symbols the moment a symbol code is scanned.

The *prefix property* — no code for a symbol is a prefix of the code for another symbol — is a necessary and sufficient condition for instantaneous decoding.

The remarkable thing is that there is a simple arithmetical condition on the length of the codes for the scheme to be instantaneously decodable. This is the *Kraft Inequality*.

The Kraft Inequality

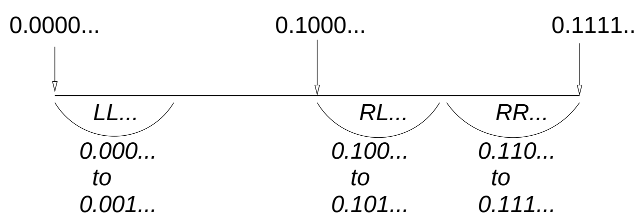
Suppose there are n symbols $s_1, \ldots, s_n$. Let them be encoded into codes $c_1, \ldots, c_n$. By $|c|$ we mean the length of code c.

$$\sum_{i=1}^{i=n} 2^{-|c_i|} \leq 1 \tag{1}$$

This is known as the *Kraft Inequality*.

Theorem: A coding scheme is instantaneously decodable implies the code lengths satisfy the Kraft Inequality. Conversely, if the Kraft Inequality holds, there exists an instantaneously decodable code of those lengths.

The 0-1 Interval and 0-1 Codes I

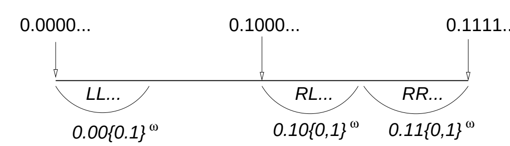


0.101101... interpret as: RLRRLR...

where 0 = Left half; 1 = Right half

The 0-1 Interval and 0-1 Codes II

A CONCISE REPRESENTATION FOR INFINITE STRINGS

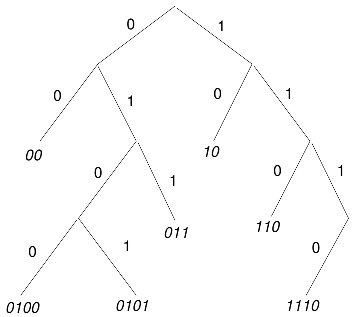


0.101101... interpret as: RLRRLR...

where 0 = Left half; 1 = Right half

$\{a,b\}^\omega$ means an infinite string of a's and b's, e.g. *aababbbaababbaaaab....*

0-1 Prefix Codes and Trees

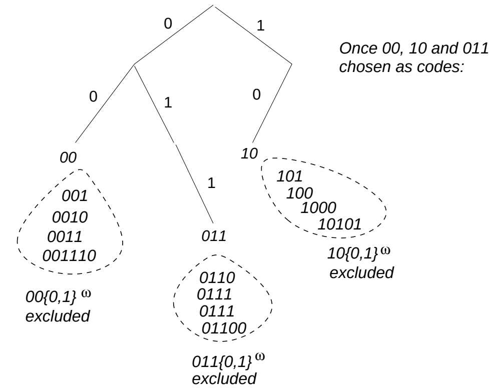


Real–time decodable code = can read codes off from the left w/o back–tracking

Implies no code is the prefix of another –– Prefix Codes

Prefix code iff all codes are in leaves of code assignment tree

Prefix Trees and Excluded Sub-trees



Each leaf of a prefix tree is a "root" of an implicit exclusion subtree!
Say that the code there "covers" the excluded codes in its subtree

Excluded Strings

Lemma A
If $a_1a_2\ldots a_k$ is a chosen prefix code, then all strings of the form $a_1a_2\ldots a_k\{0,1\}^+$ are excluded as codes.
Corollary B
In particular, if this is so, then no string whose initial segment is in the *sub-interval* $[0.a_1a_2\ldots a_kO^\omega, 0.a_1a_2\ldots a_k1^\omega]$ can be used as another code.
As the difference between the ends of this sub-interval is 0.0^k1^ω, its *length* is 2^{-k}.

Uniqueness of Exclusion

Lemma C
$a_1a_2\ldots a_k$ excludes $c_1c_2\ldots c_l$ if and only if $a_1a_2\ldots a_k$ is a *prefix* of $c_1c_2\ldots c_l$, i.e., $a_1 = c_1,\ a_2 = c_2,\ \ldots a_k = c_k$ where $k < l$.
Corollary D
No two distinct prefix codes can exclude the same code.
Proof
Suppose distinct prefix codes $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_j$ both exclude $c_1c_2\ldots a_l$. Then by the lemma, both $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_j$ are prefixes of $c_1c_2\ldots a_l$. Thus if $k \le j$, $a_i = c_i = b_i$ for $1 \le i \le k$, and therefore $a_1a_2\ldots a_k$ is a prefix of $b_1b_2\ldots b_j$, a contradiction. The other case $l \le k$ is similar.

Total Covered Length

Lemma E
If $s_1, s_2, \ldots, s_n$ are prefix $0-1$ codes, then they exclude codes whose prefixes occupy disjoint sub-intervals of lengths $|s_1|, |s_2|, \ldots, |s_n|$.
Proof
From the previous lemmas.
Corollary F — one half of the Kraft Inequality
If $s_1, s_2, \ldots, s_n$ are prefix codes, then

$$\sum_{i=1}^{i=n} 2^{-|s_i|} \leq 1 \tag{2}$$

Existence of Prefix Codes

We now argue that if there are numbers $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{i=n} 2^{-|x_i|} \leq 1 \tag{3}$$

then there is a set of prefix codes $s_1, s_2, \ldots, s_n$ such that $|s_i| = x_i$ for $1 \leq i \leq n$. Without loss of generality we may assume that $x_1 \leq x_2 \leq \ldots \leq x_n$.

Desired Code Lengths

Suppose we want codes of increasing lengths $j_1, j_2 \ldots j_k$. Let $N(j_1), N(j_2) \ldots N(j_k)$ be the number of codes corresponding to those lengths. Then the Kraft Inequality can be re-written

$$\sum_{i=1}^{i=k} N(j_i) 2^{-j_i} \leq 1 \tag{4}$$

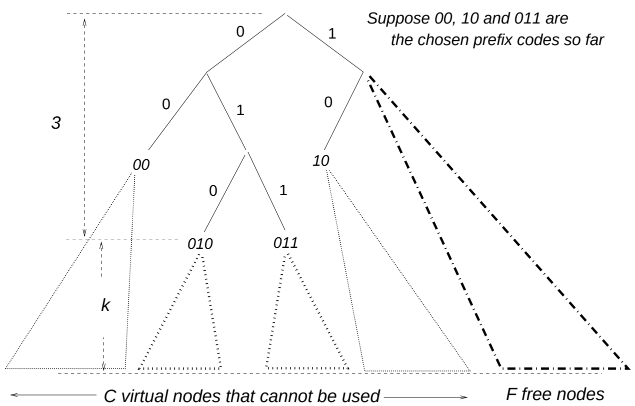
Algorithm for Trie Construction

We construct a trie for the codes as follows.
Start with a (binary) tree of depth j_1. To the leaves of this tree, assign j of them to the shortest $N(j_1)$ desired codes. Then extend the unassigned leaves as follows: make them the roots of subtrees that extend the original tree to the next depth j_2. Assign $N(j_2)$ of those new leaves to the next shortest codes.

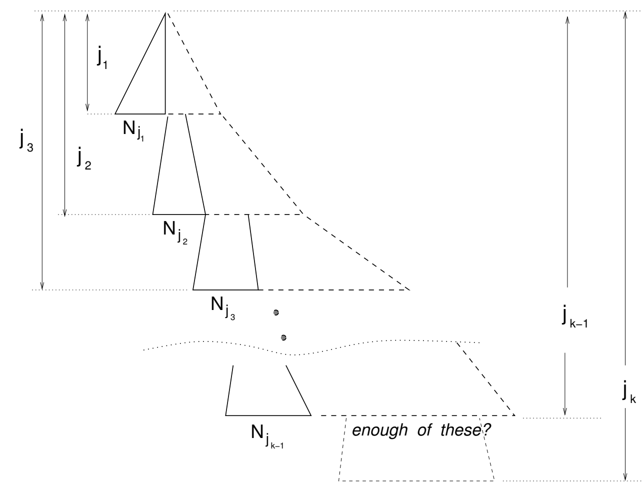
After the *i-th* stage of the algorithm the tree has been extended to a depth of j_i and at that new frontier $N(j_i)$ of its leaves have been assigned.

The feasibility (correctness) of the algorithm is proved if we can show that at any stage *it is the case that enough leaves are left over to extend for the codes of the next higher length to be assigned.*

Code Construction — A Sketch



Enough Free Nodes?



Code Construction — A Sketch

At level j_k there are 2^{j_k} virtual nodes. These are the total number of nodes available for codes *if no ancestors were "used up" at higher levels.*

At level j_1, N_{j_1} are used up by assigning codes to them. Therefore the number of "lost" virtual nodes at level j_k is $N_{j_1}.2^{j_k - j_1}$.

Generally, the number of lost virtual nodes at level j_k due to N_{j_h} assignments at higher level j_h ($j_h < j_k$) is $N_{j_h}.2^{j_k - j_h}$.

Hence the number $F(k)$ of *free* nodes at level j_k is
$2^{j_k} - 2^{j_k}(N_{j_1}.2^{-j_1} + \ldots N_{j_{k-1}}.2^{-j_{k-1}})$.

Thus, there is space for the N_{j_k} nodes at level j_k if $N_{j_k} \leq F(k)$, i.e.
$N_{j_k} \leq 2^{j_k}(1 - N_{j_1}.2^{-j_1} + \ldots N_{j_{k-1}}.2^{-j_{k-1}})$.

But the last expression is precisely the Kraft Inequality!