

Binary Relations

A binary relation R on a set S is a subset of $R \times S$.

Examples:

The edge set of a graph $G = (V, E)$ with vertices V is a binary relation on V . Write in infix form $v_1 E v_2$ to mean (v_1, v_2) is an edge in G .

The relation $Factor(N, M)$ on N defined by $Factor(x, y)$ iff x is a factor of y .

The relation $Sibling(P, Q)$, with the informal meaning “ P is a sibling of Q ”, is a binary relation on the set of all people.

Common Binary Relation Types

A binary relation R is *reflexive* if for all x , $R(x, x)$; *symmetric* if for all x, y , $R(x, y)$ implies $R(y, x)$; and *transitive* if for all x, y, z , $R(x, y)$ and $R(y, z)$ implies $R(x, z)$.

Examples:

E is symmetric for undirected graphs, irreflexive if at least one vertex does not have a self-loop, and intransitive.

$Factor$ is reflexive and transitive but asymmetric.

$Sibling$ is symmetric and transitive but not reflexive.

Transitive Closure

Given a binary relation R , the *transitive closure* R^+ of R is the *smallest transitive* binary relation S such that $R \subseteq S$.

The *reflexive transitive closure* R^* of R is $R^+ \cup Id$ where Id is the identity relation, i.e., $I(x, x)$ for all x .

If R is transitive, $R^+ = R$.

Characterization of Transitive Closure

Let R^k be defined recursively as follows:

$$R^1 = R.$$

$$R^{k+1} = R \circ R^k, \text{ where } \circ \text{ is relational composition.}$$

Proposition:

$$R^+ = \bigcup_{k=1}^{\infty} R^k.$$

Outline proof. Easily verified that RHS is transitive and contains R ; since LHS is the intersection of all transitive relations that contain R , it is contained in RHS. Then show that any proper subset of RHS cannot be transitive.

Transitive Closure Examples

$v_1 E^k v_2$ if there is a k -length path from vertex v_1 to vertex v_2 .

$Reach(G, v) = \{u | v E^+ u\}$ is the set of all vertices reachable from vertex v .

“Vertices u and v are in the same component” is the relation $u E^* v$ (where we recall $u E^0 v$ if $u = v$).

What is wrong with this “proof”?

(False) Assertion: If a binary relation R is symmetric and transitive then it is reflexive.

(Wrong) Proof:

Write in infix form. Suppose xRy . Then by symmetry yRx . Hence by transitivity xRx . But x was arbitrary, hence R is reflexive.

Diagnosis: Consider the relation R on the set of all people where xRy means “ x can see y ”. Think of a blind person x_0 .

Equivalence Relations

A binary relation R is an *equivalence* (relation) if it is reflexive, symmetric and transitive. *Fact: An equivalence relation induces a partition on its set.*

Examples:

- (i) The relation $path(u, v)$ in an undirected self-looped graph G with meaning “there is a path from vertex u to vertex v ”.
- (ii) The relation $x \equiv y \pmod{k}$ on numbers, with meaning x and y have the same remainder on division by k .

A *partition* of a set S is a disjoint collection of subsets of S whose union is S . Informally, a partition of S divides it up into disjoint pieces.

Equivalence Relations

R is an equivalence on S . Denote by $[u]$ the set of all elements of S that are R -related to u , i.e. $[u] = \{v | u \in S \wedge uRv\}$. (Notice that since R is transitive, if there is a sequence $u = x_1, x_2, \dots, x_n = v$ such that $x_1Rx_2, x_2Rx_3, \dots, x_{n-1}Rx_n$, then x_1Rx_n , i.e., uRv .)

It is not hard to show that if x and y are not R -related, then $[x] \cap [y] = \emptyset$.

Examples:

In (i) the partitions are the separate connected components of G . In (ii) the partitions are the residue classes *mod* k .