Binary Relations

A binary relation R on a set S is a subset of $R \times S$.

Examples:

The edge set of a graph $G = (V, E)$ with vertices V is a binary relation on V. Write in infix form $v_1 E v_2$ to mean (v_1, v_2) is an edge in G.

The relation $Factor(N, M)$ on N defined by $Factor(x, y)$ iff x is a factor of y.

The relation $Sibling(P, Q)$, with the informal meaning "P is a sibling of Q", is a binary relation on the set of all people.

Common Binary Relation Types

A binary relation R is *reflexive* if for all x, $R(x, x)$; *symmetric* if for all x, y, $R(x, y)$ implies $R(y, x)$; and *transitive* if for all x, y, z, $R(x, y)$ and $R(y, z)$ implies $R(x, z)$.

Examples:

E is symmetric for undirected graphs, irreflexive if at least one vertex does not have a self-loop, and intransitive.

$Factor$ is reflexive and transitive but asymmetric.

$Sibling$ is symmetric and transitive but not reflexive.

Transitive Closure

Given a binary relation R, the *transitive closure* R^+ of R is the *smallest transitive* binary relation S such that $R \subseteq S$.

The *reflexive transitive closure* R^* of R is $R^+ \cup Id$ where Id is the identity relation, i.e., $I(x, x)$ for all x.

If R is transitive, $R^+ = R$.

Characterization of Transitive Closure

Let R^k be defined recursively as follows:
$R^1 = R$.
$R^{k+1} = R \circ R^k$, where $\circ$ is relational compostion.

Proposition:
$R^+ = \bigcup_{k=1}^{\infty} R^k$.

Outline proof. Easily verified that RHS is transitive and contains R; since LHS is the intersection of all transitive relations that contain R, it is contained in RHS. Then show that any proper subset of RHS cannot be transitive.

Transitive Closure Examples

$v_1 E^k v_2$ if there is a k-length path from vertex v_1 to vertex v_2.

$Reach(G, v) = \{u | vE^+u\}$ is the set of all vertices reachable from vertex v.

"Vertices u and v are in the same component" is the relation uE^*v (where we recall uE^0v if $u = v$).

What is wrong with this "proof"?

(False) Assertion: If a binary relation R is symmetric and transitive then it is reflexive.

(Wrong) Proof:
Write in infix form. Suppose xRy. Then by symmetry yRx. Hence by transitivity xRx. But x was arbitrary, hence R is reflexive.

Diagnosis: Consider the relation R on the set of all people where xRy means "x can see y". Think of a blind person x_0.

Equivalence Relations

A binary relation R is an *equivalence* (relation) if it is reflexive, symmetric and transitive. *Fact: An equivalence relation induces a partition on its set.*

Examples:
(i) The relation $path(u, v)$ in an undirected self-looped graph G with meaning "there is a path from vertex u to vertex v".
(ii) The relation $x \equiv y(mod\ k)$ on numbers, with meaning x and y have the same remainder on division by k.

A *partition* of a set S is a disjoint collection of subsets of S whose union is S. Informally, a partition of S divides it up into disjoint pieces.

Equivalence Relations

R is an equivalence on S. Denote by $[u]$ the set of all elements of S that are R-related to u, i.e. $[u] = \{v | u \in S \wedge uRv\}$. (Notice that since R is transitive, if there is a sequence $u = x_1, x_2, \ldots, x_n = v$ such that $x_1 R x_2, x_2 R x_3, \ldots, x_{n-1} R x_n$, then $x_1 R x_n$, i.e., uRv.)

It is not hard to show that if x and y are not R-related, then $[x] \cap [y] = \emptyset$.

Examples:
In (i) the partitions are the separate connected components of G. In (ii) the partitions are the residue classes *mod k*.