Due: Sun 20 March, 18:00

Submission is through give and should be a single pdf file, maximum size 4Mb. Prose should be typed, not handwritten. Use of \LaTeX is strongly encouraged, but not required.

Late submission is allowed up to 5 days (120 hours) after the deadline. A late penalty of 5% per day will be deducted from your total mark.

Discussion of assignment material with others is permitted, but you may not exchange or view each others’ (partial) solutions. The work submitted must be your own, in line with the University’s plagiarism policy.

These problems are themed around the game of noughts and crosses (aka tic-tac-toe). Two players, X (crosses) and O (noughts), take turns putting their respective symbol in an empty square on a 3x3 board. The first player to have three in a row, either horizontally, vertically, or diagonally, wins the game. If neither player has three in a row when the board is full, the game is a draw.

\[
\begin{array}{ccc}
X & O & O \\
X & & \\
O & X & \\
\end{array}
\]

The O player didn’t show their best performance in this game.

Problem 1 (40 marks)

Suppose a noughts and crosses program represents the game board as an array of characters of length 9, stored in the global variable board.[1] The idea is that the contents of board[i] is "E" if the i:th square is empty, "X" if it contains an X, and "O" if it contains an O. The squares are numbered as follows:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
\end{array}
\]

Consider the following pseudo-code, where abort is an instruction that causes the program to abort (crash) with a run-time error.

```c
void move(int pos, char fill) {
    if(board[pos] = "E" && (fill = "X" || fill = "O")) then {
        board[pos] := fill;
    } else {
        abort;
    }
}
```

[1]Please don’t write code like this. Using global variables in this manner is considered bad style.
Mathematically, we model board as a 9-tuple $b \in \{E, X, O\}^9$, where $b_i$ represents the contents of board $[i]$.

(a) Define a relation $R \subseteq \{E, X, O\}^9 \times \{E, X, O\}^9$ that models the effect that a non-aborting call to `move` can have on the board. In particular, $(b, b') \in R$ should hold iff the following holds: a call to move can terminate (without aborting) in a state where the new board is $b'$, if the old board before calling move was $b$. (10 marks)

(b) Which of the following properties does $R$ have? Is it reflexive, antireflexive, symmetric, antisymmetric, transitive, or a function? For each of these six properties, either give a counterexample, or explain (informally and in your own words) why the property holds. (15 marks)

(c) Is $R$ an accurate characterisation of the set of legal moves in noughts and crosses? If yes, explain why; if no, explain what’s missing. *Hint: a sneak peek at question 3 may help here.* (5 marks)

(d) Recall the relation composition operator $(;)$ defined as follows:

$$R_1; R_2 = \{(a, c) \mid \text{there is a } b \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

What is the cardinality of $R; R; R; R; R; R; R; R; R$? (That’s ten $R$:s.) What does your answer tell you about the effects of repeatedly calling move? What does your answer tell you about the game of noughts and crosses? (10 marks)

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**Problem 2** (40 marks)

Let’s look at another possible way of modelling a noughts and crosses board, this time with propositional logic. We will use 18 propositional letters: $X_0, \ldots, X_8$ and $O_0, \ldots, O_8$. The idea is that $X_i$ is true if the $i$:th square contains an X, and similarly $O_i$ is true if if the $i$:th square contains an O.

(a) Write a propositional formula $\varphi_0$ that holds iff the X player has three in a row diagonally. The formula should not mention any of the $O_i$ letters. (5 marks)

(b) Let $\varphi_1$ be the analogous formula that holds iff the O player has three in a row diagonally. Show that the formula $\varphi_0 \land \varphi_1$ is satisfiable (by giving a valuation $v$ such that $[\varphi_0 \land \varphi_1]_v = \text{true}$). (5 marks)

(c) One way of making this model of noughts and crosses more accurate is to introduce a formula $\varphi_2$ which characterises the well-formed states, i.e. those that correspond to legal board positions in noughts and crosses. Write a propositional formula $\varphi_2$ such that every legal board position of noughts and crosses satisfies $\varphi_2$, but that rules out illegal game states such as the one from question 3. It’s ok if $\varphi_2$ overapproximates the legal game states, i.e. if there are some game states that satisfy $\varphi_2$ despite not being reachable in an actual game (for example, a board filled with crosses). (5 marks)

(d) Prove that the following holds

$$\{\varphi_2\} \models \neg (\varphi_0 \land \varphi_1)$$

using either of the following methods: the calculational approach from Week 1, natural deduction, or a direct semantic argument (by showing that every valuation that satisfies $\varphi_2$ must also satisfy $\neg(\varphi_0 \land \varphi_1)$). (10 marks)

(e) Extend $\varphi_0$ to a formula that is true iff the X player has three in a row anywhere. (5 marks)
(f) These formulas are getting tedious. Imagine typing all that out for a larger board than 3x3.

Make a better, more extensible model using predicate logic. This means you can use \( \forall \), \( \exists \), and introduce any predicates and functions you find useful.

Write a predicate logic formula that is true iff the X player has \( n \) in a row anywhere on an \( n \times n \) board. Informally explain the intended meaning of any predicates and function symbols you introduce.

*Hint:* You don’t need to formally specify your domain of discourse, nor do you need to give a model for your vocabulary. (10 marks)

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Problem 3 (20 marks)

Now, we will use formal languages to describe the set of valid games of noughts and crosses. Let \( \Sigma = \{X_0, \ldots, X_8, O_0, \ldots, O_8\} \). As always, we use \( w, v \) to range over words, and \( a, b, c \) to range over symbols from \( \Sigma \). Words in \( \Sigma^* \) represent sequences of moves; for example, \( X_0 O_2 X_7 \) represents this unlikely sequence of moves:

\[
\begin{array}{cccc}
X & \\
\hline \\
X & O & \\
\hline \\
X & O & X
\end{array}
\]

(a) Write an inductive definition of a language \( L \subseteq \Sigma^* \) such that \( w \in L \) iff \( w \) is a sequence of legal moves in noughts and crosses. Your definition should consists of a base case, and one or more inductive clauses which describe the circumstances under which a sequence of moves \( w \in L \) may be extended to the right with a single move \( a \) such that \( wa \in L \). Remember to account for the following:

- Games end when one player has won.
- The players need to alternate turns.
- Move sequences that correspond to unfinished games should also be included.

Feel free to write your definition in prose rather than in any formal notation. You don’t need to define auxiliary concepts in detail; for an example, feel free to use a “someone has won” predicate on words, without spelling out its exact definition, as long as the intended meaning is clear. (15 marks)

*Hint:* feel free to use this solution template (filling in the ... as appropriate).

<table>
<thead>
<tr>
<th>Solution template</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( L ) be the smallest set such that</td>
</tr>
<tr>
<td>- ( \ldots \in L )</td>
</tr>
<tr>
<td>- If ( w \in L ), and if ( w ) does not end with a character ( X_i ) for any ( i ), and ( \ldots ), then ( wX_n \in L ).</td>
</tr>
<tr>
<td>- ( \ldots )</td>
</tr>
</tbody>
</table>

(b) Describe how you would generalise from a 3x3 board to an \( n \times n \) board. (5 marks)