COMP2111 Week 7
Term 1, 2022
State machines
Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata
Motivation

In Assignment 1, we modelled programs as relations between initial and final states of successful executions. So this program:

```c
void move(int pos, char fill) {
    if (board[pos] = "E" && (fill = "X" || fill = "O")) {
        board[pos] := fill;
    } else abort;
}
```

we modelled with this relation:

\[
\{(b, b') : \exists n. \forall i. (n = i \rightarrow b_i = E \land b'_i \neq E) \land (n \neq i \rightarrow b_i = b'_i)\}
\]
Motivation

Such relational modelling is useful (spoiler alert: W8-9), but doesn’t always capture everything we care about.
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Possibility of failure is sometimes not captured. This:

```java
void incr() {
    if (Math.random() < .5) then abort; else
        x := x + 1;
}
```

can be modelled by this relation over $\mathbb{Z} \times \mathbb{Z}$:

$$\{(x, x') : x + 1 = x'\}$$
Motivation

Such relational modelling is useful (spoiler alert: W8-9), but doesn’t always capture everything we care about.

Possibility of failure is sometimes not captured. This:

```java
void incr() {
    if (Math.random() < .5) then abort else
    x := x + 1;
}
```

can also be modelled by this relation over \( \mathbb{Z} \times \mathbb{Z} \):

\[
\{(x, x') : x + 1 = x'\}
\]
Motivation

Such relational modelling is useful (spoiler alert: W8-9), but doesn’t always capture everything we care about.

Sometimes the final state isn’t what’s interesting.

```java
void yes() {
    while(true) print("y\n");
}
```

This program has no final states, so its relational model doesn’t say much:

```java
{}
```
Motivation

State machines model step-by-step processes with:

- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.
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- A set of states, possibly including a designated start state.
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Example

The semantics of a program:

- States: functions from variable names to values
- Transitions: execute a line of code.
Motivation

State machines model step-by-step processes with:

- A set of states, possibly including a designated start state.
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Example

A game of noughts and crosses
- States: Board positions
- Transitions: Legal moves
Motivation

State machines model step-by-step processes with:
- A set of states, possibly including a designated start state.
- A transition relation, detailing how to move (transition) from one state to another.

Example

Stateful communication protocols: e.g. SMTP
- States: Stages of communication
- Transitions: Determined by commands given (e.g. HELO, DATA, etc)
Motivation

State machines model step-by-step processes with:

- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.

Example

A bounded counter that counts from 0 to 99 and overflows at 100:
Motivation

State machines model step-by-step processes with:
- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.

Example

A robot that moves diagonally

States: Locations
Transitions: Moves
Motivation

State machines model step-by-step processes with:
- A set of *states*, possibly including a designated *start state*.
- A *transition relation*, detailing how to move (transition) from one state to another.

**Example**

Die Hard jug problem: Given jugs of 3L and 5L, measure out exactly 4L.
- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)
Summary

- Motivation
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- Finite automata
Definitions

A transition system is a pair \((S, \rightarrow)\) where:

- \(S\) is a set (of states), and
- \(\rightarrow \subseteq S \times S\) is a (transition) relation.

If \((s, s') \in \rightarrow\) we write \(s \rightarrow s'\).
Definitions

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- \(S\) may have designated final states, \(F \subseteq S\)
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- \(S\) may have a designated start state, \(s_0 \in S\)
- \(S\) may have designated final states, \(F \subseteq S\)
- The transitions may be labelled by elements of a set \(L\):
  - \(\rightarrow \subseteq S \times L \times S\)
  - \((s, a, s') \in \rightarrow\) is written as \(s \overset{a}{\rightarrow} s'\)
Definitions

A transition system is a pair \((S, \rightarrow)\) where:

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- The transitions may be labelled by elements of a set \(L\):
  - \(\rightarrow \subseteq S \times L \times S\)
  - \((s, a, s') \in \rightarrow\) is written as \(s \xrightarrow{a} s'\)
- If \(\rightarrow\) is a partial function we say that the system is deterministic, otherwise it is non-deterministic.
Example: Bounded counter

A bounded counter that counts from 0 to 99 and overflows at 100:

\[ S = \{0, 1, \ldots, 99, \text{overflow}\} \]
\[ \rightarrow = \{(i, i + 1) : 0 \leq i < 99\} \]
\[ \cup \{(99, \text{overflow})\} \]
\[ \cup \{(\text{overflow}, \text{overflow})\} \]
\[ s_0 = 0 \]
\[ \text{Deterministic} \]
Example: yes

```c
void yes() {
    while(true) print("y\n\n\n") ;
}
```

\[ S = \{ s_0 \} \]

\[ L = \text{the set of strings} \]

\[ s_0 \xrightarrow{"y\n\n"} s_o \]
Example: Diagonally moving robot

Example

States: Locations
Transitions: Moves
Example: Diagonally moving robot

$$S = \mathbb{Z} \times \mathbb{Z}$$

$$(x, y) \rightarrow (x \pm 1, y \pm 1)$$

Non-deterministic
Example: Diagonally moving robot

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ L = \{ \text{NW, NE, SW, SE} \} \]

\[(x, y) \xrightarrow{\text{NW}} (x - 1, y + 1)\]

\[(x, y) \xrightarrow{\text{NE}} (x + 1, y + 1)\]

\[(x, y) \xrightarrow{\text{SW}} (x - 1, y - 1)\]

\[(x, y) \xrightarrow{\text{SE}} (x + 1, y - 1)\]

Deterministic
Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)
A $3L$ and $5L$ jug problem

Example
Given jugs of 3L and 5L, measure out exactly 4L.

$S = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$

$s_0 = (0,0)$

$\rightarrow$ given by
Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- \( S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\} \)
- \( s_0 = (0, 0) \)
- \( \rightarrow \text{ given by} \)
  - \((i, j) \rightarrow (0, j)\) [empty 5L jug]
  - \((i, j) \rightarrow (i, 0)\) [empty 3L jug]
Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0, 0)$
- $\rightarrow$ given by
  - $(i, j) \rightarrow (5, j)$ [fill 5L jug]
  - $(i, j) \rightarrow (i, 3)$ [fill 3L jug]
  - $(i, j) \rightarrow (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
  - $(i, j) \rightarrow (0, i + j)$ if $i + j \leq 3$ [empty 5L jug into 3L jug]
  - $(i, j) \rightarrow (5, j - 5 + i)$ if $i + j \geq 5$ [fill 5L jug from 3L jug]
  - $(i, j) \rightarrow (i - 3 + j, 3)$ if $i + j \geq 3$ [fill 3L jug from 5L jug]
Example: Die Hard jug problem

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Given jugs of 3L and 5L, measure out exactly 4L.

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- $(i, j) \rightarrow (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
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- \( \rightarrow \) given by

- \( (i,j) \rightarrow (5,j - 5 + i)) \) if \( i + j \geq 5 \)  
  [fill 5L jug from 3L jug]
- \( (i,j) \rightarrow (i - 3 + j,3) \) if \( i + j \geq 3 \)  
  [fill 3L jug from 5L jug]
Example: Die Hard jug problem

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  - $(i, j) \rightarrow (0, j)$ [empty 5L jug]
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  - $(i, j) \rightarrow (5, j)$ [fill 5L jug]
  - $(i, j) \rightarrow (i, 3)$ [fill 3L jug]
  - $(i, j) \rightarrow (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
  - $(i, j) \rightarrow (0, i + j)$ if $i + j \leq 3$ [empty 5L jug into 3L jug]
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  - $(i, j) \rightarrow (i - 3 + j, 3)$ if $i + j \geq 3$ [fill 3L jug from 5L jug]
Runs and reachability

Given a transition system \((S, \rightarrow)\) and states \(s, s' \in S\),

- a run (or trace) from \(s\) is a (possibly infinite) sequence \(s_1, s_2, \ldots\) such that \(s = s_1\) and \(s_i \rightarrow s_{i+1}\) for all \(i \geq 1\).
Runs and reachability

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- A run is **maximal** if it cannot be extended; i.e., it is either infinite, or ends in a state from which there are no transitions.

We say \(s'\) is reachable from \(s\), written \(s \rightarrow^* s'\), if \((s, s')\) is in the reflexive and transitive closure of \(\rightarrow\).

**NB** \(s'\) is reachable from \(s\) if there is a run from \(s\) which contains \(s'\).
Runs and reachability

Given a transition system \((S, \rightarrow)\) and states \(s, s' \in S\),

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**NB**

\(s'\) is reachable from \(s\) if there is a run from \(s\) which contains \(s'\).
Reachability example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- Transition relation: $(i, j) \rightarrow (0, j)$ etc.

Is $(4, 0)$ reachable from $(0, 0)$?
Reachability example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: \( S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\} \)
- Transition relation: \((i, j) \rightarrow (0, j)\) etc.

Is \((4, 0)\) reachable from \((0, 0)\)?

Yes:

\[
\begin{align*}
(0, 0) & \rightarrow (0, 3) \rightarrow (3, 0) \\
(0, 1) & \leftarrow (5, 1) \leftarrow (3, 3) \\
(1, 0) & \rightarrow (1, 3) \rightarrow (4, 0)
\end{align*}
\]
Safety and Liveness

Transition systems can be used to study whether systems satisfy safety and liveness properties.

**Safety:** something bad will never happen.
**Liveness:** something good will happen.

Contrast this with **reachability:**

**Reachability:** something good can happen.
Safety and Liveness: Examples

Example

Suppose our transition system models a nuclear power plant.

Safety: the reactor never reaches the meltdown state.

Liveness: the power plant will keep supplying power.
Example

```c
void yes() {
    while(true){
        print("y\n");
    }
}
```

**Safety:** `yes()` never prints anything but "y\n".

**Liveness:** `yes()` will always print another "y\n".
Example

\[
y := 1;  
z := x;  
\text{while}(z \neq 0)\{  
y := y \times z;  
z := z - 1;  
\}
\]

**Safety:** If the program ever terminates, then \( y = x! \)

**Liveness:** The program will terminate

(How is that a safety property?)
Safety and Liveness

A property is a set of infinite runs. (Terminating runs can be made infinite by adding a self-loop to the final state.)

**Safety:** A safety property can be falsified by a finite prefix of a behaviour.

**Liveness:** A liveness property can always be satisfied eventually.
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge*
Properties Examples

Are they safety or liveness?

- When I come home, there must be beer in the fridge – Safety
- When I come home, I’ll drop on the couch and drink a beer
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – Safety
- *When I come home, I’ll drop on the couch and drink a beer* – Liveness
- *I’ll be home later* – Liveness
- *The program never allocates more than 100MB of memory*
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I’ll drop on the couch and drink a beer* – **Liveness**
- *I’ll be home later* – **Liveness**
- *The program never allocates more than 100MB of memory* — **Safety**
- *The program will allocate at least 100MB of memory*
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – Safety
- *When I come home, I’ll drop on the couch and drink a beer* – Liveness
- *I’ll be home later* – Liveness
- *The program never allocates more than 100MB of memory* — Safety
- *The program will allocate at least 100MB of memory* – Liveness
- *No two processes are simultaneously in their critical section*
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I’ll drop on the couch and drink a beer* – **Liveness**
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- *The program never allocates more than 100MB of memory* — **Safety**
- *The program will allocate at least 100MB of memory* – **Liveness**
- *No two processes are simultaneously in their critical section* — **Safety**
- *If a process wishes to enter its critical section, it will eventually be allowed to do so*
Properties Examples

Are they safety or liveness?

- *When I come home, there must be beer in the fridge* – **Safety**
- *When I come home, I’ll drop on the couch and drink a beer* – **Liveness**
- *I’ll be home later* – **Liveness**
- *The program never allocates more than 100MB of memory* — **Safety**
- *The program will allocate at least 100MB of memory* – **Liveness**
- *No two processes are simultaneously in their critical section* — **Safety**
- *If a process wishes to enter its critical section, it will eventually be allowed to do so* – **Liveness**
Summary

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Safety example: Diagonally moving robot

Example

Starting at (0, 0)
Can the robot get to (0, 1)?
Safety example: Diagonally moving robot

Example

Starting at \((0, 0)\)
Can the robot get to \((0, 1)\)?
Safety example: Diagonally moving robot

Example

Starting at (0, 0)
Can the robot get to (0, 1)? No
Safety example: Diagonally moving robot

Example

Starting at (0, 0)
Can the robot get to (0, 1)?  No

\[ \text{isBlue}((m, n)) := 2 | (m + n) \]
Safety example: Diagonally moving robot

Example

Starting at \((0, 0)\)

Can the robot get to \((0, 1)\)? No

\[\text{isBlue}((m, n)) := 2|(m + n)\]

if \text{isBlue}(s) and \(s \rightarrow s'\) then \text{isBlue}(s')
Safety example: Diagonally moving robot

Example

Starting at (0, 0)

Can the robot get to (0, 1)? No

\[ \text{isBlue}((m, n)) := 2|m + n| \]

if isBlue(s) and s \rightarrow s'

then isBlue(s')

isBlue((0, 0)) and \neg isBlue((0, 1))
The invariant principle

A preserved invariant of a transition system is a unary predicate \( \varphi \) on states such that if \( \varphi(s) \) holds and \( s \rightarrow s' \) then \( \varphi(s') \) holds.

**Invariant principle**

If a preserved invariant holds at a state \( s \), then it holds for all states reachable from \( s \).
The invariant principle

A preserved invariant of a transition system is a unary predicate \( \varphi \) on states such that if \( \varphi(s) \) holds and \( s \rightarrow s' \) then \( \varphi(s') \) holds.

**Invariant principle**

If a preserved invariant holds at a state \( s \), then it holds for all states reachable from \( s \).

Proof sketch: Let \( s' \) be a state reachable from \( s \). We can show \( \varphi(s') \) by induction on the length of the run from \( s \) to \( s' \).
Invariant example: Modified Die Hard problem

Example

Given jugs of 3L and 6L, measure out exactly 4L.

- States: \( S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 6 \text{ and } 0 \leq j \leq 3\} \)
- Transition relation: \((i, j) \rightarrow (0, j)\) etc.

Is \((4, 0)\) reachable from \((0, 0)\)?
Invariant example: Modified Die Hard problem

Example

Given jugs of 3L and 6L, measure out exactly 4L.

- States: \( S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 6 \text{ and } 0 \leq j \leq 3\} \)
- Transition relation: \((i, j) \rightarrow (0, j)\) etc.

Is \((4, 0)\) reachable from \((0, 0)\)?
No. Consider \(\varphi((i,j)) = (3\parallel i) \land (3\parallel j)\).
Summary

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Partial correctness

Let \((S, \rightarrow, s_0, F)\) be a transition system with start state \(s_0\) and final states \(F\) and a \(\varphi\) be a unary predicate on \(S\). We say the system is partially correct for \(\varphi\) if \(\varphi(s')\) holds for all states \(s' \in F\) that are reachable from \(s_0\).

**NB**

*Partial correctness is a safety property. It doesn’t say whether the transition system will ever reach a final state.*
Partial correctness example: Fast exponentiation

Example
Consider the following program in $\mathcal{L}$:

$$
x := m;
y := n;
r := 1;
while y > 0 do
  if 2|y then
    y := y/2
  else
    y := (y - 1)/2;
  r := r \times x
fi;
x := x \times x
od$$
Partial correctness example: Fast exponentiation

Example

- States: Functions from $\{m, n, x, y, r\}$ to $\mathbb{N}$
- Transitions:

Goal: Show partial correctness for $\phi((x, y, r)) := (r = m^n)$

Show $\psi((x, y, r)) := (rx^y = m^n)$ is a preserved invariant...

How can we show total correctness?
Partial correctness example: Fast exponentiation

Example

• States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
• Transitions: Effect of each line of code?
Partial correctness example: Fast exponentiation

**Example**

- **States:** \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
- **Transitions:** Effect of each iteration of while loop
  - \((x, y, r) \rightarrow (x^2, y/2, r)\) if \(y\) is even
  - \((x, y, r) \rightarrow (x^2, (y - 1)/2, rx)\) if \(y\) is odd
Partial correctness example: Fast exponentiation

Example

- States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
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  - \((x, y, r) \rightarrow (x^2, (y - 1)/2, rx)\) if \(y\) is odd
- Start state: \((m, n, 1)\)
Partial correctness example: Fast exponentiation

Example

- States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
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- Start state: \((m, n, 1)\)
- Final states: \:\{(x, 0, r) : x, r \in \mathbb{N}\}
Partial correctness example: Fast exponentiation

Example

• States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)

• Transitions: Effect of each iteration of while loop
  - \((x, y, r) \rightarrow (x^2, y/2, r)\) if \(y\) is even
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• Start state: \((m, n, 1)\)

• Final states: \(\{(x, 0, r) : x, r \in \mathbb{N}\}\)

Goal: Show partial correctness for \(\varphi((x, y, r)) := (r = m^n)\)
Partial correctness example: Fast exponentiation

Example

- **States:** \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
- **Transitions:** Effect of each iteration of while loop
  - \((x, y, r) \rightarrow (x^2, y/2, r)\) if \(y\) is even
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- **Start state:** \((m, n, 1)\)
- **Final states:** \\{\((x, 0, r) : x, r \in \mathbb{N}\)\}

**Goal:** Show partial correctness for \(\varphi((x, y, r)) := (r = m^n)\)

Show \(\psi((x, y, r)) := (rx^y = m^n)\) is a preserved invariant...
Partial correctness example: Fast exponentiation

Example

- States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
- Transitions: Effect of each iteration of while loop
  - \((x, y, r) \rightarrow (x^2, y/2, r)\) if \(y\) is even
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Goal: Show partial correctness for \(\varphi((x, y, r)) := (r = m^n)\)

Show \(\psi((x, y, r)) := (rx^y = m^n)\) is a preserved invariant...

How can we show total correctness?
Total correctness = safety + liveness

A transition system \((S, \rightarrow)\) terminates from a state \(s \in S\) if all runs from \(s\) have finite length.

A transition system is **totally correct for a unary predicate** \(\varphi\), if it terminates (from \(s_0\)) and \(\varphi\) holds in the last state of every run.
In a transition system \((S, \to)\), a **measure** is a function \(f : S \to \mathbb{N}\).

A measure is **strictly decreasing** if \(s \to s'\) implies \(f(s') < f(s)\).
In a transition system \((S, \rightarrow)\), a **measure** is a function \(f : S \rightarrow \mathbb{N}\).

A measure is **strictly decreasing** if \(s \rightarrow s'\) implies \(f(s') < f(s)\).

**Theorem**

If \(f\) is a strictly decreasing measure, then the length of any run from \(s\) is at most \(f(s)\).
Termination example: Fast exponentiation

Example

- States: \((x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}\)
- Transitions: Effect of each iteration of while loop:
  - \((x, y, r) \rightarrow (x^2, y/2, r)\) if \(y\) is even
  - \((x, y, r) \rightarrow (x^2, (y - 1)/2, rx)\) if \(y\) is odd

Measure:
Termination example: Fast exponentiation

Example

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Measure: \(f((x, y, r)) = y\)
Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata
Interaction with the environment

We can model the system interacting with an external entity via inputs ($\Sigma$) and outputs ($\Gamma$) by using **labelled transitions**: 

$$\rightarrow \subseteq S \times L \times S \text{ where } L = \Sigma \times \Gamma$$

We’ll look at categories of input/output transition systems:

**Acceptors:** Accept/reject a sequence of inputs

**Transducers:** Take a sequence of inputs and produce a sequence of outputs
We can model the system interacting with an external entity via inputs ($\Sigma$) and outputs ($\Gamma$) by using **labelled transitions**:

$$\rightarrow \subseteq S \times L \times S \text{ where } L = \Sigma \times \Gamma$$

We’ll look at categories of input/output transition systems:

**Acceptors:** Accept/reject a sequence of inputs (Relations)

**Transducers:** Take a sequence of inputs and produce a sequence of outputs (Functions)
Acceptor example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[ (x, y) \xrightarrow{NW} (x - 1, y + 1) \]

\[ (x, y) \xrightarrow{NE} (x + 1, y + 1) \]

\[ (x, y) \xrightarrow{SW} (x - 1, y - 1) \]

\[ (x, y) \xrightarrow{SE} (x + 1, y - 1) \]

Accept if \((2, 2)\) reached
Acceptor example: Diagonally moving robot

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Accept if \((2, 2)\) reached

Accepted sequences:

\[NE, NE\]
Acceptor example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[
(x, y) \xrightarrow{\text{NW}} (x - 1, y + 1) \\
(x, y) \xrightarrow{\text{NE}} (x + 1, y + 1) \\
(x, y) \xrightarrow{\text{SW}} (x - 1, y - 1) \\
(x, y) \xrightarrow{\text{SE}} (x + 1, y - 1)
\]

Accept if \((2, 2)\) reached

Accepted sequences:

\[ \text{NE}, \text{NE} \]
\[ \text{NE}, \text{SE}, \text{NE}, \text{NW} \]
Acceptor example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[ (x, y) \xrightarrow{NW} (x - 1, y + 1) \]

\[ (x, y) \xrightarrow{NE} (x + 1, y + 1) \]

\[ (x, y) \xrightarrow{SW} (x - 1, y - 1) \]

\[ (x, y) \xrightarrow{SE} (x + 1, y - 1) \]

Accept if \((2, 2)\) reached

Accepted sequences:

\[ NE, NE \]

\[ NE, SE, NE, NW \]

\[ NE, NE, NE, SW \ldots \]
Transducer example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[ (x, y) \xrightarrow{NW/x} (x - 1, y + 1) \]

\[ (x, y) \xrightarrow{NE/x} (x + 1, y + 1) \]

\[ (x, y) \xrightarrow{SW/x} (x - 1, y - 1) \]

\[ (x, y) \xrightarrow{SE/x} (x + 1, y - 1) \]

Input direction
Output \(x\)-coordinate
Transducer example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[ (x, y) \xrightarrow{NW/x} (x - 1, y + 1) \]

\[ (x, y) \xrightarrow{NE/x} (x + 1, y + 1) \]

\[ (x, y) \xrightarrow{SW/x} (x - 1, y - 1) \]

\[ (x, y) \xrightarrow{SE/x} (x + 1, y - 1) \]

Input direction

Output \( x \)-coordinate

Input: \( NE, SE, NE, NW \)

Output: 1, 2, 3, 2
Transducer example: Diagonally moving robot

Example

\[ S = \mathbb{Z} \times \mathbb{Z} \]

\[ s_0 = (0, 0) \]

\[
(x, y) \xrightarrow{NW/y} (x - 1, y + 1)
\]

\[
(x, y) \xrightarrow{NE/y} (x + 1, y + 1)
\]

\[
(x, y) \xrightarrow{SW/y} (x - 1, y - 1)
\]

\[
(x, y) \xrightarrow{SE/y} (x + 1, y - 1)
\]

Input direction
Output \( y \)-coordinate

Input: \( NE, SE, NE, NW \)
Output: \( 1, 0, 1, 2 \)
Acceptor example: Die Hard jug problem

Example

- $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0, 0)$
- $\rightarrow$ given by

  - $(i, j) \xrightarrow{E_5} (0, j)$ [empty 5L jug]
  - $(i, j) \xrightarrow{E_3} (i, 0)$ [empty 3L jug]
  - $(i, j) \xrightarrow{F_5} (5, j)$ [fill 5L jug]
  - $(i, j) \xrightarrow{F_3} (i, 3)$ [fill 3L jug]
  - $(i, j) \xrightarrow{E_{35}} (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
  - $(i, j) \xrightarrow{E_{53}} (0, i + j)$ if $i + j \leq 3$ [empty 5L jug into 3L jug]
  - $(i, j) \xrightarrow{F_{53}} (5, j - 5 + i)$ if $i + j \geq 5$ [fill 5L jug from 3L jug]
  - $(i, j) \xrightarrow{F_{35}} (i - 3 + j, 3)$ if $i + j \geq 3$ [fill 3L jug from 5L jug]

- Accept if $(4, 0)$ is reached:
Acceptor example: Die Hard jug problem

Example

- \( S = \{ (i, j) \in \mathbb{N} \times \mathbb{N} : 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3 \} \)
- \( s_0 = (0, 0) \)
- \( \rightarrow \) given by
  - \((i, j) \xrightarrow{E_5} (0, j)\) [empty 5L jug]
  - \((i, j) \xrightarrow{E_3} (i, 0)\) [empty 3L jug]
  - \((i, j) \xrightarrow{F_5} (5, j)\) [fill 5L jug]
  - \((i, j) \xrightarrow{F_3} (i, 3)\) [fill 3L jug]
  - \((i, j) \xrightarrow{E_{35}} (i + j, 0)\) if \( i + j \leq 5 \) [empty 3L jug into 5L jug]
  - \((i, j) \xrightarrow{E_{53}} (0, i + j)\) if \( i + j \leq 3 \) [empty 5L jug into 3L jug]
  - \((i, j) \xrightarrow{F_{53}} (5, j - 5 + i)\) if \( i + j \geq 5 \) [fill 5L jug from 3L jug]
  - \((i, j) \xrightarrow{F_{35}} (i - 3 + j, 3)\) if \( i + j \geq 3 \) [fill 3L jug from 5L jug]

- Accept if \((4, 0)\) is reached: e.g. F3, E35, F3, F53, E5, E35, F3, E35
It can be useful to allow the system to transition without taking input or producing output. We use the special symbol $\epsilon$ to denote such transitions.
An acceptor is a \( \Sigma \cup \{\epsilon\} \)-labelled transition system \( A = (S, \rightarrow, \Sigma, s_0, F) \) with a start state \( s_0 \in S \) and a set of final states \( F \subseteq S \).

A transducer is a \( (\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\}) \)-labelled transition system \( T = (S, \rightarrow, \Sigma, s_0, F) \) with a start state \( s_0 \in S \) and a set of final states \( F \subseteq S \).
Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata
Finite state transition systems

State transition systems with a finite set of states are particularly useful in Computer Science.

**Acceptors:** Finite state automata

**Transducers:** Mealy machines