1 Relations and Functions

You need to gain familiarity with the various types of relations used in B, as these will dominate the models you will be building. There are a confusingly large number of arrows that you will need to master.

A relation is simply a set of pairings between two sets, for example between FRIENDS and their PHONE numbers. We might have a set of such pairings in the set phone, which we might declare as

\[ phone \in \text{FRIENDS} \leftrightarrow \text{PHONE} \]

In B a pair is denoted using the maps to symbol →, for example

\[ phone = \{ \text{Jim} \mapsto 0456123456, \text{Lisa} \mapsto 0423456234, \text{Jim} \mapsto 0293984321, \ldots \} \]

Notice that relations can be many to many, there can be many friends mapping to telephone numbers, but also each friend may have many telephone numbers.

Functions are many to one, meaning that there are many things, but each thing can map to only one value. You’ve met functions in mathematics and maybe other places.

There are two basic sort of functions:

- **Partial functions:** \( f \in X \mapsto Y \), a function that may not be defined everywhere in \( X \), for example a function between friend and their partner.

- **Total functions:** \( f \in X \rightarrow Y \), a function that is defined everywhere in \( X \), for example the function that maps each number to its square.

Having got that far we don’t leave it there. We further restrict functions as follows:

- **Injective functions:** \( f \in X \mapsto Y \), or \( f \in X \rightarrow Y \), one to one functions, where \( f(x) = f(y) \) only if \( x = y \), for example a function from person and their licence number, assuming that each person is uniquely identified.

- **Surjective function:** \( f \in X \mapsto Y \), or \( f \in X \rightarrow Y \), onto functions, where each value of \( Y \) is equal to \( f(x) \) for some value of \( x \) in \( X \).
**Bijective functions:** \( f \in X \mapsto Y \), one-to-one and onto functions.

It is important to recognise and use these relationships when developing a model.

**Exercises**

Investigate the relationships for the following:

1. the *sibling* relationship between people;
2. the *brother* and *sister* relationships between people;
3. the relationship between people and their cars;
4. the relationship between people and registration plates;
5. relationships in student enrolment at UNSW;
6. the relationship between coin denominations and their value;
7. the relationship that describes the coins you have in your pocket;
8. relationships concerning products on a supermarket shelf;
9. the relationship between courses and lecturers.

## 2 More on Sets

These exercises are intended to familiarise you with set concepts and the way EventB uses sets to model mathematical concepts. The tutorial also introduces EventB notation.

It is recommended that these exercise should be done in conjunction with the *B Concise Summary*. Also, while notation needs to be understood and this involves semantics, it is recommended that the reasoning about expressions should be conducted syntactically.

In this tutorial we also use B’s *joker* notation and we utilise the notation of EventB proof theories (rules) for expressing properties. In the joker notation we use a single letter to denote an arbitrary expression. Thus when we say, “let S be a set”, S is an expression, which in this case must be a set expression, for example \( \text{members} \cup \{\text{newmember}\} \). You should not think of single letters as being variables.

A rule has the form \( P \Rightarrow Q \), stating that if we know \( P \) is true, then \( Q \) is true. For example,

\[
A \subseteq \text{EventB} \land a \in A \Rightarrow a \in B.
\]

Notice that while rules look like predicates, the elements of the rule are not typed, for example in the above rule \( A \) and \( B \) are both sets and their types must be compatible, otherwise \( A \subseteq B \) would not be defined. The rules are higher order logic, not first-order as used in EventB machines.

The use of the joker and proof rule notation allows us to say things about arbitrary expressions so long as they are well-typed.
Simple sets  The basis of EventB is simple sets. A set is an unordered collection of things, without multiplicity. The only property of sets is membership: we can evaluate \( x \in X \), “\( x \) is a member of \( X \)”. Finite sets have cardinality, \( \text{card}(S) \), the number of elements in \( S \). Infinite sets do not have cardinality; EventB does not have an infinity.

Powersets  From a simple set \( S \) we can form the powerset of \( S \), written \( \mathbb{P}(S) \), which is the set of all subsets of \( S \). We can define \( \mathbb{P}(S) \) using set comprehension:

\[
\mathbb{P}(S) = \{ s \mid s \subseteq S \}
\]

We could also use a symmetric rule to express a property of powersets

\[
p \in \mathbb{P}(P) \Rightarrow p \subseteq P
\]

\[
p \subseteq P \Rightarrow p \in \mathbb{P}(P)
\]

Also,

\[
S \in \mathbb{P}(S)
\]

Products  Given two sets \( S \) and \( T \) we can form the product of \( S \) and \( T \), sometimes called the Cartesian product denoted \( S \times T \). The product is the set of ordered pairs taken respectively from \( S \) and \( T \):

\[
S \times T = \{ x, y \mid x \in S \land y \in T \}
\]

A rule for products is

\[
a \mapsto b \in A \times B \Rightarrow a \in A \land b \in EventB
\]

The following sets are used in the exercises:

\[
\text{NAMES} = \{ \text{Jack, Jill} \};
\]

\[
\text{PHONE} = \{ 123, 456, 789 \}
\]

1. In this question you will be dealing with products, or sets of pairs. Instead of writing a pair as \((a, b)\), which is probably what you would normally do, write them as \( a \mapsto b \), where \( \mapsto \) is pronounced “maps to”.

(a) What is \( \text{NAMES} \times \text{PHONE} \)?

(b) What might it represent (model)?

(c) What is \( \text{card}(\text{NAMES} \times \text{PHONE}) \)?

(d) What is \( \text{card}(\text{NAMES} \times \{ \}) \)?

(e) What is \( \mathbb{P}(S) \)?

(f) Given \( \text{card}(S) = N \), what is \( \text{card}(\mathbb{P}(S)) \)?

(g) What is \( \text{card}(\mathbb{P}(\text{NAMES} \times \text{PHONE})) \)?

(h) What does \( \text{card}(\mathbb{P}(\text{NAMES} \times \text{PHONE})) \) give you?

(i) Is \( \text{NAMES} \times \text{PHONE} \) a function?

(j) Give a functional subset.
(k) Give a total functional subset.

(l) If a subset S is described as a partial functional set, which of the following is correct?
   i. S is not a total functional set.
   ii. S might not be a total functional set

All of the following classes of functions may be total or not total.

(m) Give an injective functional subset.

(n) Give a surjective functional subset.

(o) Give a (total) bijective functional subset.

Relations Any subset of $X \times Y$ is called a (many-to-many) relation. The set of all relations between $X$ and $Y$ is denoted $X \leftrightarrow Y$. Since each relation is an element of $X \times Y$, it follows that $X \leftrightarrow Y = \mathcal{P}(X \times Y)$. This could be expressed by a rule:

$$r \in X \leftrightarrow Y \Rightarrow r \in \mathcal{P}(X \times Y)$$

Domain and range Given a relation $r$, where $r \in X \leftrightarrow Y$ then the domain of $r$, $\text{dom}(r)$, is the subset of $X$ for which a relation is defined. The range of $r$, $\text{ran}(r)$ is the subset of $Y$ onto which the $\text{dom}(r)$ is mapped. Here are some rules:

$$r \in X \leftrightarrow Y \Rightarrow \text{dom}(r) \subseteq X \land \text{ran}(r) \subseteq Y$$

$$r \in X \leftrightarrow Y \land x \mapsto y \in r \Rightarrow x \in \text{dom}(r) \land y \in \text{ran}(r)$$

2. Given phonebook $\in$ NAMES $\leftrightarrow$ PHONE,

   (a) Give some examples of phonebook.
   (b) Give NAMES $\leftrightarrow$ PHONE.
   (c) What is $\text{card}(\text{NAMES} \leftrightarrow \text{PHONE})$?

Relational inverse The relational inverse of $r$, $r^{-1}$, is the relation produced by inverting the mappings within $r$

$$r \in X \leftrightarrow Y \land x \mapsto y \in r \Rightarrow y \mapsto x \in r^{-1}$$

Domain and range restriction Domain (range) restriction restricts the domain (range) of a relation. $s \triangleleft r$ is the relation $r$ domain restricted to $s$. This gives a subset of the relation $r$ whose domain is a subset of $s$:

$$r \in X \leftrightarrow Y \land s \subseteq X \Rightarrow s \triangleleft r \subseteq r \land \text{dom}(s \triangleleft r) \subseteq s$$

$r \triangleright s$ is the relation $r$ range restricted to $s$. This gives a subset of the relation $r$ whose range is a subset of $s$:

$$r \in X \leftrightarrow Y \land s \subseteq Y \Rightarrow r \triangleright s \subseteq r \land \text{ran}(r \triangleright s) \subseteq s$$
Relational image The relational image \( r[s] \) give the image of a set \( s \) under the relation \( r \): the mapping of all elements of \( s \) according to the maplets in \( r \):
\[
r[s] = \text{ran}(s \circ r)
\]
Relational image is the counterpart for relations of functional application for functions; the former being many-to-many and the latter many-to-one.

3. Given \( \text{phonebook} = \{\text{Jack} \mapsto 123, \text{Jack} \mapsto 789, \text{Jill} \mapsto 456, \text{Jill} \mapsto 789\} \)
   
   (a) What is \( \text{dom}(\text{phonebook}) \)?
   (b) What is \( \text{ran}(\text{phonebook}) \)?
   (c) What is \( \text{phonebook} \preceq \{\text{Jack} \mapsto 123\} \)?
   (d) What is \( \{\text{Jack}\} \triangleright \text{phonebook} \)?
   (e) What is \( \{\text{Jack}\} \preceq \text{phonebook} \)?
   (f) What is \( \text{phonebook} \triangleright \{123, 789\} \)?
   (g) What is \( \text{phonebook} \triangleright \{123, 789\} \)?
   (h) What is \( \text{phonebook}[\{\text{Jack}\}] \)?

Functions Functions are *many-to-one relations*. \( X \leftrightarrow Y \) is the set of all partial functions formed from \( X \) and \( Y \). A many-to-one relation is one where each element of the domain maps to only one value in the range, as illustrated by the following rule:
\[
f \in X \leftrightarrow Y \wedge x \mapsto u \in f \wedge x \mapsto v \Rightarrow u = v
\]
Partial functions are the most general form of function. For every \( x \) in the domain of a function \( f \ (x \in \text{dom}(f)) \) we can write \( f(x) \) to obtain the value \( x \) maps to under \( f \), that is
\[
f \in X \leftrightarrow Y \wedge x \mapsto y \Rightarrow f(x) = y
\]
4. (a) Give \( \text{NAMES} \leftrightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \leftrightarrow \text{PHONE}) \)?

Total functions \( X \rightarrow Y \) is the set of all total functions formed from \( X \) and \( Y \). Total functions are (partial) functions with maximal domains:
\[
f \in X \rightarrow Y \Rightarrow \text{dom}(f) = X
\]
5. (a) Give \( \text{NAMES} \rightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \rightarrow \text{PHONE}) \)?

Partial injective functions \( X \leftrightarrow Y \) is the set of all partial, injective functions formed from \( X \) and \( Y \). An injective function is a one-to-one relation:
\[
f \in X \leftrightarrow Y \wedge u \mapsto y \in f \wedge v \mapsto y \in f \Rightarrow u = v
\]
6. (a) Give \( \text{NAMES} \leftrightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \leftrightarrow \text{PHONE}) \)?
Total injective functions \( X \rightarrow Y \) is the set of all total, injective functions formed from \( X \) and \( Y \). A total injective function is both total and injective:

\[
f \in X \rightarrow Y \Rightarrow f \in X \rightarrow Y \land f \in X \twoheadrightarrow Y
\]

7. (a) Give \( \text{NAMES} \rightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \rightarrow \text{PHONE})? \)

Surjective functions \( X \twoheadrightarrow Y \) is the set of all partial, surjective functions formed from \( X \) and \( Y \). A surjective function is a functional onto relations; a function whose range is maximal:

\[
f \in X \twoheadrightarrow Y \Rightarrow \text{ran}(f) = Y
\]

8. (a) Give \( \text{NAMES} \twoheadrightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \twoheadrightarrow \text{PHONE})? \)

Total surjective functions \( X \rightarrow Y \) is the set of all total, surjective functions formed from \( X \) and \( Y \). A total surjective function is both total and surjective:

\[
f \in X \rightarrow Y \Rightarrow f \in X \rightarrow Y \land f \in X \twoheadrightarrow Y
\]

9. (a) Give \( \text{NAMES} \twoheadrightarrow \text{PHONE} \).
   (b) What is \( \text{card}(\text{NAMES} \twoheadrightarrow \text{PHONE})? \)

   (c) Why is a partial bijection unnecessary?

11. Suppose \( \text{STUDENTS} \) is the set of all students that could be enrolled in a particular course. Students pass a course if they gain at least 50 marks in the final examination. Given a function \( \text{results} \in \text{STUDENTS} \rightarrow \mathbb{N} \), that yields the examination result for a particular student, specify

   (a) the set of students that pass;
   (b) the set of students that fail.

12. If we were modelling a taxi fleet company we might have three variables, \( \text{drivers} \), \( \text{taxis} \) and \( \text{assigned} \) constrained by

\[
\begin{align*}
\text{drivers} & \in \mathbb{P}(\text{DRIVERS}) \\
\text{taxis} & \in \mathbb{P}(\text{TAXIS}) \\
\text{assigned} & \in \text{drivers} \twoheadrightarrow \text{taxis}
\end{align*}
\]
where \( \text{DRIVERS} \) is the set of possible drivers, \( \text{TAXIS} \) is the set of possible taxis, \( \text{drivers} \) is the set of drivers working for the company, \( \text{taxis} \) is the set of taxis owned by the company, and \( \text{assigned} \) is a function recording the assignment of drivers to taxis.

The arrow \( \text{inj} \) denotes a \text{partial injective} function. An injective function is a one-to-one function.

(a) Why is \( \text{assigned} \) a function?
(b) Why is \( \text{assigned} \) a partial function?
(c) Why is \( \text{assigned} \) an injective function?
(d) Specify the drivers who are currently assigned.
(e) Specify the drivers who are currently unassigned.
(f) Specify the taxis that are currently assigned.
(g) Specify the taxis that are currently unassigned.

13. Are the following rules correct or incorrect?

\begin{enumerate}
\item \( f \in X \rightarrow Y \Rightarrow f \in X \mapsto Y \)
\item \( f \in X \mapsto Y \Rightarrow f \in X \mapsto Y \)
\item \( f \in X \mapsto Y \Rightarrow f \in X \mapsto Y \)
\item \( f \in X \mapsto Y \Rightarrow f \in X \rightarrow Y \)
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\item \( f \in X \mapsto Y \Rightarrow f \in X \rightarrow Y \)
\item \( f \in X \mapsto Y \Rightarrow f \in X \rightarrow Y \)
\end{enumerate}

14. \( \text{union}(S) \) is the generalised union of the elements of \( S \), that is, if \( S \) is a set of sets, then \( \text{union}(S) \) is the union of all of the sets that are contained in \( S \). What is \( \text{union}(\{\}) \)?

15. \( \text{inter}(S) \) is the generalised intersection of the elements of \( S \), that is, if \( S \) is a set of sets, then \( \text{inter}(S) \) is the intersection of all of the sets that are contained in \( S \). What is \( \text{inter}(\{\}) \)?
16. Two subsets of a set \( S \) are said to be *disjoint* if and only if they have no elements in common. Define a binary relation \( \text{disjoint} \) that holds between a pair of subsets of \( S \) exactly when they are *disjoint*.

17. A set of subsets of \( S \) is said to be *pairwise disjoint* if and only if every pair of distinct sets in it is disjoint (in the sense of (c)). A partition of a set \( S \) is a pairwise disjoint set of subsets of \( S \) whose generalised union is equal to \( S \).

(a) Define the set of all partitions of \( S \).

(b) Which of the subsets of \( \{a, b\} \) are partitions of \( \{a, b\} \)?