Compositionality and Asynchrony

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Where we are at

Last lecture, we looked at *proof methods for termination*. This lecture, we will conclude our examination of proof methods with *compositional* techniques, and *asynchronous systems*. 
**Synchronous Transition Diagrams**

**Definition**

A *synchronous transition diagram* is a parallel composition $P_1 \parallel \ldots \parallel P_n$ of $n$ (sequential) transition diagrams $P_1, \ldots, P_n$ called *processes*.

The processes $P_i$

- do not share variables
- communicate along channels $C, D, \ldots$ connecting processes by way of
  - *output* statements $C \leftarrow e$ for sending the value of expression $e$ along channel $C$
  - *input* statements $C \Rightarrow x$ for receiving a value along channel $C$ into variable $x$
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NB

Today, we will assume that all communication channels are unidirectional, and shared between at most 2 processes.
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Analysis of AFR and L&G

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- Both are only applicable to *closed* systems.
- So we have to reason about the system as a whole, even including users modelled as processes.
- In other words: we can’t reason *compositionally*. Typically, non-compositional proof methods don’t scale, and prevent proof re-use.
Quotes on Compositionality

**de Roever et al.**

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Compositionality is a red herring.
Lamport (1997) – “Composition: a way to make proofs harder”

Systems are complicated. We master their complexity by building them from simpler components. This suggests that to master the complexity of reasoning about systems, we should prove properties of the separate components and then combine those properties to deduce properties of the entire system. In concurrent systems, the obvious choice of component is the process. So, compositional reasoning has come to mean deducing properties of a system from properties of its processes. I have long felt that this whole approach is rather silly. You don’t design a mutual exclusion algorithm by first designing the individual processes and then hoping that putting them together guarantees mutual exclusion.
Compositionally-Inductive Assertion Network

Key Idea
Handle communication with a special logical variable $h$, containing the history of all communication, i.e. a sequence of pairs of channels and messages $\langle C, x \rangle$. Programs shouldn’t write to $h$. 
Compositionality

Asynchrony

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A local assertion network $Q$ is *compositionally-inductive* for a sequential synchronous transition diagram $P = (L, T, s, t)$, written $P \vdash Q$, if

- $Q_\ell \land b \implies Q_{\ell'} \circ f$ for each $\ell \xrightarrow{b;f} \ell' \in T$.
- $Q_\ell \land b \implies Q_{\ell'} \circ (f \circ [h \leftarrow h \cdot \langle C, e \rangle])$, for each $\ell \xrightarrow{b;C=e;f} \ell' \in T$.
- $Q_\ell \land b \implies \forall x \ (Q_{\ell'} \circ (f \circ [h \leftarrow h \cdot \langle C, x \rangle]))$, for each $\ell \xrightarrow{b;C=x;f} \ell' \in T$. 
Partial Correctness

Let $Q$ be an assertion network for a process $P$ and $Q_s$ and $Q_t$ be the assertions at the start and end states. We have the **Basic diagram rule**:

\[
P \vdash Q
\]

\[
\{Q_s\} \quad P \quad \{Q_t\}
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\end{align*}
$$

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\begin{align*}
\{\phi \land h = \varepsilon\} & P \{\psi\} \\
\{\phi\} & P \{\psi\}
\end{align*}
$$

The Consequence rule allows pre/post-conditions to be strengthened/weakened:

$$
\begin{align*}
\phi \Rightarrow \phi' \quad \{\phi'\} & P \{\psi'\} \quad \psi' \Rightarrow \psi \\
\{\phi\} & P \{\psi\}
\end{align*}
$$
Parallel composition rule

Provided $\psi_i$ only makes assertions about (a) local variables in $P_i$, and (b) the history that directly involves channels used by $P_i$, we get this compositional Parallel composition rule:

\[
\begin{align*}
\{\phi_1\} & P_1 \{\psi_1\} & \{\phi_2\} & P_2 \{\psi_2\} \\
\{\phi_1 \land \phi_2\} & P_1 \parallel P_2 & \{\psi_1 \land \psi_2\}
\end{align*}
\]

Observe that we don’t need to prove anything like interference freedom or generate a proof obligation about each possible communication.
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\end{array}$$

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Notation

Define $h|_H$ as the history $h$ filtered to only contain those pairs $\langle C, x \rangle$ where $C \in H$. 
Example 2 once more

\[ s_1 \xrightarrow{C \leftarrow 1} l_1 \xrightarrow{C \leftarrow 2} t_1 \]

\[ s_2 \xrightarrow{C \Rightarrow x} l_2 \xrightarrow{C \Rightarrow x} t_2 \]
Example 2 once more

\[ s_1 \xrightarrow{C \leftarrow 1} l_1 \xrightarrow{C \leftarrow 2} t_1 \]

\[ h|\{C\} = \varepsilon \quad h|\{C\} = \langle C, 1 \rangle \quad h|\{C\} = \langle C, 1 \rangle \cdot \langle C, 2 \rangle \]

\[ h|\{C\} = \varepsilon \quad h|\{C\} = \langle C, x \rangle \quad \exists y. \ h|\{C\} = \langle C, y \rangle \cdot \langle C, x \rangle \]
Example 2 once more cont’d

For the two output transitions we need to show

\[ \models h|_\{C\} = \varepsilon \implies h|_\{C\} = \langle C, 1 \rangle \circ [h \leftarrow h \cdot \langle C, 1 \rangle] \]  
\[ \models h|_\{C\} = \langle C, 1 \rangle \implies h|_\{C\} = \langle C, 1 \rangle \cdot \langle C, 2 \rangle \circ [h \leftarrow h \cdot \langle C, 2 \rangle] \]  

which is obvious; and for the two input transitions

\[ \models h|_\{C\} = \varepsilon \implies \forall x \ (h|_\{C\} = \langle C, x \rangle \circ [h \leftarrow h \cdot \langle C, x \rangle]) \]  
\[ \models h|_\{C\} = \langle C, x \rangle \implies \forall x \exists y \ (h|_\{C\} = \langle C, y \rangle \cdot \langle C, x \rangle \circ [h \leftarrow h \cdot \langle C, x \rangle]) \]  

which also works out nicely.
Example 2 once more cont’d

Using the **Basic diagram rule** we may now deduce

\[
\begin{align*}
\{ h\|_C = \varepsilon \} & \quad C \leftarrow 1; C \leftarrow 2 \quad \{ h\|_C = \langle C, 1 \rangle \cdot \langle C, 2 \rangle \} \\
\{ h\|_C = \varepsilon \} & \quad C \Rightarrow x; C \Rightarrow x \quad \{ \exists y. \; h\|_C = \langle C, y \rangle \cdot \langle C, x \rangle \}
\end{align*}
\]
Example 2 once more cont’d

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\]

before applying the **parallel composition rule** to obtain

\[
\begin{align*}
\{ h|_C = \varepsilon \} & \quad P \ \{ h|_C = \langle C, 1 \rangle \cdot \langle C, 2 \rangle \land \exists y. \ h|_C = \langle C, y \rangle \cdot \langle C, x \rangle \}
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Example 2 once more cont’d

Using the **Basic diagram rule** we may now deduce

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before applying the **parallel composition rule** to obtain

\[
\{ h|_C = \varepsilon \} \quad P \quad \{ h|_C = \langle C, 1 \rangle \cdot \langle C, 2 \rangle \wedge \exists y. \quad h|_C = \langle C, y \rangle \cdot \langle C, x \rangle \}
\]

which implies (via the rule of consequence):

\[
\{ h = \varepsilon \} \quad P \quad \{ x = 2 \}
\]
Example 2 once more cont’d

Using the **Basic diagram rule** we may now deduce

\[
\{h|_c = \varepsilon\} \quad C \leftarrow 1; \ C \leftarrow 2 \ \{h|_c = \langle C, 1 \rangle \cdot \langle C, 2 \rangle\}
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\]

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\[
\{h = \varepsilon\} \ P \ \{x = 2\}
\]

and finally the **initialisation rule** takes us to

\[
\{\text{TRUE}\} \ P \ \{x = 2\}\]
Asynchrony

Consider a process $P$ that sends a file $a$ on the channel $C$ to the process $Q$, which saves it to $b$.

- $a[i] \neq EOF; C \leftarrow a[i]; i \leftarrow i + 1$
- $C \Rightarrow b[j]; j \leftarrow j + 1$

How do we verify this if $C$ is asynchronous?
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How do we verify this if $C$ is asynchronous?
Convert to Synchronous

\[ a[i] \neq \text{EOF}; A \leftarrow a[i]; i \leftarrow i + 1 \]

\[ B \Rightarrow b[j]; j \leftarrow j + 1 \]

\[ a[i] = \text{EOF}; A \leftarrow \text{EOF}; i \leftarrow i + 1 \]

\[ j > 0 \land b[j - 1] = \text{EOF} \]
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\[ j > 0 \land b[j - 1] = \text{EOF} \]

\[ A \Rightarrow x; \ q \leftarrow q \cdot x \]

\[ q \neq \varepsilon; \ B \leftarrow \text{head}(q); \ q \leftarrow \text{tail}(q) \]
Compositionally

By adding an extra process with two synchronous channels to explicitly manage the queue, we convert this asynchronous system to a synchronous one. We can now use AFR, Levin and Gries or the compositional method.
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Compositionally

By adding an extra process with two synchronous channels to explicitly manage the queue, we convert this asynchronous system to a synchronous one. We can now use AFR, Levin and Gries or the compositional method. Using the compositional method, we have the desired postcondition:

\[ \exists i. \ a[i] = EOF \land a[0 \ldots i] = b[0 \ldots i] \]

And the following assertion network:

\[
\begin{align*}
Q(p_s) & \equiv \hat{h}_{\{A\}} = a[0 \ldots i] \land EOF \notin a[0 \ldots i] \\
Q(p_t) & \equiv \hat{h}_{\{A\}} = a[0 \ldots i] \land EOF \notin a[0 \ldots i - 1] \land a[i - 1] = EOF \\
Q(q_s) & \equiv \hat{h}_{\{B\}} = b[0 \ldots j] \\
Q(q_t) & \equiv \hat{h}_{\{B\}} = b[0 \ldots j] \land b[j - 1] = EOF \\
Q(C) & \equiv \hat{h}_{\{A\}} = \hat{h}_{\{B\}} \cdot q
\end{align*}
\]

Proof obligations will be informally described.
What Now?

If time allows, we’ll take a brief detour into the world of *process algebra*, a high level formalism for describing concurrent systems.

Either way, we’ll then discuss *distributed algorithms*. 