Ambiguity and Simultaneous Definitions

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Semester 2 2018

Arithmetic

Example (Arithmetic Expression)

$$\textbf{Arith} ::= i \mid \textbf{Arith} \times \textbf{Arith} \mid \textbf{Arith} + \textbf{Arith} \mid (\textbf{Arith}) \quad (i \in \mathbb{Z})$$

$$\frac{i \in \mathbb{Z}}{i\ \textbf{Arith}}L \qquad \frac{a\ \textbf{Arith} \qquad b\ \textbf{Arith}}{a \times b\ \textbf{Arith}}P \qquad \frac{a\ \textbf{Arith} \qquad b\ \textbf{Arith}}{a + b\ \textbf{Arith}}S \qquad \frac{a\ \textbf{Arith}}{(a)\ \textbf{Arith}}$$

Infer $1 + 2 \times 3$ **Arith** (both ways) to whiteboard

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Ambiguity

Arith is *ambiguous*, which means that there are multiple ways to derive the same judgement.

For syntax, this is a big problem, as different interpretations of syntax can lead to semantic inconsistency:

$$\dfrac{\dfrac{1 \in \mathbb{Z}}{1 \ \textbf{Arith}} \quad \dfrac{\dfrac{2 \in \mathbb{Z}}{2 \ \textbf{Arith}} \quad \dfrac{3 \in \mathbb{Z}}{3 \ \textbf{Arith}}}{2 \times 3 \ \textbf{Arith}}}{1 + 2 \times 3 \ \textbf{Arith}} \qquad \dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1 \ \textbf{Arith}} \quad \dfrac{2 \in \mathbb{Z}}{2 \ \textbf{Arith}}}{1 + 2 \ \textbf{Arith}} \quad \dfrac{3 \in \mathbb{Z}}{3 \ \textbf{Arith}}}{1 + 2 \times 3 \ \textbf{Arith}}$$

Second Attempt

We want to specify **Arith** in such a way that enforces order of operations.

Here we will use multiple judgements:

Example (Arithmetic Expression)

$$\begin{array}{lcl}
\textbf{Atom} & ::= & i \mid (\textbf{SExp}) \quad (i \in \mathbb{Z}) \\
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\end{array}$$

$$\frac{i \in \mathbb{Z}}{i\ \textbf{Atom}} \qquad \frac{a\ \textbf{SExp}}{(a)\ \textbf{Atom}} \qquad \frac{e\ \textbf{Atom}}{e\ \textbf{PExp}} \qquad \frac{e\ \textbf{PExp}}{e\ \textbf{SExp}}$$

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Consider: Is there still any ambiguity here?

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More ambiguity

$$\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \textbf{Atom}}}{1\ \textbf{PExp}} \qquad \dfrac{\dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \textbf{Atom}}}{2\ \textbf{PExp}} \qquad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \textbf{Atom}}}{3\ \textbf{PExp}}}{2 \times 3\ \textbf{PExp}}}{1 \times 2 \times 3\ \textbf{PExp}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \textbf{Atom}}}{1\ \textbf{PExp}} \qquad \dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \textbf{Atom}}}{2\ \textbf{PExp}}}{1 \times 2\ \textbf{PExp}} \qquad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \textbf{Atom}}}{3\ \textbf{PExp}}}{1 \times 2 \times 3\ \textbf{PExp}}$$

This ambiguity seems harmless, but it would not be harmless for some other operations. Which ones? Operators that are not *associative*.

We have to specify the *associativity* of operators. How?

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Associativities

Operators have various *associativity* constraints:

Associative	All associativities are equal.
Left-Associative	$A \odot B \odot C = (A \odot B) \odot C$
Right-Associative	$A \odot B \odot C = A \odot (B \odot C)$

Try to think of some examples!

Enforcing associativity

We force the grammar to accept a smaller set of expressions on one side of the operator only. Show how this works on the whiteboard.

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$$

$$
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Here we made multiplication and addition right associative. How would we do left?

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Bring Back Parentheses

The Parenthetical Language

$$\boxed{s \ \mathbf{M}}$$

$$\frac{}{\varepsilon \ \mathbf{M}} M_E \qquad \frac{s \ \mathbf{M}}{(s) \ \mathbf{M}} M_N \qquad \frac{s_1 \ \mathbf{M} \qquad s_2 \ \mathbf{M}}{s_1 s_2 \ \mathbf{M}} M_J$$

Is this language ambiguous? to whiteboard

Ambiguity in Parentheses

Not only is it ambiguous, it is infinitely so. Strings like ()()() could be split at two different locations by rule M_J, but if we use ε, then even the string () is ambiguous:

$$\frac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E}{()\ \mathbf{M}}\,M_N \qquad \frac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E}{()\ \mathbf{M}}\,M_N}{()\ \mathbf{M}}\,M_J$$

$$\frac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathbf{M}}\,M_E}{()\ \mathbf{M}}\,M_N}{()\ \mathbf{M}}\,M_J}{()\ \mathbf{M}}\,M_J$$

We will eliminate the ambiguity by once again splitting **M** into two judgements, **N** and **L**.

The crucial observation is that terms in **M** are a list (**L**) of terms nested within parentheses (**N**).

Example (Unambiguous Parentheses)

$$\boxed{s\ \mathbf{L}} \qquad \boxed{s\ \mathbf{N}}$$

$$\frac{}{\varepsilon\ \mathbf{L}}L_E \qquad \frac{s\ \mathbf{L}}{(s)\ \mathbf{N}}N_N \qquad \frac{s_1\ \mathbf{N} \quad s_2\ \mathbf{L}}{s_1 s_2\ \mathbf{L}}L_J$$

Proving Equivalence

Now we shall prove $\mathbf{M} = \mathbf{L}$. There are two cases, each dispatched with rule induction:

$$\frac{s\ \mathbf{M}}{s\ \mathbf{L}} \qquad \frac{s\ \mathbf{L}}{s\ \mathbf{M}}$$

The first case requires proving a *lemma*. The second requires *simultaneous induction*.
These proofs will be carried out on the "board" (iPad). A properly typeset PDF of the proof will also be uploaded.