COMP3161/COMP9161
Preliminaries Exercises

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1. **Strange Loops:** The following system, based on a system called Miu, is perhaps famously mentioned in Douglas Hofstadter's book, *Gödel, Escher, Bach.*

$$\frac{}{\texttt{MI}\ \textsc{Miu}}1 \quad \frac{x\texttt{I}\ \textsc{Miu}}{x\texttt{IU}\ \textsc{Miu}}2 \quad \frac{\texttt{M}x\ \textsc{Miu}}{\texttt{M}xx\ \textsc{Miu}}3 \quad \frac{x\texttt{III}y\ \textsc{Miu}}{x\texttt{U}y\ \textsc{Miu}}4 \quad \frac{x\texttt{UU}y\ \textsc{Miu}}{xy\ \textsc{Miu}}5$$

(a) [$\star$] Is `MUII` Miu derivable? If so, show the derivation tree. If not, explain why not.

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MI}\ \textsc{Miu}}1}{\texttt{MII}\ \textsc{Miu}}3}{\texttt{MIIII}\ \textsc{Miu}}3}{\texttt{MIIIIIIII}\ \textsc{Miu}}3}{\texttt{MIIIIIIIIU}\ \textsc{Miu}}2}{\texttt{MIIIIIUU}\ \textsc{Miu}}4}{\texttt{MIIIII}\ \textsc{Miu}}5}{\texttt{MUII}\ \textsc{Miu}}4$$

(b) [$\star\star$] Is $\dfrac{x\texttt{IU}\ \textsc{Miu}}{x\texttt{I}\ \textsc{Miu}}$ admissible? Is it derivable? Justify your answer.

Solution: It is not derivable, but it is admissible. It is not derivable because there is no way to construct a tree that looks like this:

$$\dfrac{\dfrac{x\texttt{IU}\ \textsc{Miu}}{\vdots}}{x\texttt{I}\ \textsc{Miu}}$$

I believe it is, however, *admissible* because it does not change the language Miu. I think there is no string x that could be judged x Miu with this rule that could not be so judged without it, although I welcome any counterexample or proof.

(c) [$\star\star\star\star$] Perhaps famously, `MU` Miu is not admissible. Prove this using rule induction. *Hint*: Try proving something related to the number of `I`s in the string.

Solution: We will prove that the number of `I`s in any string in Miu is not divisible by three. Seeing as `MU` has zero `I`s (a multiple of three), if we prove the above, we prove that `MU` is not admissible.

Base Case (From rule 1). We see that the string `MI` has only one `I`, which is not a multiple of three, hence we have shown our goal.

Inductive case (From rule 2). Given that the number of `I`s in x`I` is not divisible by three (our inductive hypothesis), we can easily see that the number of `I`s in x`IU` is identical and therefore is similarly not divisible by three.

Inductive case (From rule 3). Let n be the number of `I`s in `M`x. Our inductive hypothesis is that $3 \nmid n$. The number of `I`s in `M`xx, clearly $2n$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid 2n$.

Inductive case (From rule 4). Let n be the number of Is in $x\texttt{III}y$. Our inductive hypothesis is that $3 \nmid n$. The number of Is in $x\texttt{U}y$, clearly $n-3$, is similarly indivisible, i.e $3 \nmid n \implies 3 \nmid (n-3)$
Inductive case (From rule 5). Given that the number of Is in $x\texttt{UU}y$ is the same as the number of Is in xy, our inductive hypothesis trivially proves our goal.

Thus, by induction, no string in Miu has a number of Is divisible by three. Therefore, MU Miu is not admissible. □

(d) Here is another language, which we'll call Mi:

$$\frac{}{\texttt{MI}\ \textsc{Mi}}A \qquad \frac{\texttt{M}x\ \textsc{Mi}}{\texttt{M}xx\ \textsc{Mi}}B \qquad \frac{x\texttt{IIIIII}y\ \textsc{Mi}}{xy\ \textsc{Mi}}C$$

i. [★★★] Prove using rule induction that all strings in Mi could be expressed as follows, for some k and some i, where $2^k - 6i > 0$ (where $\texttt{C}^n$ is the character $\texttt{C}$ repeated n times):

$$\texttt{M}\,\texttt{I}^{2^k-6i}$$

Solution:

Base case (From rule A). $\texttt{MI} = \texttt{M}\,\texttt{I}^{2^k-6i}$ when $2^k - 6i = 1$, i.e when $k = 0$ and $i = 0$.

Inductive case (From rule B) Given that $\texttt{M}x = \texttt{MI}^{2^a-6b}$ (our inductive hypothesis), we must show that $\texttt{M}xx = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some k and some i. As $x = \texttt{I}^{2^a-6b}$ (from I.H), it is easy to see that $xx = \texttt{I}^{2(2^a-6b)} = \texttt{I}^{2^{a+1}-6(2b)} = \texttt{I}^{2^k-6i}$ for $k = a+1$ and $i = 2b$.

Inductive case (from rule C) Given that $x\texttt{IIIIII}y = \texttt{M}\,\texttt{I}^{2^a-6b}$ (our inductive hypothesis). We must show that $xy = \texttt{M}\,\texttt{I}^{2^k-6i}$ for some k and i. It should be clear to see that this rule simply subtracts six $\texttt{I}$ characters, and therefore $xy = \texttt{MI}^{2^a-6(b+1)}$, hence $k = a$ and $i = b+1$.

Thus, all strings in Mi can be expressed as $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$ □

ii. We will now prove the opposite claim that, for all k and i, assuming $2^k - 6i > 0$:

$$\texttt{M}\,\texttt{I}^{2^k-6i}\ \textsc{Mi}$$

To prove this we will need a few lemmas which we will prove separately.

α) [★★] Prove, using induction on the natural number k (i.e when $k = 0$ and when $k = k'+1$), that $\texttt{M}\,\texttt{I}^{2^k}$ Mi

Solution:

Base case (when $k = 0$). We have to show $\texttt{MI}$ Mi, which is true by rule A.
Inductive case (when $k = k'+1$) We have to show $\texttt{MI}^{2^{k'+1}}$ Mi, with the inductive hypothesis that $\texttt{MI}^{2^{k'}}$ Mi. Equivalently, we have to show $\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}$ Mi, as follows:

$$\cfrac{\cfrac{}{\texttt{MI}^{2^{k'}}\ \textsc{Mi}}I.H}{\texttt{MI}^{2^{k'}}\texttt{I}^{2^{k'}}\ \textsc{Mi}}B$$

Therefore, by induction on the natural number k, we have shown $\forall k.\texttt{M}\,\texttt{I}^{2^k}$ Mi. □

β) [★★] Prove, using induction on the natural number i, that $\texttt{M}\,\texttt{I}^k$ Mi implies $\texttt{M}\,\texttt{I}^{k-6i}$ Mi, assuming $k - 6i > 0$.

Solution:

Base case (when $i = 0$). We must show that $\texttt{M}\,\texttt{I}^k$ Mi implies $\texttt{M}\,\texttt{I}^{k-0}$ Mi, which is obviously a tautology.

Inductive case (when $i = i' + 1$) We must show that $\texttt{M}\,\texttt{I}^k$ Mɪ implies $\texttt{M}\,\texttt{I}^{k-6(i'+1)}$ Mɪ, given the inductive hypothesis that $\texttt{M}\,\texttt{I}^{k-6i'}$ Mɪ. Note that our I.H can be restated as $\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}$ Mɪ due to our assumption that $k - 6(i' + 1) > 0$. With this, we can prove our goal as shown:

$$\dfrac{\dfrac{}{\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \text{M}\textsc{i}}I.H}{\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \text{M}\textsc{i}}C$$

□

Therefore, our goal is shown by induction.

Hence, as we know $\texttt{M}\,\texttt{I}^{2^k}$ Mɪ for all k from lemma α, we can conclude from lemma β that $\texttt{M}\,\texttt{I}^{2^k-6i}$ Mɪ for all k and all i where $2^k - 6i > 0$ by modus ponens.

These two parts prove that the language Mɪ is exactly characterised by the formulation $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$. A very useful result!

iii. [⋆] Hence prove or disprove that the following rule is admissible in Mɪ:

$$\dfrac{\texttt{M}xx\ \text{M}\textsc{i}}{\texttt{M}x\ \text{M}\textsc{i}}\textsc{Lem}_1$$

Solution: We know from part i that $\texttt{M}xx$ Mɪ $\implies x^2 = \texttt{I}^{2^k-6i}$ for some k and some i where $2^k - 6i > 0$.
This rule is *not admissible* as it adds strings to the language. As $2^4 - 6 = 10$, we know $\texttt{MI}^{10}$ is in the language. This rule would make $\texttt{MI}^5$ a string in the language which it is not as there is no k and i such that $2^k - 6i = 5$.

iv. [⋆] Why is the following rule **not** admissible in Mɪ?

$$\dfrac{xy\ \text{M}\textsc{i}}{x\texttt{IIIIII}y\ \text{M}\textsc{i}}\textsc{Lem}_2$$

Solution: The rule is not admissible as it adds strings to the language. This allows us to *add* six I characters to any string in Mɪ and judge it in Mɪ, which results in additional strings. For example, applying the rule to MI (which is in Mɪ), gives us $\texttt{M}\,\texttt{I}^7$, when our existing formulation of Mɪ ($\texttt{M}\,\texttt{I}^{2^k-6i}$) clearly only allows for even amounts of Is.

v. [⋆⋆⋆] Prove that, for all s, s Mɪ $\implies$ s Mɪᴜ. Note that using straightforward rule induction appears to necessitate $\textsc{Lem}_2$ above, which we know is not admissible. Try proving using the characterisation we have already developed.

Solution: We shall show that all strings in Mɪ, characterised by $\texttt{M}\,\texttt{I}^{2^k-6i}$ where $2^k - 6i > 0$, are also in Mɪᴜ. That is, we shall show that $\texttt{M}\,\texttt{I}^{2^k-6i}$ Mɪᴜ.
To start, we shall prove inductively on k that $\texttt{M}\,\texttt{I}^{2^k}$ Mɪᴜ for all k.

Base case (Where $k = 0$). We must show MI Mɪᴜ, which we know trivially from rule 1.

Inductive case (where $k = k' + 1$). We must show $\texttt{M}\,\texttt{I}^{2^{k'+1}}$ Mɪᴜ, given the inductive hypothesis that $\texttt{M}\,\texttt{I}^{2^{k'}}$ Mɪᴜ. Note we can restate our proof goal as $\texttt{M}\,\texttt{I}^{2^{k'}}\,\texttt{I}^{2^{k'}}$ Mɪᴜ

$$\dfrac{\dfrac{}{\texttt{M}\,\texttt{I}^{2^{k'}}\ \text{M}\textsc{iu}}I.H}{\texttt{M}\,\texttt{I}^{2^{k'}}\,\texttt{I}^{2^{k'}}\ \text{M}\textsc{iu}}B$$

Thus we have shown by induction that $\texttt{M}\,\texttt{I}^{2^k}$ Mɪᴜ for all k.

Next we must prove that $\texttt{M}\,\texttt{I}^k$ Mɪᴜ implies $\texttt{M}\,\texttt{I}^{k-6i}$ Mɪᴜ for all i, assuming $k - 6i > 0$.

Base case (where $i = 0$), we must show that $\texttt{M}\,\texttt{I}^k$ MIU implies $\texttt{M}\,\texttt{I}^{k-0}$ MIU, which is trivially a tautology.
Inductive case (where $i = i' + 1$) we must show that $\texttt{M}\,\texttt{I}^{k-6(i'+1)}$ MIU given the inductive hypothesis $\texttt{M}\,\texttt{I}^{k-6i'}$ MIU. As we know $k-6(i'+1) > 0$, we can restate our inductive hypothesis as $\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}$ MIU, and easily prove our goal:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\texttt{MIIIIII}\,\texttt{I}^{k-6(i'+1)}\ \text{MIU}}\,I.H}{\texttt{MUIII}\,\texttt{I}^{k-6(i'+1)}\ \text{MIU}}\,4}{\texttt{MUU}\,\texttt{I}^{k-6(i'+1)}\ \text{MIU}}\,4}{\texttt{M}\,\texttt{I}^{k-6(i'+1)}\ \text{MIU}}\,5$$

Thus, by modus ponens we can see that $\texttt{M}\,\texttt{I}^{2^k-6i}$ MIU for all k and i where $2^k - 6i > 0$. As this is the exact characterisation of MI, we have proven that s MI $\implies$ s MIU for all s. □

2. **Counting Sticks:** The following language (also presented in a similar form by Douglas Hofstadter, but the original invention is not his) is called the $\Phi\Psi$ system. Unlike the MIU language discussed above, this language is not comprised of a single judgement, but of a ternary *relation*, written $x\ \Phi\ y\ \Psi\ z$, where x, y and z are strings of hyphens (i.e '-'), which may be empty (ϵ). The system is defined as follows:

$$\dfrac{}{\epsilon\ \Phi\ x\ \Psi\ x}\,B \qquad \dfrac{x\ \Phi\ y\ \Psi\ z}{\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z}\,I$$

(a) [★] Prove that -- Φ --- Ψ -----.

Solution:

$$\dfrac{\dfrac{\dfrac{}{\epsilon\ \Phi\ \texttt{---}\ \Psi\ \texttt{---}}\,B}{\texttt{-}\ \Phi\ \texttt{---}\ \Psi\ \texttt{----}}\,I}{\texttt{--}\ \Phi\ \texttt{---}\ \Psi\ \texttt{-----}}\,I$$

(b) [★] Is the following rule admissible? Is it derivable? Explain your answer

$$\dfrac{\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z}{x\ \Phi\ y\ \Psi\ z}\,I'$$

Solution: It is not derivable (as it cannot be shown with a proof tree), but it is admissible. We know this because the language definition for $\Phi\Psi$ is unambiguous, so the only way for $\texttt{-}x\ \Phi\ y\ \Psi\ \texttt{-}z$ to hold is if this was established by rule I. Therefore, we can deduce that $x\ \Phi\ y\ \Psi\ z$, as this is the premise of rule I. We can often "flip" or invert rules in this way, but *only if the language definition is unambiguous.*

(c) [★★] Show that $x\ \Phi\ \epsilon\ \Psi\ x$, for all hyphen strings x, by doing induction on the length of the hyphen string (where $x = \epsilon$ and $x = \texttt{-}x'$).

Solution:

Base case (where $x = \epsilon$). We must show that $\epsilon\ \Phi\ \epsilon\ \Psi\ \epsilon$, which is trivially true by rule B.

Inductive case (where $x = \texttt{-}x'$) We have the inductive hypothesis $x'\ \Phi\ \epsilon\ \Psi\ x'$, and must show that $\texttt{-}x'\ \Phi\ \epsilon\ \Psi\ \texttt{-}x'$. Our goal trivially reduces to our induction hypothesis by rule I.

Therefore we have shown $x\ \Phi\ \epsilon\ \Psi\ x$ for all x by induction on x. □

(d) [★★★] Show that if $\texttt{-}x\ \Phi\ y\ \Psi\ z$ then $x\ \Phi\ \texttt{-}y\ \Psi\ z$, for all hyphen strings x, y and z, by doing induction on the size of x.

Solution: We shall do induction on x where we keep y and z as arbitrary.

Base case. (where $x = \epsilon$). We must show that if $\texttt{-}\ \Phi\ y\ \Psi\ z$ then $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ z$. Observe that the only way for $\texttt{-}\ \Phi\ y\ \Psi\ z$ to hold is if $z = \texttt{-}y$. Therefore we must show that $\epsilon\ \Phi\ \texttt{-}y\ \Psi\ \texttt{-}y$ which is true by rule B.

Inductive case. (From rule I, where $x = \texttt{-}x'$). We have the inductive hypothesis that $\texttt{-}x' \;\Phi\; y' \;\Psi\; z'$ implies $x' \;\Phi\; \texttt{-}y' \;\Psi\; z'$ for any y', z'.
We must show that $\texttt{--}x' \;\Phi\; y \;\Psi\; z$ implies $\texttt{-}x' \;\Phi\; \texttt{-}y \;\Psi\; z$. Observe that the only way that $\texttt{--}x' \;\Phi\; y \;\Psi\; z$ could hold is if $z = \texttt{-}k$ for some k, then by the rule I' which we have already shown to be admissible, we know that $\texttt{-}x' \;\Phi\; y \;\Psi\; k$. Using our induction hypothesis where $y' = y$ and $z' = k$, we can establish that $x' \;\Phi\; \texttt{-}y \;\Psi\; k$, and therefore by rule I we can finally conclude that $\texttt{-}x' \;\Phi\; \texttt{-}y \;\Psi\; \texttt{-}k$ as required. ☐

(e) [★★] Show that $x \;\Phi\; y \;\Psi\; z$ implies $y \;\Phi\; x \;\Psi\; z$.

Solution: We show this by rule induction on the premise with the rules of $\Phi\Psi$.

Base case. (From rule B, where $\epsilon \;\Phi\; y \;\Psi\; y$). We must show that $y \;\Phi\; \epsilon \;\Psi\; y$. We proved this, most fortunately, above in part (c).

Inductive case. (From rule I, where $\texttt{-}x' \;\Phi\; y \;\Psi\; -z'$). We have the inductive hypothesis that

$$y \;\Phi\; x' \;\Psi\; z'$$

We must show that $y \;\Phi\; \texttt{-}x' \;\Psi\; \texttt{-}z'$.

$$\dfrac{\dfrac{\dfrac{}{y \;\Phi\; x' \;\Psi\; z'}\,I.H}{\texttt{-}y \;\Phi\; x' \;\Psi\; \texttt{-}z'}\,I}{y \;\Phi\; \texttt{-}x' \;\Psi\; \texttt{-}z'}\,(d)$$

Thus we have shown by induction that $x \;\Phi\; y \;\Psi\; z$ implies $y \;\Phi\; x \;\Psi\; z$. ☐

(f) [★★] Have you figured out what the $\Phi\Psi$ system actually is? Prove that if $\texttt{-}^x \;\Phi\; \texttt{-}^y \;\Psi\; \texttt{-}^z$, then $z = \texttt{-}^{x+y}$ (where $\texttt{-}^x$ is a hyphen string of length x).

Solution: We proceed by rule induction on the premise.

Base case. (From rule B, where $\texttt{-}^0 \;\Phi\; \texttt{-}^y \;\Psi\; \texttt{-}^y$), we must show that $0 + y = y$, which holds trivially.
Inductive case (From rule I, where $\texttt{-}^{x'+1} \;\Phi\; \texttt{-}^y \;\Psi\; \texttt{-}^{z'+1}$), we have the inductive hypothesis that $z' = x' + y$. We must show that $z' + 1 = (x' + 1) + y$, or, equivalently, that $z' = x' + y$, which is just our I.H.
Thus we have shown by rule induction that the $\Phi\Psi$ system is in fact unary addition. ☐

3. **Ambiguity and Simultaneity**: Here is a simple grammar for a functional programming language [1]:

$$\dfrac{x \in \mathbb{N}}{x \; Expr}\,\textsc{Var.} \qquad \dfrac{e_1 \; Expr \quad e_2 \; Expr}{e_1 e_2 \; Expr}\,\textsc{Appl.} \qquad \dfrac{e \; Expr}{\lambda e \; Expr}\,\textsc{Abst.} \qquad \dfrac{e \; Expr}{(e) \; Expr}\,\textsc{Paren.}$$

(a) [★] Is this grammar ambiguous? If not, explain why not. If so, give an example of an expression that has multiple parse trees.

Solution: Yes, the expression 1 2 3 could be parsed two different ways, i.e:

$$\dfrac{\dfrac{\dfrac{}{1 \; Expr}\,\textsc{Var.} \quad \dfrac{}{2 \; Expr}\,\textsc{Var.}}{1\;2 \; Expr}\,\textsc{Appl.} \quad \dfrac{}{3 \; Expr}\,\textsc{Var.}}{1\;2\;3 \; Expr}\,\textsc{Appl.}$$

Or:

$$\dfrac{\dfrac{}{1 \; Expr}\,\textsc{Var.} \quad \dfrac{\dfrac{}{2 \; Expr}\,\textsc{Var.} \quad \dfrac{}{3 \; Expr}\,\textsc{Var.}}{2\;3 \; Expr}\,\textsc{Appl.}}{1\;2\;3 \; Expr}\,\textsc{Appl.}$$

[1] if you're interested, it's called *lambda calculus*, with *de Bruijn indices* syntax, not that it's relevant to the question!

(b) [⋆⋆] Develop a new (unambiguous) grammar that encodes the left associativity of application, that is 1 2 3 4 should be parsed as ((1 2) 3) 4 (modulo parentheses). Furthermore, lambda expressions should extend as far as possible, i.e $\lambda 1\ 2$ is equivalent to $\lambda(1\ 2)$ not $(\lambda 1)2$.

Solution:

$$\frac{x \in \mathbb{N}}{x\ \mathit{AExpr}}\text{AVar.} \quad \frac{e_1\ \mathit{PExpr} \quad e_2\ \mathit{AExpr}}{e_1 e_2\ \mathit{PExpr}}\text{AAppl.} \quad \frac{e\ \mathit{LExpr}}{\lambda e\ \mathit{LExpr}}\text{AAbs.}$$

$$\frac{e\ \mathit{LExpr}}{(e)\ \mathit{AExpr}}\text{AParen.}y \quad \frac{e\ \mathit{PExpr}}{e\ \mathit{LExpr}}\text{Shunt}_1 \quad \frac{e\ \mathit{AExpr}}{e\ \mathit{PExpr}}\text{Shunt}_2$$

(c) [⋆⋆⋆] Prove that all expressions in your grammar are representable in *Expr*, that is, that your grammar describes only strings that are in *Expr*.

Solution: We shall prove the following simultaneously:

- $x\ \mathit{LExpr} \Rightarrow x\ \mathit{Expr}$
- $x\ \mathit{PExpr} \Rightarrow x\ \mathit{Expr}$
- $x\ \mathit{AExpr} \Rightarrow x\ \mathit{Expr}$

Proof. Base case (From rule AVar., where $x\ \mathit{AExpr}$ for some $x \in \mathbb{N}$). We must show $x\ \mathit{Expr}$, trivial by rule Var.

Inductive case. (From rule AAppl., where $e_1 e_2\ \mathit{PExpr}$). We know $e_1\ \mathit{PExpr}$, and $e_2\ \mathit{AExpr}$ which give rise to inductive hypotheses $e_1\ \mathit{Expr}$ (I.H_1) and $e_2\ \mathit{Expr}$ (I.H_2). We must show that $e_1 e_2\ \mathit{Expr}$.

$$\frac{\dfrac{}{e_1\ \mathit{Expr}}IH_1 \quad \dfrac{}{e_2\ \mathit{Expr}}IH_2}{e_1 e_2\ \mathit{Expr}}$$

Inductive case. (From rule AAbs., where $\lambda x\ \mathit{LExpr}$). We know that $x\ \mathit{LExpr}$, giving inductive hypothesis $x\ \mathit{Expr}$. The rule Abs. then proves our goal: $\lambda x\ \mathit{Expr}$.

Inductive case. (From rule AParen., where $(x)\ \mathit{AExpr}$). We know that $x\ \mathit{LExpr}$, giving inductive hypothesis $x\ \mathit{Expr}$. Then by rule Paren. we show our goal $(x)\ \mathit{Expr}$.

The inductive case for the rules Shunt_1 and Shunt_2 are trivial as they do not alter the expression.

Thus, by induction, $s\ \mathit{LExpr} \vee s\ \mathit{PExpr} \vee s\ \mathit{AExpr} \implies s\ \mathit{Expr}$. We can state this more succinctly thanks to the Shunt rules as $s\ \mathit{LExpr} \implies s\ \mathit{Expr}$.

□