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## Where we're at

- **Syntax Foundations** ✓

Concrete/Abstract Syntax, Ambiguity, HOAS, Binding, Variables, Substitution

- **Semantics Foundations** ✓

Static Semantics, Dynamic Semantics (Small-Step/Big-Step), Abstract Machines, Environments (**Assignment 1**)

- **Features**

- Algebraic Data Types ✓

- **Polymorphism**

- Polymorphic Type Inference (**Assignment 2**)

- Linear Types

- Overloading

- Subtyping

- Modules

- Concurrency

# A Swap Function

Consider the humble `swap` function in Haskell:

$$\begin{aligned} \text{swap} &:: (t_1, t_2) \rightarrow (t_2, t_1) \\ \text{swap } (a, b) &= (b, a) \end{aligned}$$

In our MinHS with algebraic data types from last lecture, we can't define this function.

## Monomorphic

In MinHS, we're stuck copy-pasting our function over and over for every different type we want to use it with:

```
recfun swap1 :: ((Int × Bool) → (Bool × Int))
                p = (snd p, fst p)
```

```
recfun swap2 :: ((Bool × Int) → (Int × Bool))
                p = (snd p, fst p)
```

```
recfun swap3 :: ((Bool × Bool) → (Bool × Bool))
                p = (snd p, fst p)
```

...

This is an acceptable state of affairs for some domain-specific languages, but not for general purpose programming.

## Solutions

We want some way to specify that we **don't care** what the types of the tuple elements are.

$$\text{swap} :: (\forall a\ b. (a \times b) \rightarrow (b \times a))$$

This is called *parametric polymorphism* (or just *polymorphism* in functional programming circles). In Java and some other languages, this is called *generics* and polymorphism refers to something else. Don't be confused.

## How it works

There are two main components to parametric polymorphism:

- 1 *Type abstraction* is the ability to define functions regardless of specific types (like the `swap` example before). In MinHS, we will write using **type** expressions like so: (the literature uses  $\Lambda$ )

*swap* = **type** *a*.

**type** *b*.

**recfun** *swap* :: (*a* × *b*) → (*b* × *a*)

*p* = (snd *p*, fst *p*)

- 2 *Type application* is the ability to *instantiate* polymorphic functions to specific types. In MinHS, we use @ signs.

*swap*@Int@Bool (3, True)

## Analogies

The reason they're called type abstraction and application is that they behave analogously to  $\lambda$ -calculus.

We have a  $\beta$ -reduction principle, but for types:

$$(\mathbf{type} \ a. \ e)@_{\tau} \quad \mapsto_{\beta} \quad (e[a := \tau])$$

### Example (Identity Function)

```

      (type a. recfun f :: (a → a) x = x)@Int 3
  ↪ (recfun f :: (Int → Int) x = x) 3
  ↪ 3

```

This means that **type** expressions can be thought of as **functions** from types to values.

## Type Variables

What is the type of this?

**(type a. recfun f :: (a → a) x = x)**

$\forall a. a \rightarrow a$

Types can mention *type variables* now<sup>1</sup>.

If  $id : \forall a. a \rightarrow a$ , what is the type of  $id@Int$ ?

$(a \rightarrow a)[a := Int] = (Int \rightarrow Int)$

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<sup>1</sup>Technically, they already could with recursive types.



# Typing Rules Sketch

We would like rules that look something like this:

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathbf{type} \ a. e : \forall a. \tau}$$

$$\frac{\Gamma \vdash e : \forall a. \tau}{\Gamma \vdash e@{\rho} : \tau[a := \rho]}$$

But these rules don't account for what **type variables** are available or in scope.

## Type Wellformedness

With variables in the picture, we need to check our types to make sure that they only refer to well-scoped variables.

$$\begin{array}{c}
 \frac{t \text{ bound} \in \Delta}{\Delta \vdash t \text{ ok}} \quad \frac{}{\Delta \vdash \text{Int ok}} \quad \frac{}{\Delta \vdash \text{Bool ok}} \\
 \frac{\Delta \vdash \tau_1 \text{ ok} \quad \Delta \vdash \tau_2 \text{ ok}}{\Delta \vdash \tau_1 \rightarrow \tau_2 \text{ ok}} \quad \frac{\Delta \vdash \tau_1 \text{ ok} \quad \Delta \vdash \tau_2 \text{ ok}}{\Delta \vdash \tau_1 \times \tau_2 \text{ ok}} \\
 \text{(etc.)}
 \end{array}$$

$$\frac{\Delta, a \text{ bound} \vdash \tau \text{ ok}}{\Delta \vdash \forall a. \tau \text{ ok}}$$

## Typing Rules, Properly

We add a **second context** of type variables that are bound.

$$\frac{a \text{ bound}, \Delta; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \text{type } a. e : \forall a. \tau}$$

$$\frac{\Delta; \Gamma \vdash e : \forall a. \tau \quad \Delta \vdash \rho \text{ ok}}{\Delta; \Gamma \vdash e@{\rho} : \tau[a := \rho]}$$

(the other typing rules just pass  $\Delta$  through)

# Dynamic Semantics

First we evaluate the LHS of a type application as much as possible:

$$\frac{e \mapsto_M e'}{e@_{\tau} \mapsto_M e'@_{\tau}}$$

Then we apply our  $\beta$ -reduction principle:

$$\frac{}{(\mathbf{type} \ a. \ e)@_{\tau} \mapsto_M e[a := \tau]}$$

# Curry-Howard

Previously we noted the correspondence between types and logic:

$\times$	$\wedge$
$+$	$\vee$
$\rightarrow$	$\Rightarrow$
<b>1</b>	$\top$
<b>0</b>	$\perp$
$\forall$	$\exists$

## Curry-Howard

The type quantifier  $\forall$  corresponds to a universal quantifier  $\forall$ , but it is **not** the same as the  $\forall$  from first-order logic. What's the difference?

First-order logic quantifiers range over a set of *individuals* or values, for example the natural numbers:

$$\forall x. x + 1 > x$$

These quantifiers range over **propositions** (types) themselves. It is analogous to *second-order logic*, not first-order:

$$\forall A. \forall B. A \wedge B \Rightarrow B \wedge A$$

$$\forall A. \forall B. A \times B \rightarrow B \times A$$

The first-order quantifier has a type-theoretic analogue too (type indices), but this is not nearly as common as polymorphism.

# Generality

If we need a function of type  $\text{Int} \rightarrow \text{Int}$ , a polymorphic function of type  $\forall a. a \rightarrow a$  will do just fine, we can just instantiate the type variable to  $\text{Int}$ . But the reverse is not true. This gives rise to an ordering.

## Generality

A type  $\tau$  is *more general* than a type  $\rho$ , often written  $\rho \sqsubseteq \tau$ , if type variables in  $\tau$  can be instantiated to give the type  $\rho$ .

## Example (Functions)

$$\text{Int} \rightarrow \text{Int} \quad \sqsubseteq \quad \forall z. z \rightarrow z \quad \sqsubseteq \quad \forall x y. x \rightarrow y \quad \sqsubseteq \quad \forall a. a$$

# Implementation Strategies

Our simple dynamic semantics belies a complex implementation headache.

While we can easily define functions that operate uniformly on multiple types, when this is compiled to machine code the results may differ depending on the **size** of the type in question.

There are two main approaches to solve this problem.



## Template Instantiation

### Key Idea

Automatically generate a monomorphic copy of each polymorphic functions based on the types applied to it.

For example, if we defined our polymorphic swap function:

$$\begin{aligned} \text{swap} &= \text{type } a. \text{ type } b. \\ &\quad \text{recfun } \text{swap} :: (a \times b) \rightarrow (b \times a) \\ &\quad \quad p = (\text{snd } p, \text{fst } p) \end{aligned}$$

Then a type application like `swap@Int@Bool` would be replaced **statically** by the compiler with the monomorphic version:

$$\begin{aligned} \text{swap}_{\text{IB}} &= \text{recfun } \text{swap} :: (\text{Int} \times \text{Bool}) \rightarrow (\text{Bool} \times \text{Int}) \\ &\quad \quad p = (\text{snd } p, \text{fst } p) \end{aligned}$$

A new copy is made for each unique type application.

## Evaluating Template Instatiation

This approach has a number of advantages:

- 1 Little to no run-time cost
- 2 Simple mental model
- 3 Allows for custom specialisations (e.g. list of booleans into bit-vectors)
- 4 Easy to implement

However the downsides are just as numerous:

- 1 Large binary size if many instantiations are used
- 2 This can lead to long compilation times
- 3 Restricts the type system to **statically** instantiated type variables.

**Languages that use Template Instantiation:** Rust, C++, Cogent, some ML dialects

## Polymorphic Recursion

Consider the following Haskell data type:

```
data Dims a = Step a (Dims [a]) | Epsilon
```

This describes a list of matrices of increasing dimensionality, e.g:

```
Step 1 (Step [1, 2] (Step [[1, 2], [3, 4]] Epsilon)) :: Dims Int
```

We can write a sum function like this:

$$\text{sumDims} :: \forall a. (a \rightarrow \text{Int}) \rightarrow \text{Dims } a \rightarrow \text{Int}$$

$$\text{sumDims } f \text{ Epsilon} = 0$$

$$\text{sumDims } f (\text{Step } a \ t) = (f \ a) + \text{sumDims } (\text{sum } f) \ t$$

How many different instantiations of the type variable  $a$  are there?

**We'd have to run the program to find out.**

# HM Types

Automatically generating a copy for each instantiation is great but can't handle all polymorphic programs.

In practice a statically determined subset can be carved out by **restricting** what sort of programs can be written:

- 1 Only allow  $\forall$  quantifiers on the **outermost** part of a type declaration (not inside functions or type constructors).
- 2 Recursive functions **cannot** call themselves with different type parameters.

This restriction is sometimes called *Hindley-Milner* polymorphism. This is also the subset for which *type inference* is both complete and tractable.

## Boxing

An alternative to our copy-paste-heavy template instantiation approach is to make **all types** represented the same way. Thus, a polymorphic function only requires one function in the generated code.

Typically this is done by **boxing** each type. That is, all data types are represented as a **pointer** to a data structure on the heap. If everything is a pointer, then all values use exactly 32 (or 64) bits of stack space.

The extra indirection has a run-time penalty, and it can make garbage collection more necessary, but it results in smaller binaries and unrestricted polymorphism.

**Languages that use boxing:** Haskell, Java, C#, OCaml

# Constraining Implementations

How many possible implementations are there of a function of the following type?

$$\text{Int} \rightarrow \text{Int}$$

How about this type?

$$\forall a. a \rightarrow a$$

Polymorphic type signatures constrain implementations.

# Parametricity

## Definition

The principle of **parametricity** states that the result of polymorphic functions cannot depend on **values** of an abstracted type.

More formally, suppose I have a polymorphic function  $g$  that takes a type parameter. If run any arbitrary function  $f : \tau \rightarrow \tau$  on some values of type  $\tau$ , then run the function  $g@_{\tau}$  on the result, that will give the same results as running  $g@_{\tau}$  first, then  $f$ .

## Example

$$foo :: \forall a. [a] \rightarrow [a]$$

We know that **every** element of the output occurs in the input. The parametricity theorem we get is, for all  $f$ :

$$foo \circ (map f) = (map f) \circ foo$$

## More Examples

$$\text{head} :: \forall a. [a] \rightarrow a$$

What's the parametricity theorems?

### Example (Answer)

For any  $f$ :

$$f (\text{head } \ell) = \text{head } (\text{map } f \ell)$$



## More Examples

$$(++) :: \forall a. [a] \rightarrow [a] \rightarrow [a]$$

What's the parametricity theorem?

### Example (Answer)

$$\text{map } f (a ++ b) = \text{map } f a ++ \text{map } f b$$

## More Examples

$concat :: \forall a. [[a]] \rightarrow [a]$

What's the parametricity theorem?

### Example (Answer)

$map\ f\ (concat\ ls) = concat\ (map\ (map\ f)\ ls)$

# Higher Order Functions

$$\text{filter} :: \forall a. (a \rightarrow \text{Bool}) \rightarrow [a] \rightarrow [a]$$

What's the parametricity theorem?

## Example (Answer)

$$\text{filter } p (\text{map } f \text{ } l\text{s}) = \text{map } f (\text{filter } (p \circ f) \text{ } l\text{s})$$

# Parametricity Theorems

Follow a similar structure. In fact it can be mechanically derived, using the *relational parametricity* framework invented by John C. Reynolds, and popularised by Wadler in the famous paper, “Theorems for Free!”<sup>2</sup>.

Upshot: We can ask `lambdabot` on the Haskell IRC channel for these theorems.

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<sup>2</sup><https://people.mpi-sws.org/~dreyer/tor/papers/wadler.pdf>