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## Motivation

Suppose we added `Float` to `MinHS`.

Ideally, the same functions should be able to work on both `Int` and `Float`.

`4 + 6 :: Int`

`4.3 + 5.1 :: Float`

Similarly, a numeric literal should take on whatever type is inferred from context.

`(5 :: Int) mod 3`

`sin(5 :: Float)`

## Without Overloading

We effectively have two functions:

$$(+_{\text{Int}}) :: \text{Int} \rightarrow \text{Int} \rightarrow \text{Int}$$
$$(+_{\text{Float}}) :: \text{Float} \rightarrow \text{Float} \rightarrow \text{Float}$$

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We would like to refer to both of these functions by the **same name** and have the specific implementation chosen based on the **type**.

Such type-directed name resolution is called *ad-hoc polymorphism* or *overloading*.

# Type Classes

Type classes are a common approach to ad-hoc polymorphism, and exist in various languages under different names:

- Type Classes in Haskell
- Traits in Rust
- Implicits in Scala
- Protocols in Swift
- Contracts in Go 2
- Concepts in C++
- Other languages approximate with *subtype polymorphism* (coming)

# Type Classes

A *type class* is a *set of types* for which implementations (*instances*) have been provided for various functions, called *methods*<sup>1</sup>.

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## Example (Numeric Types)

In Haskell, the types `Int`, `Float`, `Double` etc. are all instances of the type class `Num`, which has methods such as `(+)`, `negate`, etc.

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## Example (Numeric Types)

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## Example (Equality)

In Haskell, the `Eq` type class contains methods `(==)` and `(/=)` for computable equality. What types cannot be an instance of `Eq`?

---

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# Notation

We write:

$$f :: \forall a. P \Rightarrow \tau$$

To indicate that  $f$  has the type  $\tau$  where  $a$  can be instantiated to any type **under the condition** that the constraint  $P$  is satisfied.

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## Example

- $(+) :: \forall a. (\text{Num } a) \Rightarrow a \rightarrow a \rightarrow a$
- $(==) :: \forall a. (\text{Eq } a) \Rightarrow a \rightarrow a \rightarrow \text{Bool}$

Is  $(1 :: \text{Int}) + 4.4$  a well-typed expression?

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Is  $(1 :: \text{Int}) + 4.4$  a well-typed expression?

**No.** The type of  $(+)$  requires its arguments to have the same type.

# Extending MinHS

Extending implicitly typed MinHS with type classes:

Predicates  $P ::= C \tau$   
 Polytypes  $\pi ::= \tau \mid \forall a. \pi \mid P \Rightarrow \pi$   
 Monotypes  $\tau ::= \text{Int} \mid \text{Bool} \mid \tau + \tau \mid \dots$   
 Class names  $C$

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 Class names  $C$

Our typing judgement  $\Gamma \vdash e : \pi$  now includes a set of type class **axiom schema**:

$$A \mid \Gamma \vdash e : \pi$$

This set contains predicates for all type class instances known to the compiler.

# Typing Rules

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In order to use an overloaded type one must first show that the predicate is **satisfied** by the known axioms:

$$\frac{\mathcal{A} \mid \Gamma \vdash e : P \Rightarrow \pi \quad \mathcal{A} \Vdash P}{e : \pi} \text{INST}$$

Right now,  $\mathcal{A} \Vdash P$  iff  $P \in \mathcal{A}$ , but we will complicate this situation later.



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Right now,  $\mathcal{A} \Vdash P$  iff  $P \in \mathcal{A}$ , but we will complicate this situation later.

If, adding a predicate to the known axioms, we can conclude a typing judgement, then we can overload the expression with that predicate:

$$\frac{P, \mathcal{A} \mid \Gamma \vdash e : \pi}{\mathcal{A} \mid \Gamma \vdash e : P \Rightarrow \pi} \text{GEN}$$

## Example

Suppose we wanted to show that  $3.2 + 4.4 :: \text{Float}$ .

- 1  $(+) :: \forall a. (\text{Num } a) \Rightarrow a \rightarrow a \rightarrow a \in \Gamma$ .
- 2  $\text{Num Float} \in \mathcal{A}$ .

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- 3 Using ALLE (from previous lecture), we can conclude  $(+) :: (\text{Num Float}) \Rightarrow \text{Float} \rightarrow \text{Float} \rightarrow \text{Float}$ .

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- 4 Using INST (on previous slide) and 2, we can conclude  $(+) :: \text{Float} \rightarrow \text{Float} \rightarrow \text{Float}$

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- 4 Using INST (on previous slide) and 2, we can conclude  $(+) :: \text{Float} \rightarrow \text{Float} \rightarrow \text{Float}$
- 5 By the function application rule, we can conclude  $3.2 + 4.4 :: \text{Float}$  as required.

## Dictionaries and Resolution

This is called *ad-hoc polymorphism* because the type checker removes it — it is not a fundamental language feature, but merely a naming convenience.

The type checker will convert ad-hoc polymorphism to parametric polymorphism.

Type classes are converted to types:

```
class Eq a where  
  (==) : a → a → Bool  
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```

becomes

```
type EqDict a = (a → a → Bool × a → a → Bool)
```

A *dictionary* contains all the method implementations of a type class for a specific type.

## Dictionaries and Resolution

Instances become **values** of the dictionary type:

```
instance Eq Bool where
  True  == True  = True
  False == False = True
  _     == _     = False
  a /= b = not (a == b)
```



## Dictionaries and Resolution

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instance Eq Bool where
  True  == True  = True
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becomes

```
True  ==_Bool True  = True
False ==_Bool False = True
  _    ==_Bool  _    = False
a /=_Bool b = not (a ==_Bool b)

eqBoolDict = ((==_Bool), (/=_Bool))
```

## Dictionaries and Resolution

Programs that rely on overloading now take dictionaries as parameters:

$$\text{same} :: \forall a. (\text{Eq } a) \Rightarrow [a] \rightarrow \text{Bool}$$
$$\text{same } [] = \text{True}$$
$$\text{same } (x : []) = \text{True}$$
$$\text{same } (x : y : xs) = x == y \wedge \text{same } (y : xs)$$

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Becomes:

$$\begin{aligned} \text{same} &:: \forall a. (\text{EqDict } a) \rightarrow [a] \rightarrow \text{Bool} \\ \text{same eq } [] &= \text{True} \\ \text{same eq } (x : []) &= \text{True} \\ \text{same eq } (x : y : xs) &= (\text{fst eq}) x y \wedge \text{same eq } (y : xs) \end{aligned}$$

## Generative Instances

We can make **instances** also predicated on some constraints:

```
instance (Eq a) => (Eq [a]) where
  []      == []      = True
  (x : xs) == (y : ys) = x == y & (xs == ys)
  _       == _       = False
  a /= b  = not (a == b)
```

Such instances are transformed into **functions**:

$$eqList :: EqDict\ a \to EqDict\ [a]$$

Our set of axiom schema  $\mathcal{A}$  now includes **implications**, like  $(Eq\ a) \Rightarrow (Eq\ [a])$ . This makes the relation  $\mathcal{A} \Vdash P$  much more complex to solve.

# Coherence

Some languages (such as Haskell and Rust) insist that there is only **one instance per class per type** in the entire program. It achieves this by requiring that all instances are either:

- Defined along with the definition of the type class, or
- Defined along with the definition of the type.

This rules out so-called *orphan* instances.

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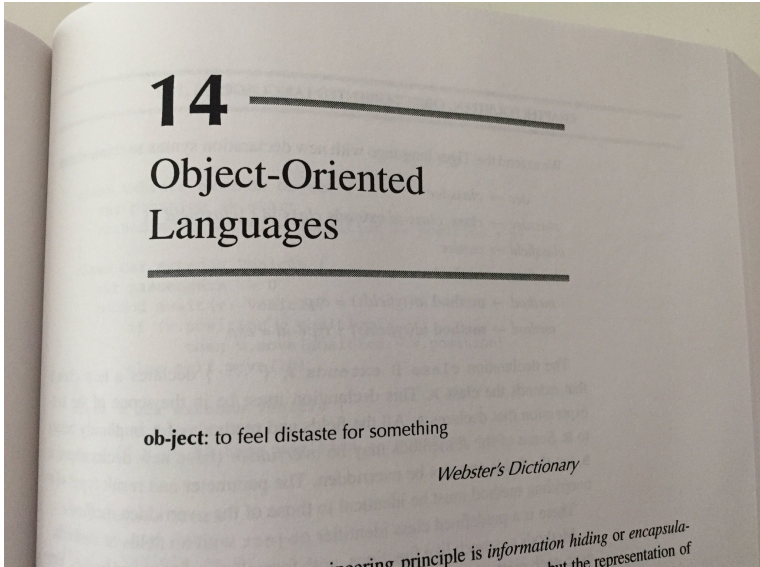
There are a number of trade-offs with this decision:

- Modularity has been compromised but,
- Types like `Data.Set` can exploit this coherence to enforce invariants.

## Static Dispatch

Typically, the compiler can *inline* all dictionaries to their usage sites, thus eliminating all run-time cost for using type classes. This is only not possible if the exact type being used cannot be determined at compile-time, such as with *polymorphic recursion* etc.

# Subtyping





## Subtyping

To add subtyping to a language, we define a *partial order*<sup>2</sup> on types  $\tau \leq \rho$  and a *rule of subsumption*:

$$\frac{\Gamma \vdash e : \tau \quad \tau \leq \rho}{\Gamma \vdash e : \rho}$$

To aid in type inference (as type inference with subtyping is undecidable in general), sometimes subsumptions (called *upcasts*) are made explicit (e.g. in OCaml):

$$\frac{\Gamma \vdash e : \tau \quad \tau \leq \rho}{\Gamma \vdash \text{upcast } \rho \ e : \rho}$$

---

<sup>2</sup>Remember discrete maths, or check the glossary.

# What is Subtyping?

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- **Most common:** where upcasts do not have dynamic behaviour, i.e. `upcast v ↦ v`. This requires that any **value** of type  $\tau$  could also be judged to have type  $\rho$ . If types are viewed as sets, this could be viewed as a **subset** relation.

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- **Uncommon**: where upcasts cause a **coercion** to occur, actually converting the value from  $\tau$  to  $\rho$  at runtime.

**Observation**: By using an identity function as a coercion, the coercion view is **more general**.

## Desirable Properties

The coercion approach is the most general, but we might have some confusing results.

### Example

Suppose  $\text{Int} \leq \text{Float}$ ,  $\text{Float} \leq \text{String}$  and  $\text{Int} \leq \text{String}$ .  
There are now two ways to coerce an `Int` to a `String`:

- 1 Directly: "3"
- 2 via `Float`: "3.0"

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There are now two ways to coerce an `Int` to a `String`:

- 1 Directly: "3"
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Typically, we would enforce that the subtype coercions are **coherent**, such that no matter which coercion is chosen, the same result is produced.

## Behavioural Subtyping

Another constraint is that the **syntactic** notion of subtyping should correspond to something **semantically**. In other words, if we know  $\tau \leq \rho$ , then it should be reasonable to replace any value of type  $\rho$  with an value of type  $\tau$  without any observable difference.

### Liskov Substitution Principle

Let  $\varphi(x)$  be a property provable about objects  $x$  of type  $\rho$ . Then  $\varphi(y)$  should be true for objects  $y$  of type  $\tau$  where  $\tau \leq \rho$ .

Languages such as Java and C++, which allow for user-defined subtyping relationships (**inheritance**), put the onus on the user to ensure this condition is met.

## Product Types

Assuming a basic rule  $\text{Int} \leq \text{Float}$ , how do we define subtyping for our compound data types?

What is the relationship between these types?

- $(\text{Int} \times \text{Int})$
- $(\text{Float} \times \text{Float})$
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- $(\text{Int} \times \text{Float})$

$$\frac{\tau_1 \leq \rho_1 \quad \tau_2 \leq \rho_2}{(\tau_1 \times \tau_2) \leq (\rho_1 \times \rho_2)}$$

# Sum Types

What is the relationship between these types?

- $(\text{Int} + \text{Int})$
- $(\text{Float} + \text{Float})$
- $(\text{Float} + \text{Int})$
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# Sum Types

What is the relationship between these types?

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- (Float + Float)
- (Float + Int)
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Any other **compound** types?

# Functions

What is the relationship between these types?

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- $(\text{Float} \rightarrow \text{Float})$
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$$\frac{\rho_1 \leq \tau_1 \quad \tau_2 \leq \rho_2}{(\tau_1 \rightarrow \tau_2) \leq (\rho_1 \rightarrow \rho_2)}$$

# Variance

The way a *type constructor* (such as  $+$ ,  $\times$ , Maybe or  $\rightarrow$ ) interacts with subtyping is called its *variance*.



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**Examples:** Products (both arguments), Sums (both arguments), Function return type, ...

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**Examples:** Products (both arguments), Sums (both arguments), Function return type, ...
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**Examples:** Function argument type, ...
- If it is neither *covariant* nor *contravariant* then it is (confusingly) called *invariant*.  
**Examples:** `data Endo a = E (a → a)`

## Stuffing it up

Many languages have famously stuffed this up, at the expense of type safety.

### 19 Types

Dart supports optional typing based on interface types.

*The type system is unsound, due to the covariance of generic types. This is a deliberate choice (and undoubtedly controversial). Experience has shown that sound type rules for generics fly in the face of programmer intuition. It is easy for tools to provide a sound type analysis if they choose, which may be useful for tasks like refactoring.*

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A few years later...

## Language and libraries

- Dart's type system is now sound.
  - Fixing common type problems

# Java too

Java (and its Seattle-based cousin, C<sup>#</sup>) also broke type safety with incorrect variance in **arrays**.

Liam will demonstrate how this violates **preservation**, time permitting.