Syntax

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All the syntax we have seen so far is *concrete syntax*. Concrete syntax is described by judgements on *strings*.
Abstract Syntax

Working with concrete syntax directly is *unsuitable* for both compiler implementation and proofs. Consider:

- $3 + (4 \times 5)$
- $3 + 4 \times 5$
- $(3 + (4 \times 5))$

\(^1\)“There is more than one way to do it”. 
Abstract Syntax

Working with concrete syntax directly is *unsuitable* for both compiler implementation and proofs. Consider:

- $3 + (4 \times 5)$
- $3 + 4 \times 5$
- $(3 + (4 \times 5))$

**TIMTOWTDI**\(^1\) makes life harder for us. Different derivations represent the same semantic program. We would like a representation of programs that is as simple as possible, removing any extraneous information. Such a representation is called *abstract syntax*.

\(^1\)“There is more than one way to do it”.
Typically, the *abstract syntax* of a program is represented as a *tree* rather than as a string.

\[(3 + (4 \times 5)) \leftrightarrow 3 + 4 \times 5\]

Writing trees in our inference rules would become unwieldy. We shall define a *term* language in which to express trees.
In this course, a term is a structure that can either be a symbol, like Plus or Times or 3; or a compound, which consists of a symbol followed by one or more argument subterms, all in parentheses.

\[ t ::= \text{Symbol} | (\text{Symbol} \; t_1 \; t_2 \; \ldots) \]

These particular terms are also known as s-expressions. Terms can equivalently be thought of a subset of Haskell where the only kinds of expressions allowed are literals and data constructors.
Armed with an appropriate Haskell data declaration, this can be implemented straightforwardly:

```
data Exp = Plus Exp Exp  
           | Times Exp Exp  
           | Num Int
```
Concrete Syntax

\[
\begin{align*}
  i & \in \mathbb{Z} & a & \text{SExp} & e & \text{Atom} & e & \text{PExp} \\
  i & \text{Atom} & (a) & \text{Atom} & e & \text{PExp} & e & \text{SExp} \\
  a & \text{Atom} & b & \text{PExp} & a & \text{PExp} & b & \text{SExp} \\
  a \times b & \text{PExp} & a + b & \text{SExp}
\end{align*}
\]

Abstract Syntax

\[
\begin{align*}
  i & \in \mathbb{Z} & a & \text{AST} & b & \text{AST} & a & \text{AST} & b & \text{AST} \\
  (\text{Num } i) & \text{AST} & (\text{Plus } a \ b) & \text{AST} & (\text{Times } a \ b) & \text{AST}
\end{align*}
\]

Now we have to specify a \textit{relation} to connect the two!
Relations

Up until now, most judgements we have used have been unary — corresponding to a set of satisfying objects. A judgement can also express a relationship between two objects (a binary judgement) or a number of objects (an $n$-ary judgement).

Example (Relations)

- $4 \textit{ divides } 16$ (binary)
- mail is an anagram of liam (binary)
- $3 \textit{ plus } 5 \textit{ equals } 8$ (ternary)

$n$-ary judgements where $n \geq 2$ are sometimes called $\textit{relations}$, and correspond to an $n$-tuple of satisfying objects.
Abstract Syntax
Parsing
Bindings
First Order Abstract Syntax
Higher Order Abstract Syntax

Parsing Relation

3 + (4 × 5)

3 + 4 × 5

(3 + (4 × 5))

(Plus (Num 3) (Times (Num 4) (Num 5)))

\[ i \in \mathbb{Z} \]

\[ i \text{ Atom} \]

\[ a \text{ Atom} \]
\[ b \text{ PExp} \]

\[ a \times b \text{ PExp} \]

\[ a \text{ PExp} \]
\[ b \text{ SExp} \]

\[ a + b \text{ SExp} \]

\[ e \text{ SExp} \]

\[ (e) \text{ Atom} \]

\[ e \text{ Atom} \]

\[ e \text{ PExp} \]

\[ e \text{ SExp} \]
Parsing Relation

\[ 3 + (4 \times 5) \]

\[ 3 + 4 \times 5 \]

\[ (3 + (4 \times 5)) \]

\[ (\text{Plus } (\text{Num } 3) (\text{Times } (\text{Num } 4) (\text{Num } 5))) \]

\[ i \in \mathbb{Z} \]

\[ i \text{ Atom } \leftrightarrow (\text{Num } i) \text{ AST} \]

\[ \begin{array}{c}
    a \text{ Atom} \\
    b \text{ PExp}
\end{array} \quad \frac{}{a \times b \text{ PExp}} \]

\[ \begin{array}{c}
    a \text{ PExp} \\
    b \text{ SExp}
\end{array} \quad \frac{}{a + b \text{ SExp}} \]

\[ \begin{array}{c}
    e \text{ SExp}
\end{array} \quad \frac{}{(e) \text{ Atom}} \]

\[ \begin{array}{c}
    e \text{ PExp}
\end{array} \quad \frac{}{e \text{ ATom}} \]

\[ \begin{array}{c}
    e \text{ SExp}
\end{array} \quad \frac{}{e \text{ PExp}} \]

\[ \begin{array}{c}
    e \text{ PExp}
\end{array} \quad \frac{}{e \text{ SExp}} \]
Parsing Relation

\[ 3 + (4 \times 5) \]
\[ 3 + 4 \times 5 \]
\[ (3 + (4 \times 5)) \]

\[
\begin{align*}
  i \in \mathbb{Z} \\
  i \text{ Atom} & \iff (\text{Num } i) \text{ AST} \\
  a \text{ Atom} & \iff a' \text{ AST} \\
  b \text{ PExp} & \iff b' \text{ AST} \\
  a \times b \text{ PExp} \\
  a \text{ PExp} & \iff a' \text{ AST} \\
  b \text{ SExp} & \iff b' \text{ AST} \\
  a + b \text{ SExp} \\
  e \text{ SExp} & \iff (e) \text{ Atom} \\
  e \text{ Atom} & \iff e \text{ PExp} \\
  e \text{ PExp} & \iff e \text{ SExp}
\end{align*}
\]
Parsing Relation

\[
3 + (4 \times 5) \\
3 + 4 \times 5 \\
(3 + (4 \times 5))
\]

\[
i \in \mathbb{Z}
\]

\[
i \text{ Atom} \leftrightarrow (\text{Num } i) \text{ AST}
\]

\[
a \text{ Atom} \leftrightarrow a' \text{ AST} \quad b \text{ PExp} \leftrightarrow b' \text{ AST}
\]

\[
a \times b \text{ PExp} \leftrightarrow (\text{Times } a' b') \text{ AST}
\]

\[
a \text{ PExp} \quad b \text{ SExp}
\]

\[
a + b \text{ SExp}
\]

\[
e \text{ SExp} \quad e \text{ Atom} \quad e \text{ PExp}
\]

\[
(e) \text{ Atom} \quad e \text{ PExp} \quad e \text{ SExp}
\]
### Parsing Relation

3 + (4 × 5)

3 + 4 × 5

(3 + (4 × 5))

\[ i \in \mathbb{Z} \]

\[ i \text{ Atom} \leftrightarrow (\text{Num } i) \text{ AST} \]

\[ a \text{ Atom} \leftrightarrow a' \text{ AST} \quad b \text{ PExp} \leftrightarrow b' \text{ AST} \]

\[ a \times b \text{ PExp} \leftrightarrow (\text{Times } a' \ b') \text{ AST} \]

\[ a \text{ PExp} \leftrightarrow a' \text{ AST} \quad b \text{ SExp} \leftrightarrow b' \text{ AST} \]

\[ a + b \text{ SExp} \leftrightarrow (\text{Plus } a' \ b') \text{ AST} \]

\[ e \text{ SExp} \]

\[ (e) \text{ Atom} \]

\[ e \text{ PExp} \]

\[ e \text{ SExp} \]
Parsing Relation

\[
3 + (4 \times 5) \\
3 + 4 \times 5 \quad \rightarrow \quad (\text{Plus} (\text{Num} \ 3) (\text{Times} (\text{Num} \ 4) (\text{Num} \ 5))) \\
(3 + (4 \times 5))
\]

\[
i \in \mathbb{Z} \\
i \ \text{Atom} \quad \leftrightarrow \quad (\text{Num} \ i) \ \text{AST} \\
a \ \text{Atom} \quad \leftrightarrow \quad a' \ \text{AST} \quad b \ \text{PExp} \quad \leftrightarrow \quad b' \ \text{AST} \quad \\
\quad \quad \rightarrow \quad (\text{Times} a' b') \ \text{AST} \\
a \ \text{PExp} \quad \leftrightarrow \quad a' \ \text{AST} \quad b \ \text{SExp} \quad \leftrightarrow \quad b' \ \text{AST} \quad \\
\quad \quad \rightarrow \quad (\text{Plus} a' b') \ \text{AST} \\
e \ \text{SExp} \quad \leftrightarrow \quad a' \ \text{AST} \quad e \ \text{Atom} \quad \rightarrow \quad e \ \text{PExp} \quad \\
(e) \ \text{Atom} \quad \leftrightarrow \quad a' \ \text{AST} \quad e \ \text{PExp} \quad \rightarrow \quad e \ \text{SExp}
\]
Parsing Relation

\[
3 + (4 \times 5)
\]

\[
3 + 4 \times 5
\]

\[
(3 + (4 \times 5))
\]

\[
i \in \mathbb{Z}
\]

\[
i \text{ Atom} \leftrightarrow (\text{Num } i) \text{ AST}
\]

\[
a \text{ Atom} \leftrightarrow a' \text{ AST} \quad b \text{ PExp} \leftrightarrow b' \text{ AST}
\]

\[
a \times b \text{ PExp} \leftrightarrow (\text{Times } a' b') \text{ AST}
\]

\[
a \text{ PExp} \leftrightarrow a' \text{ AST} \quad b \text{ SExp} \leftrightarrow b' \text{ AST}
\]

\[
a + b \text{ SExp} \leftrightarrow (\text{Plus } a' b') \text{ AST}
\]

\[
e \text{ SExp} \leftrightarrow a' \text{ AST}
\]

\[
(e) \text{ Atom} \leftrightarrow a' \text{ AST}
\]

\[
e \text{ Atom} \leftrightarrow a \text{ AST}
\]

\[
e \text{ PExp} \leftrightarrow a \text{ AST}
\]

\[
e \text{ SExp} \leftrightarrow a \text{ AST}
\]
## Relations as Algorithms

The *parsing relation* $\leftrightarrow$ is an extension of our existing concrete syntax rules. Therefore it is **unambiguous**, just as those rules are. Furthermore, the abstract syntax can be **unambiguously** determined **solely** by looking at the left hand side of $\leftrightarrow$. 

### An Algorithm

To determine the term corresponding to a particular string:

1. Derive the left hand side of the $\leftrightarrow$ (the concrete syntax) bottom-up until reaching axioms.
2. Fill in the right hand side of the $\leftrightarrow$ (the abstract syntax) top-down, starting at the axioms.

This process is called **parsing**.
Relations as Algorithms

The *parsing relation* $\iff$ is an extension of our existing concrete syntax rules. Therefore it is unambiguous, just as those rules are. Furthermore, the abstract syntax can be unambiguously determined solely by looking at the left hand side of $\iff$.

**An Algorithm**

To determine the term corresponding to a particular string:

1. Derive the left hand side of the $\iff$ (the concrete syntax) bottom-up until reaching axioms.
2. Fill in the right hand side of the $\iff$ (the abstract syntax) top-down, starting at the axioms.

This process is called *parsing*. 
Example

Rules

\[
\begin{align*}
i \in \mathbb{Z} & \quad \Rightarrow \quad i A \\
a S & \quad \Rightarrow \quad (a) A \\
e A & \quad \Rightarrow \quad e P \\
e P & \quad \Rightarrow \quad e S \\
a A & \quad b P & \quad a P & \quad b S \\
\Rightarrow \quad a \times b P & \quad \Rightarrow \quad a + b S & \quad \Rightarrow \quad 1 + 2 \times 3 S
\end{align*}
\]
Example

Rules

\[
\begin{align*}
    i \in \mathbb{Z} & \\
    i A & \\
    a S & \\
    (a) A & \\
    e A & \\
    e P & \\
    e P & \\
    e S & \\
    a A & \\
    b P & \\
    a \times b P & \\
    a P & \\
    b S & \\
    a + b S & \\
\end{align*}
\]

\[
\begin{align*}
    1 P & \\
    2 \times 3 S & \\
    1 + 2 \times 3 S & 
\end{align*}
\]
Example

Rules

\[
\begin{align*}
  i & \in \mathbb{Z} \\
  i A & \Rightarrow (a) A \\
  a S & \Rightarrow e A \\
  b P & \Rightarrow e P \\
  a \times b P & \Rightarrow e S \\
  a P & \Rightarrow a + b S \\
  b S & \Rightarrow a \times b P \\
\end{align*}
\]

\[
\begin{align*}
  1 A & \Rightarrow 2 \times 3 S \\
  1 P & \Rightarrow 1 + 2 \times 3 S \\
\end{align*}
\]
Example

Rules

\[
\begin{align*}
& i \in \mathbb{Z} \\
& \quad \quad i \ A \\
& a \ S & \quad \quad (a) \ A \\
& e \ A & \quad \quad e \ P \\
& e \ P & \quad \quad e \ S \\
& a \ A & \quad \quad b \ P \\
& a \times b \ P \\
& a \ A & \quad \quad b \ S \\
& a + b \ S
\end{align*}
\]

\[
\begin{align*}
1 \ A & \quad \quad 2 \times 3 \ P \\
1 \ P & \quad \quad 2 \times 3 \ S \\
1 + 2 \times 3 \ S
\end{align*}
\]
Example

Rules

\[
\begin{align*}
  i & \in \mathbb{Z} \\
  a S & \quad (a) A \\
  e A & \quad e P \\
  e P & \quad e S \\
  a A & \quad b P & a A & \quad b P & a P & \quad b S \\
  a \times b & \quad P & a \times b & \quad P & a \times b & \quad P \\
  a + b S & \quad a + b S \\
  3 A & \quad 3 A \\
  3 P & \quad 3 P \\
  2 A & \quad 2 A \\
  2 \times 3 P & \quad 2 \times 3 P \\
  2 \times 3 S & \quad 2 \times 3 S \\
  1 + 2 \times 3 S & \quad 1 + 2 \times 3 S
\end{align*}
\]
Example

Rules

\[
i \in \mathbb{Z} \\
i A \leftrightarrow (\text{Num } i) \\
(a) A \leftrightarrow a' \\
a A \leftrightarrow a' \\
a \times b P \leftrightarrow (\text{Times } a' b') \\
a \times b P \leftrightarrow (\text{Times } a' b') \\
a + b S \leftrightarrow (\text{Plus } a' b') \\
\]

\[
\frac{1 A}{2 A} \\
\frac{1 P}{2 \times 3 P} \\
\frac{1 + 2 \times 3 S}{2 \times 3 S} \\
\frac{3 A}{3 P} \\
\]
Example

Rules

\[
\begin{align*}
  i & \in \mathbb{Z} \\
  i \cdot A & \leftarrow (\text{Num } i) \\
  a \cdot S & \leftarrow a' \\
  e \cdot A & \leftarrow a \\
  e \cdot P & \leftarrow a \\
  e \cdot S & \leftarrow a \\
  a \cdot A & \leftarrow a' \\
  b \cdot P & \leftarrow b' \\
  a \cdot P & \leftarrow a' \\
  b \cdot S & \leftarrow b' \\
  a \times b \cdot P & \leftarrow (\text{Times } a' \ b') \\
  a + b \cdot S & \leftarrow (\text{Plus } a' \ b')
\end{align*}
\]
Example

Rules

\[
\begin{align*}
i \in \mathbb{Z} & \quad a \mathcal{S} \leftrightarrow a' \\
i A \leftrightarrow (\text{Num } i) & \quad (a) A \leftrightarrow a' \\
a A \leftrightarrow a' & \quad b \mathcal{P} \leftrightarrow b' \\
a \times b \mathcal{P} \leftrightarrow (\text{Times } a' b') & \\
a + b \mathcal{S} \leftrightarrow (\text{Plus } a' b')
\end{align*}
\]

1 A \leftrightarrow (\text{Num 1}) \text{ AST} \quad 2 A \leftrightarrow 3 \text{ P} \quad 3 A \\
1 P \leftrightarrow (\text{Num 1}) \text{ AST} \quad 2 \times 3 \text{ P} \\
1 + 2 \times 3 \text{ S} \quad 2 \times 3 \text{ S} \quad 3 \text{ P}
Example

Rules

\[
\begin{align*}
 i \in \mathbb{Z} & \quad a \ A \leftrightarrow (\text{Num } i) & \quad e \ A \leftrightarrow a & \quad e \ P \leftrightarrow a & \quad e \ S \leftrightarrow a \\
 i \ A \leftrightarrow (\text{Num } i) & \quad (a) \ A \leftrightarrow a' & a \ A \leftrightarrow a' & b \ P \leftrightarrow b' & a \ P \leftrightarrow a' & b \ S \leftrightarrow b' \\
 a \times b \ P \leftrightarrow (\text{Times } a' \ b') & \quad a + b \ S \leftrightarrow (\text{Plus } a' \ b')
\end{align*}
\]
**Example**

**Rules**

\[
\begin{align*}
  i \in \mathbb{Z} & \quad \frac{}{i \ A \leftrightarrow (\text{Num } i)} \\
  \quad \frac{}{(a) \ A \leftrightarrow a'} \\
  \quad \frac{}{a \ A \leftrightarrow a'} \quad \frac{}{b \ P \leftrightarrow b'} \\
  \quad \frac{}{a \times b \ P \leftrightarrow (\text{Times } a' \ b')} \\
  \quad \frac{}{a + b \ S \leftrightarrow (\text{Plus } a' \ b')} \\
  \quad \frac{}{1 \ + \ 2 \times 3 \ S} \\
\end{align*}
\]
Example

Rules

\[
\begin{align*}
& i \in \mathbb{Z} \\
& i A \quad \mapsto \quad (\text{Num } i) \\
& a S \quad \mapsto \quad a' \\
& (a) A \quad \mapsto \quad a' \\
& e A \quad \mapsto \quad a \\
& e P \quad \mapsto \quad a \\
& e S \quad \mapsto \quad a \\
& a A \quad \mapsto \quad a' \\
& b P \quad \mapsto \quad b' \\
& a \times b P \quad \mapsto \quad (\text{Times } a' \ b') \\
& a P \quad \mapsto \quad a' \\
& b S \quad \mapsto \quad b' \\
& a + b S \quad \mapsto \quad (\text{Plus } a' \ b') \\
\end{align*}
\]

1 A \quad \mapsto \quad (\text{Num } 1) \quad \text{AST} \\
1 P \quad \mapsto \quad (\text{Num } 1) \quad \text{AST} \\
2 A \quad \mapsto \quad (\text{Num } 2) \quad \text{AST} \\
3 A \quad \mapsto \quad (\text{Num } 3) \quad \text{AST} \\
3 P \quad \mapsto \quad (\text{Num } 3) \quad \text{AST} \\
2 \times 3 \ P \\
2 \times 3 \ S \\
1 + 2 \times 3 \ S
### Example

#### Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i \in \mathbb{Z}$</td>
<td>$i A \leftrightarrow (\text{Num } i)$</td>
</tr>
<tr>
<td>$a S \leftrightarrow a'$</td>
<td>$(a) A \leftrightarrow a'$</td>
</tr>
<tr>
<td>$e A \leftrightarrow a$</td>
<td>$e P \leftrightarrow a$</td>
</tr>
<tr>
<td>$e P \leftrightarrow a$</td>
<td>$e S \leftrightarrow a$</td>
</tr>
<tr>
<td>$a A \leftrightarrow a'$</td>
<td>$b P \leftrightarrow b'$</td>
</tr>
<tr>
<td>$b P \leftrightarrow b'$</td>
<td>$a P \leftrightarrow a'$</td>
</tr>
<tr>
<td>$b S \leftrightarrow b'$</td>
<td>$a + b S \leftrightarrow (\text{Plus } a' \ b')$</td>
</tr>
<tr>
<td>$a \times b P \leftrightarrow (\text{Times } a' \ b')$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example</th>
<th>Syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 A \leftrightarrow (\text{Num } 1)$ AST</td>
<td>$3 A \leftrightarrow (\text{Num } 3)$ AST</td>
</tr>
<tr>
<td>$1 P \leftrightarrow (\text{Num } 1)$ AST</td>
<td>$2 \times 3 P \leftrightarrow (\text{Times } (\text{Num } 2) (\text{Num } 3))$ AST</td>
</tr>
<tr>
<td>$1 + 2 \times 3 S$</td>
<td>$2 \times 3 S$</td>
</tr>
</tbody>
</table>
### Example

#### Rules

\[
\begin{align*}
i \in \mathbb{Z} & \quad i \; A \quad \leftrightarrow \quad (\text{Num } i) \\
i A & \quad \leftrightarrow \quad (\text{Num } i) \\
(a) \; A & \quad \leftrightarrow \quad a' \\
(a) \; A & \quad \leftrightarrow \quad a' \\
e \; A & \quad \leftrightarrow \quad a \\
e \; P & \quad \leftrightarrow \quad a \\
e \; S & \quad \leftrightarrow \quad a \\
e \; P & \quad \leftrightarrow \quad a \\
e \; S & \quad \leftrightarrow \quad a \\
a \; A & \quad \leftrightarrow \quad a' \\
b \; P & \quad \leftrightarrow \quad b' \\
a \; P & \quad \leftrightarrow \quad a' \\
b \; S & \quad \leftrightarrow \quad b' \\
a \; P & \quad \leftrightarrow \quad a' \\
b \; S & \quad \leftrightarrow \quad b' \\
a \times b \; P & \quad \leftrightarrow \quad (\text{Times } a' \; b') \\
a \times b \; P & \quad \leftrightarrow \quad (\text{Times } a' \; b') \\
a + b \; S & \quad \leftrightarrow \quad (\text{Plus } a' \; b') \\
a + b \; S & \quad \leftrightarrow \quad (\text{Plus } a' \; b') \\
\end{align*}
\]

#### Example Expressions

\[
\begin{align*}
1 \; A & \quad \leftrightarrow \quad (\text{Num } 1) \; \text{AST} \\
2 \; A & \quad \leftrightarrow \quad (\text{Num } 2) \; \text{AST} \\
3 \; A & \quad \leftrightarrow \quad (\text{Num } 3) \; \text{AST} \\
1 \; P & \quad \leftrightarrow \quad (\text{Num } 1) \; \text{AST} \\
2 \times 3 \; P & \quad \leftrightarrow \quad (\text{Times } (\text{Num } 2) \; (\text{Num } 3)) \; \text{AST} \\
2 \times 3 \; P & \quad \leftrightarrow \quad (\text{Times } (\text{Num } 2) \; (\text{Num } 3)) \; \text{AST} \\
\end{align*}
\]

\[
1 + 2 \times 3 \; S
\]
# Example

## Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i \in \mathbb{Z}$</td>
<td>$i \mathrel{A} \iff (\text{Num } i)$</td>
</tr>
<tr>
<td>$a \mathrel{S} \iff a'$</td>
<td>$(a) \mathrel{A} \iff a'$</td>
</tr>
<tr>
<td>$e \mathrel{A} \iff a$</td>
<td>$e \mathrel{P} \iff a$</td>
</tr>
<tr>
<td>$e \mathrel{S} \iff a$</td>
<td>$a \mathrel{P} \iff a'$</td>
</tr>
<tr>
<td>$b \mathrel{P} \iff b'$</td>
<td>$b \mathrel{S} \iff b'$</td>
</tr>
<tr>
<td>$a \times b \mathrel{P} \iff (\text{Times } a' b')$</td>
<td>$a + b \mathrel{S} \iff (\text{Plus } a' b')$</td>
</tr>
</tbody>
</table>

## Example Calculations

<table>
<thead>
<tr>
<th>Expression</th>
<th>AST</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \mathrel{A} \iff (\text{Num } 1) \mathrel{AST}$</td>
<td>$1 \mathrel{+} 2 \times 3 \mathrel{S} \iff (\text{Plus } (\text{Num } 1) (\text{Times } (\text{Num } 2) (\text{Num } 3))) \mathrel{AST}$</td>
</tr>
<tr>
<td>$2 \mathrel{A} \iff (\text{Num } 2) \mathrel{AST}$</td>
<td>$2 \times 3 \mathrel{P} \iff (\text{Times } (\text{Num } 2) (\text{Num } 3)) \mathrel{AST}$</td>
</tr>
<tr>
<td>$3 \mathrel{P} \iff (\text{Num } 3) \mathrel{AST}$</td>
<td>$2 \times 3 \mathrel{S} \iff (\text{Times } (\text{Num } 2) (\text{Num } 3)) \mathrel{AST}$</td>
</tr>
<tr>
<td>$3 \mathrel{A} \iff (\text{Num } 3) \mathrel{AST}$</td>
<td>$3 \mathrel{P} \iff (\text{Num } 3) \mathrel{AST}$</td>
</tr>
</tbody>
</table>
The Inverse

What about the inverse operation to parsing?

Unparsing

Unparsing, also called *pretty-printing*, is the process of starting with the term on the right hand side of \( \leftarrow \rightarrow \) and attempting to synthesise a string on the left.
The Inverse

What about the inverse operation to parsing?

**Unparsing**

Unparsing, also called *pretty-printing*, is the process of starting with the term on the right hand side of $\leftrightarrow$ and attempting to synthesise a string on the left.

**Problem**

There are many concrete strings for a given abstract syntax term. The algorithm is *non-deterministic*.

While it is desirable to have:

$$parse \circ unparse = id$$

It is not usually true that:

$$unparse \circ parse = id$$
Example

Going from right to left requires some formatting guesswork to produce readable code.

Algorithms to do this can get quite involved!

Let’s implement a parser for arithmetic.
Adding Let

Let us extend our arithmetic expression language with variables, including a `let` construct to give them values.

Concrete Syntax

$$
\frac{x \ \text{Ident}}{x \ \text{Atom}} \quad \frac{x \ \text{Ident} \quad e_1 \ \text{SExp} \quad e_2 \ \text{SExp}}{\text{let } x = e_1 \ \text{in } e_2 \ \text{end} \ \text{Atom}}
$$

Example

```
let x = 3 in
  x + 4
end
```

```
let x = 3 in
  let y = 4 in x + y end
end
```
The process of finding the binding occurrence of each used variable is called **scope resolution**. Usually this is done statically. If no binding can be found, an *out of scope* error is raised.
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The process of finding the binding occurrence of each used variable is called *scope resolution*. Usually this is done statically. If no binding can be found, an *out of scope* error is raised.
Shadowing

What does this program evaluate to?

```plaintext
let x = 5 in
  let x = 2 in
    x + x
  end
+ x end
```
This program results in 9.
What does this program evaluate to?

```
let x = 5 in
  let x = 2 in
    x + x
  end
+ x end
```

This program results in 9.
What is the difference between these two programs?

let $x = 5$ in
  let $x = 2$ in
    $x + x$
  end
end

let $a = 5$ in
  let $y = 2$ in
    $y + y$
  end
end

$\alpha$-equivalence
What is the difference between these two programs?

\[
\begin{align*}
\text{let } x &= 5 \text{ in } \ \\
&\quad \text{let } x = 2 \text{ in } x + x \ \\
&\quad \text{end} \\
\text{end} \\
\end{align*}
\]

\[
\begin{align*}
\text{let } a &= 5 \text{ in } \ \\
&\quad \text{let } y = 2 \text{ in } y + y \ \\
&\quad \text{end} \\
\text{end} \\
\end{align*}
\]

They are semantically identical, but differ in the choice of bound variable names. Such expressions are called \(\alpha\)-equivalent. We write \(e_1 \equiv_\alpha e_2\) if \(e_1\) is \(\alpha\)-equivalent to \(e_2\). The relation \(\equiv_\alpha\) is an equivalence relation. That is, it is reflexive, transitive and symmetric.

The process of consistently renaming variables that preserves \(\alpha\)-equivalence is called \(\alpha\)-renaming.
A variable $x$ is **free** in an expression $e$ if $x$ occurs in $e$ but is not bound in $e$.

**Example (Free Variables)**

The variable $x$ is free in $x + 1$, but not in `let x = 3 in x + 1 end`. 
Substitution

A variable $x$ is \textit{free} in an expression $e$ if $x$ occurs in $e$ but is not bound in $e$.

\textbf{Example (Free Variables)}

The variable $x$ is free in $x + 1$, but not in $\text{let } x = 3 \text{ in } x + 1 \text{ end}$.

A \textit{substitution}, written $e[x := t]$ (or $e^{t/x}$ in some other courses), is the replacement of all free occurrences of $x$ in $e$ with the term $t$.

\textbf{Example (Simple Substitution)}

$(5 \times x + 7)[x := y \times 4]$ is the same as $(5 \times (y \times 4) + 7)$. 
Problems with substitution

Consider these two $\alpha$-equivalent expressions.

\[
\text{let } y = 5 \text{ in } y \times x + 7 \text{ end}
\]

and

\[
\text{let } z = 5 \text{ in } z \times x + 7 \text{ end}
\]

What happens if you apply the substitution $[x := y \times 3]$ to both expressions?
Problems with substitution

Consider these two $\alpha$-equivalent expressions.

\[
\text{let } y = 5 \text{ in } y \times x + 7 \text{ end}
\]

and

\[
\text{let } z = 5 \text{ in } z \times x + 7 \text{ end}
\]

What happens if you apply the substitution $[x := y \times 3]$ to both expressions? You get two non-$\alpha$-equivalent expressions!

\[
\text{let } y = 5 \text{ in } y \times (y \times 3) + 7 \text{ end}
\]

and

\[
\text{let } z = 5 \text{ in } z \times (y \times 3) + 7 \text{ end}
\]

This problem is called capture.
**Variable Capture**

*Capture* can occur for a substitution $e[x := t]$ when a bound variable in $e$ clashes with a free variable occurring in $t$.

**Fortunately**

It is *always possible* to avoid capture.

- $\alpha$-rename the offending bound variable to an unused name, or
- If you have access to the free variable’s definition, renaming the free variable, or
- Use a different abstract syntax representation that makes capture impossible (More on this later).
Abstract Syntax for Variables

We shall extend our AST and parsing relation to include a definition for let and variables.

Let Syntax

\[
\begin{align*}
\text{x Ident} & \quad \Rightarrow \quad \text{x Atom} \quad \rightarrow \quad \text{(Var x) AST} \\
\text{x Ident} \quad e_1 \quad \text{SExp} & \quad \rightarrow \quad a_1 \quad \text{AST} \\
\text{let} \quad x \quad = \quad e_1 \quad \text{in} \quad e_2 \quad \text{end} \quad \text{Atom} & \quad \rightarrow \quad \text{(Let x a_1 a_2) AST}
\end{align*}
\]
**First Order Abstract Syntax**

Consider the following two pieces of abstract syntax:

\[
\text{(Let "x" (Num 5) (Plus (Num 4) (Var "x")))}
\]

\[
\text{(Let "y" (Num 5) (Plus (Num 4) (Var "y")))}
\]

This demonstrates some problems with our abstract syntax approach.
First Order Abstract Syntax

Consider the following two pieces of abstract syntax:

$$(\text{Let } "x" \ (\text{Num} \ 5) \ (\text{Plus} \ (\text{Num} \ 4) \ (\text{Var} "x")))$$

$$(\text{Let } "y" \ (\text{Num} \ 5) \ (\text{Plus} \ (\text{Num} \ 4) \ (\text{Var} "y")))$$

This demonstrates some problems with our abstract syntax approach.

1. Substitution capture is a problem.
First Order Abstract Syntax

Consider the following two pieces of abstract syntax:

\[
(\text{Let } "x" \ (\text{Num } 5) \ (\text{Plus } (\text{Num } 4) \ (\text{Var } "x")))
\]

\[
(\text{Let } "y" \ (\text{Num } 5) \ (\text{Plus } (\text{Num } 4) \ (\text{Var } "y")))
\]

This demonstrates some problems with our abstract syntax approach.

1. Substitution capture is a problem.
2. $\alpha$-equivalent expressions are not equal. Determining if an expression is $\alpha$-equivalent requires us to search for a consistent $\alpha$-renaming of variables.
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\[(\text{Let } "x" \ (\text{Num } 5) (\text{Plus} \ (\text{Num } 4) \ (\text{Var } "x")))\]

\[(\text{Let } "y" \ (\text{Num } 5) (\text{Plus} \ (\text{Num } 4) \ (\text{Var } "y")))\]

This demonstrates some problems with our abstract syntax approach.

1. Substitution capture is a problem.
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3. No distinction is made between binding and usage occurrences of variables. This means that we must define substitution by hand on each type of expression we introduce.
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1. Substitution capture is a problem.
2. α-equivalent expressions are not equal. Determining if an expression is α-equivalent requires us to search for a consistent α-renaming of variables.
3. No distinction is made between binding and usage occurrences of variables. This means that we must define substitution by hand on each type of expression we introduce.
4. Scoping errors cannot be easily detected — malformed syntax is easy to write.
One popular approach to address the first issue is *de Bruijn indices*.

**Key Idea**

1. Remove all identifiers from binding expressions like Let.
2. Replace the identifier in a Var with a number indicating how many binders we must skip in order to find the binder for that variable.

(Let "a" (Num 5)
 (Let "y" (Num 2)
  (Plus (Var "a") (Var "y"))))
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**Key Idea**

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(Let "a" (Num 5) (Let "y" (Num 2) (Plus (Var "a") (Var "y"))))

(Let (Num 5) (Let (Num 2) (Plus (Var 1) (Var 0))))
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**Key Idea**

1. Remove all identifiers from binding expressions like `Let`.
2. Replace the identifier in a `Var` with a number indicating how many binders we must skip in order to find the binder for that variable.

\[
\begin{align*}
\text{(Let } "a" \text{ (Num 5)} & \quad \text{ (Let (Num 5))} \\
\text{(Let } "y" \text{ (Num 2)} & \quad \text{(Let (Num 2))} \\
\text{(Plus (Var } "a" \text{) (Var } "y" \text{))}) & \quad \text{(Plus (Var 1) (Var 0))})
\end{align*}
\]
Debruijnification

**Algorithm**

Given a piece of *first order abstract syntax* with explicit variable names, we can convert to de Bruijn indices by keeping a stack of variable names, pushing onto the stack at each Let and popping after the variable goes out of scope. When a usage occurrence is encountered, replace the variable name with its first position in the stack (starting at the top of the stack).

This approach naturally handles shadowing. It’s also possible, but harder, to have de Bruijn indices going in the other direction (from the bottom of the stack, upwards).
de Bruijn Substitution

Substitution is now capture avoiding by definition.

\[
\begin{align*}
(\text{Num } i)[n := t] &= (\text{Num } i) \\
(\text{Plus } a \ b)[n := t] &= (\text{Plus } a[n := t] \ b[n := t]) \\
(\text{Times } a \ b)[n := t] &= (\text{Times } a[n := t] \ b[n := t])
\end{align*}
\]
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$$(\text{Times } a \ b)[n := t] = (\text{Times } a[n := t] \ b[n := t])$$

$$(\text{Var } m)[n := t] = \begin{cases} 
  t & \text{if } n = m \\
  (\text{Var } (m-1)) & \text{if } m > n \\
  (\text{Var } m) & \text{otherwise}
\end{cases}$$
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(\text{Times } a \ b)[n := t] & = (\text{Times } a[n := t] \ b[n := t]) \\
& \quad \begin{cases} t \quad \text{if } n = m \\ (\text{Var } (m-1)) \quad \text{if } m > n \\ (\text{Var } m) \quad \text{otherwise} \end{cases} \\
(\text{Var } m)[n := t] & = \\
(\text{Let } e_1 \ e_2)[n := t] & = (\text{Let } e_1[n := t] \ e_2[n + 1 := t^
\uparrow 0])
\end{align*}
\]
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Substitution is now capture avoiding by definition.

\[
\begin{align*}
(\text{Num } i)[n := t] &= (\text{Num } i) \\
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(\text{Var } m)[n := t] &= \begin{cases} 
  t & \text{if } n = m \\
  (\text{Var } (m - 1)) & \text{if } m > n \\
  (\text{Var } m) & \text{otherwise}
\end{cases} \\
(\text{Let } e_1 \ e_2)[n := t] &= (\text{Let } e_1[n := t] \ e_2[n + 1 := t^0])
\end{align*}
\]

Where \(e^\uparrow_n\) is an up-shifting operation defined as follows:

\[
\begin{align*}
(\text{Num } i)^\uparrow_n &= (\text{Num } i) \\
(\text{Plus } a b)^\uparrow_n &= (\text{Plus } a^\uparrow_n \ b^\uparrow_n) \\
(\text{Times } a b)^\uparrow_n &= (\text{Times } a^\uparrow_n \ b^\uparrow_n) \\
(\text{Var } m)^\uparrow_n &= \begin{cases} 
  (\text{Var } (m + 1)) & \text{if } m \geq n \\
  (\text{Var } m) & \text{otherwise}
\end{cases} \\
(\text{Let } e_1 \ e_2)^\uparrow_n &= (\text{Let } e_1^\uparrow_n \ e_2^\uparrow_{n+1})
\end{align*}
\]
How do de Bruijn indices stack up against explicit names?

1. Substitution capture solved.
Examining de Bruijn indices

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Examining de Bruijn indices

How do de Bruijn indices stack up against explicit names?

1. Substitution capture solved.
2. $\alpha$-equivalent expressions are now equal.
3. We still must define substitution machinery by hand for each type of expression.
4. It is still possible to make malformed syntax – indices that overflow the stack, for example.

Two out of four isn’t bad, but can we do better by changing the term language?
Higher Order Terms

We shall change our term language to include built-in notions of variables and binding.

\[
t ::= \text{Symbol} \quad (\text{symbols}) \\
| \quad x \quad (\text{variables}) \\
| \quad t_1 \ t_2 \quad (\text{application}) \\
| \quad x \cdot t \quad (\text{binding or abstraction})
\]

As in Haskell, we shall say that application is left-associative, so

\[(\text{Plus (Num 3) (Num 4)}) = ((\text{Plus (Num 3)}) \ (\text{Num 4}))\]

Now the binding and usage occurrences of variables are distinguished from regular symbols in our term language. Let's see what this lets us do...
Representing Let

\[
\frac{a_1 \text{ AST} \quad a_2 \text{ AST}}{(\text{Let } a_1 (x. a_2)) \text{ AST}}
\]

We no longer need a rule for variables, because they’re baked into the structure of terms.
Representing Let

\[
\frac{a_1 \text{ AST} \quad a_2 \text{ AST}}{(\text{Let } a_1 (x. \ a_2)) \text{ AST}}
\]

We no longer need a rule for variables, because they're baked into the structure of terms.

How would we represent this AST in Haskell?

```haskell
data AST = Num Int
          | Plus AST AST
          | Times AST AST
          | Let AST ???
```
Representing Let

\[
\frac{a_1 \text{ AST} \quad a_2 \text{ AST}}{(\text{Let } a_1 (x. a_2)) \text{ AST}}
\]

We no longer need a rule for variables, because they’re baked into the structure of terms.

How would we represent this AST in Haskell?

```haskell
data AST = Num Int
          | Plus AST AST
          | Times AST AST
          | Let AST (AST → AST)
```

So \(\text{let } x = 3 \text{ in } x + 2 \text{ end}\) becomes, in Haskell:

\[(\text{Let } (\text{Num } 3) (\lambda x \rightarrow \text{Plus } x (\text{Num } 2)))\]
Substitution

We can now define substitution across all terms in the meta-logic:

\[
\begin{align*}
\text{Symbol}[x := e] &= \text{Symbol} \\
y[x := e] &= \begin{cases} 
e & \text{if } y = x \\y & \text{otherwise} \end{cases} \\
(t_1 \, t_2)[x := e] &= t_1[x := e] \, t_2[x := e] \\
(y \, . \, t)[x := e] &= \begin{cases} (y \, . \, t) & \text{if } x = y \\
(y \, . \, t[x := e]) & \text{if } y \notin \text{FV}(e) \\
\text{undefined} & \text{otherwise} \end{cases}
\end{align*}
\]

Where \( \text{FV}(\cdot) \) is the set of all free variables in a term:

\[
\begin{align*}
\text{FV}(\text{Symbol}) &= \emptyset \\
\text{FV}(x) &= \{x\} \\
\text{FV}(t_1 \, t_2) &= \text{FV}(t_1) \cup \text{FV}(t_2) \\
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y[x := e] &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\
(t_1 t_2)[x := e] &= t_1[x := e] t_2[x := e] \\
(y. t)[x := e] &= \begin{cases} (y. t) & \text{if } x = y \\ (y. t[x := e]) & \text{if } y \not\in \text{FV}(e) \\ \text{undefined} & \text{otherwise} \end{cases}
\end{align*}
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\text{FV}(x. t) &= \\
\end{align*}
\]
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\begin{align*}
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y[x := e] &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\
(t_1 \ t_2)[x := e] &= t_1[x := e] \ t_2[x := e] \\
(y \cdot t)[x := e] &= \begin{cases} (y \cdot t) & \text{if } x = y \\ (y \cdot t[x := e]) & \text{if } y \not\in \text{FV}(e) \\ \text{undefined} & \text{otherwise} \end{cases}
\end{align*}
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\text{FV}(t_1 \ t_2) &= \text{FV}(t_1) \cup \text{FV}(t_2) \\
\text{FV}(x \cdot t) &= \text{FV}(t) \setminus \{x\}
\end{align*}
\]
Substitution capture is still a problem in HOAS. But it is not our problem. Because substitution is defined in the meta-language, it’s the job of the implementors of the meta-language (if any) to deal with capture issues.

- When Haskell is our meta-language, it’s the job of the GHC developers.
- When we are doing proofs in our meta-logic, there is no implementation, so we can just say that we assume $\alpha$-equivalent terms to be equal, and therefore assume that variables are always renamed to avoid capture.

So, we have solved the problem by making it someone else’s problem. Outrageous cheating!
Evaluating All Approaches

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In embedded languages and in pen and paper proofs, HOAS is very common.
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In conventional language implementations and machine-checked formalisations, de Bruijn indices are more popular.
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In conventional language implementations and machine-checked formalisations, de Bruijn indices are more popular.

In your assignments, strings will be used 😊