COMP 3221

Microprocessors and Embedded Systems

Lectures 19 : Floating Point Number Representation – I

http://www.cse.unsw.edu.au/~cs3221

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Overview

- Floating Point Numbers
- Motivation: Decimal Scientific Notation
  - Binary Scientific Notation
- Floating Point Representation inside computer (binary)
  - Greater range, precision
- IEEE-754 Standard

Review of Numbers

- Computers are made to deal with numbers
- What can we represent in N bits?
  - Unsigned integers:

    0 to $2^N - 1$
  - Signed Integers (Two's Complement)

    $-2^{(N-1)}$ to $2^{(N-1)} - 1$

Other Numbers

- What about other numbers?
  - Very large numbers? (seconds/century)

    $3{,}155{,}760{,}000_{10}$ ($3.15576_{10} \times 10^9$)
  - Very small numbers? (atomic diameter)

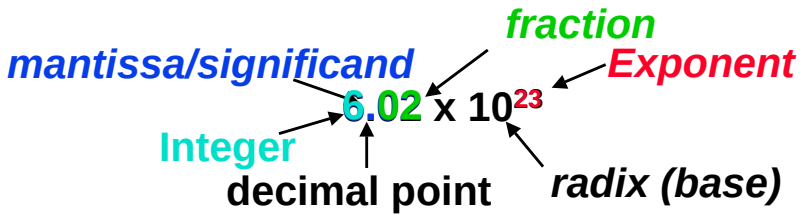
    0.00000001_{10} ($1.0_{10} \times 10^{-8}$)
  - Rationals (repeating pattern)

    2/3 (0.666666666. . .)
  - Irrationals

    $2^{1/2}$ (1.414213562373. . .)
  - Transcendentals

    e (2.718...), π (3.141...)
- All represented in scientific notation

Scientific Notation Review



- Normalized form: no leadings 0s (exactly one digit to left of decimal point)
- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

How to represent 0 in Normalized form?

Interesting Properties

- Finite precision
  - i.e. the number of bits in which to represent the significand and exponent is limited.
  - Thus not all non-integer values can be represented exactly.
  - Example: represent 1.32 as $d_1d_2.f_1 \times 10^{d_3}$ (no digit after the decimal point)

Floating Point Number Range vs Precision (#1/4)

- For simplicity, assume integer Representation for Significand.
- Represent N by $d_1d_2d_3$, where

$N = d_1d_2 \times 10^{d_3}$

$N = d_1 \times 10^{d_2d_3}$

$N = d_1d_2 \times 100^{d_3}$

Floating Point Number Range vs Precision (#2/4)

- Which representation can represent the largest value?

$N = d_1d_2 \times 10^{d_3}$ $\quad N = 99 \times 10^{9} = 9.9 \times 10^{10}$

$N = d_1 \times 10^{d_2d_3}$

$N = d_1d_2 \times 100^{d_3}$ $\quad N = 99 \times 100^{9} = 9.9 \times 10^{19}$

Floating Point Number Range vs Precision (#3/4)

° In which representation can the most different values be represented?

$N = d_1 d_2 \times 10^{d_3}$

$N = d_1 \times 10^{d_2 d_3}$

$N = d_1 d_2 \times 100^{d_3}$

°All can represent the same number of different values.

°Two of the systems can represent an equal number of values, more than the third.

°#3 can represent more values than the other two.

°#2 can represent more values than the other two.

°#1 can represent more values than the other two

Floating Point Number Range vs Precision (#4/4)

° In which representation can the most different values be represented?

$N = d_1 d_2 \times 10^{d_3}$
$99 \times 10 + 1 = 991$

$N = d_1 \times 10^{d_2 d_3}$
$9 \times 100 + 1 = 901$

$N = d_1 d_2 \times 100^{d_3}$
$99 \times 10 + 1 = 991$

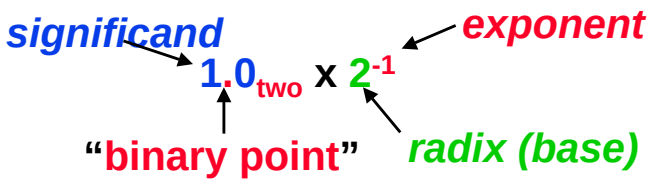
°All can represent the same number of different values.

°#3 can represent more values than the other two.

°#2 can represent more values than the other two.

°#1 can represent more values than the other two

Scientific Notation for Binary Numbers



° Computer arithmetic that supports it called floating point, because it represents numbers where binary point is not fixed, as it is for integers

- Declare such variable in C as `float`

Properties of a good FP Number rep.

° Represents many useful numbers
- most of the 2^N possible are useful
- How many LARGE? How many small?

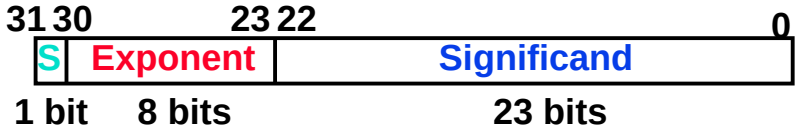
° Easy to do arithmetic (+, -, *, /)

° Easy to do comparison (==, <, >)

° Nice mathematical properties
- A != B => A - B != 0

Floating Point Representation (#1/2)

- Normal format: $+1.xxxxxxxxxx_{two} * 2^{yyyy_{two}}$
- Multiple of Word Size (32 bits)



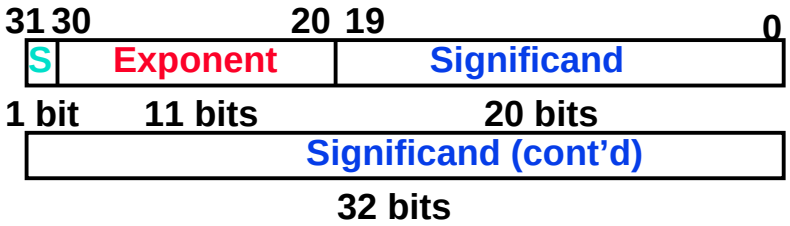
- S represents Sign
- Exponent represents y's
- Significand represents x's
- Leading 1 in Significand is implied
- Represent numbers as small as 2.0×10^{-38} to as large as 2.0×10^{38}

Floating Point Representation (#2/2)

- What if result too large? (> 2.0×10^{38})
  - Overflow!
  - Overflow => Exponent larger than represented in 8-bit Exponent field
- What if result too small? (<0, < 2.0×10^{-38})
  - Underflow!
  - Underflow => Negative exponent larger than represented in 8-bit Exponent field
- How to reduce chances of overflow or underflow?

Double Precision Fl. Pt. Representation

- Next Multiple of Word Size (64 bits)



- Double Precision (vs. Single Precision)
  - C variable declared as `double`
  - Represent numbers almost as small as 2.0×10^{-308} to almost as large as 2.0×10^{308}
  - But primary advantage is greater accuracy due to larger significand

Fl. Pt. Hardware

- Microprocessors do floating point computation using a special coprocessor.
  - works under the processor supervision
  - Has its own set of registers
- Most low end processors do not have Ft. Pt. Coprocessors
  - Ft. Pt. Computation by software emulation
  - ARM processor on DSLMU board does not have Ft. Pt. Coprocessor
  - Some high end ARM processors do

IEEE 754 Floating Point Standard (#1/6)

- Single Precision, (DP similar)
- Sign bit: 1 means negative
  0 means positive
- Significand:
  - To pack more bits, leading 1 implicit for normalized numbers. (Hidden Bit)
  - 1 + 23 bits single, 1 + 52 bits double
  - always true: 0 < Significand < 1 (for normalized numbers)
- What about Zero?
  Next Lecture

IEEE 754 Floating Point Standard (#2/6)

- Kahan* wanted FP numbers to be used even if no FP hardware; e.g., sort records with FP numbers using integer compares
- Wanted Compare to be faster, by means of a single compare operation, used for integer numbers, especially if positive FP numbers
- How to order 3 parts (Sign, Significand and Exponent) to simplify compare?
- The Author of IEEE 754 FP Standard
  http://www.cs.berkeley.edu/~wkahan/ieee754status/754story.html

IEEE 754 Floating Point Standard (#3/6)

- How to order 3 Fields in a Word?
  $+1.xxxxxxxxxx_{two} * 2^{yyyy}{}_{two}$
- "Natural": Sign, Fraction, Exponent?
  - Problem: If want to sort using integer compare operations, won't work:
  - 1.0×2^{20} vs. 1.1×2^{10} ; latter looks bigger!

0	10000	10100

0	11000	01010

- Exponent, Sign, Fraction?
  - Need to get sign first, since negative < positive
- Therefore order is Sign | Exponent | Fraction
  $1.0 \times 10^{20} > 1.1 \times 10^{10}$

S	Exponent	Significand

IEEE 754 Floating Point Standard (#4/6)

- Want compare Fl.Pt. numbers as if integers, to help in sort
  - Sign first part of number
  - Exponent next, so big exponent => bigger No.
- Negative Exponent?
  - 2's comp? 1.0×2^{-1} vs $1.0 \times 2^{+1}$ (1/2 vs 2)

1/2	0	1111 1111	000 0000 0000 0000 0000 0000
2	0	0000 0001	000 0000 0000 0000 0000 0000

- This notation using integer compare of 1/2 vs 2 makes 1/2 > 2!

IEEE 754 Floating Point Standard (#5/6)

° Instead, pick notation: 0000 0000 is most negative, 1111 1111 is most positive

- Called Biased Notation; bias subtracted to get number
- 127 in Single Prec. (1023 D.P.)

-127 0000 0000
-126 0000 0001
...
-1 0111 1110
0 0111 1111
+1 1000 0000
...
+127 1111 1110
+128 1111 1111

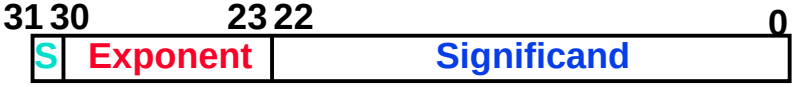
° 2's comp? 1.0×2^{-1} vs $1.0 \times 2^{+1}$ (1/2 vs 2)

1/2	0	0111 1110	000 0000 0000 0000 0000 0000
2	0	1000 0000	000 0000 0000 0000 0000 0000

- This notation using integer compare of 1/2 vs 2 makes 2 look greater than 1/2

IEEE 754 Floating Point Standard (#6/6)

° Summary (single precision):

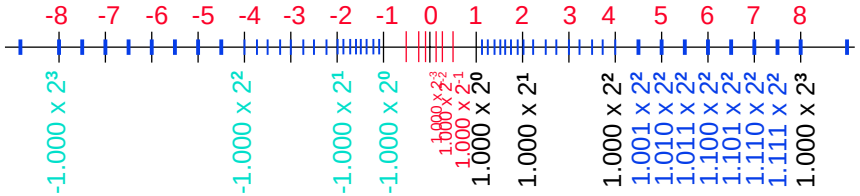


° $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$

° Double precision identical, except with exponent bias of 1023

Floating Point Number Distribution

- Which numbers can be represented?



Using mantissa of 1.000 and positive exponents

and Sign bit

and negative exponents

in each of the intervals of exponentially increasing size can represent 2^S (s=3 here) numbers of uniform difference

But how do we represent 0? Next Lecture!

"And in Conclusion.."

° Number of digits allocated to significand and exponent, and choice of base, can affect both the number of different representable values and the range of values.

° Finite precision means we have to cope with roundoff error (arithmetic with inexact values) and truncation error (large values overwhelming small ones).

° IEEE 754 Standard allows Single Precision (1 word) and Double Precision (2-word) representation of FP. Nos.