1 The Fourier Transform

What happens if \( f(t) \) is not periodic, but decays with time? In this case a representation of a signal in a way which is somewhat analogous to the Fourier series for the periodic functions, is accomplished using the Fourier transform. Since the rigorous mathematical development is somewhat tricky, rather than doing things rigorously, we will only motivate the Fourier transform heuristically.

So assume that \( f(t) \) is vanishingly small outside \([−L,L]\), where \( L \) is chosen sufficiently large, and then extend \( f(t) \) periodically outside \((-L,L]\), simply by making shifted copies of \( f(t) \) over \((-L,L]\), thus obtaining \( f_L(t) \) which is \( 2L \) periodic; see Figure 1.

![Figure 1.1: Turning a function which is small outside \([-L,L]\) (between the red lines) into a periodic function, by making its shifted copies.](image)

We can now expand \( f_L(t) \) into Fourier series

\[
f_L(t) = \sum_{k=-\infty}^{\infty} c_k^L e^{i\frac{2\pi}{L}kt} \tag{1.1}
\]

with

\[
c_k^L = \frac{1}{2L} \int_{-L}^{L} f_L(t) e^{-i\frac{2\pi}{L}kt} dt \approx \frac{1}{2L} \int_{-\infty}^{\infty} f(t) e^{-i\frac{2\pi}{L}kt} dt, \tag{1.2}
\]

because \( f(t) \) is small outside \([-L,L]\). Let us now define the Fourier transform \( \hat{f}(\omega) \) of \( f(t) \) as

\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{1.3}
\]

Then, comparing (1.2) and (1.3), we see that \( c_k^L \approx \frac{1}{2\pi} \hat{f}(k\pi/L) \), and substituting in (1.1) we get

\[
f_L(t) \approx \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \hat{f}(k\pi/L) e^{i\frac{2\pi}{L}kt} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\pi}{L} \hat{f}(k\pi/L) e^{i\frac{2\pi}{L}kt}.
\]

1
The last sum is the Riemann sum of an integral:

\[ f_L(t) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega, \]

with approximations becoming better as \( L \) becomes larger; thus, we can expect that in the limit

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega. \]  

(1.4)

Unfortunately, things are a bit more complicated than that. We mentioned that the Fourier series of functions with a finite \( L^2 \)-norm, i.e., such that

\[ \| f \|_2 = \sqrt{\frac{1}{2a} \int_{-a}^{a} |f(t)|^2 dt} < \infty \]

converge almost everywhere to \( f(t) \), but that for functions with a finite \( L^1 \)-norm,

\[ \| f \|_1 = \frac{1}{2a} \int_{-a}^{a} |f(t)| dt < \infty \]

the Fourier series might diverge everywhere.¹

When it comes to the convergence of the Fourier transform of aperiodic functions, the situation is somewhat reversed: the Fourier integral converges for all \( L^1 \) functions, i.e., for all functions such that

\[ \| f \|_1 = \int_{-\infty}^{\infty} |f(t)| dt < \infty \]

but it does not necessarily converge for all \( L^2 \) functions, i.e., functions which satisfy

\[ \| f \|_2 = \sqrt{\int_{-\infty}^{\infty} |f(t)|^2 dt} < \infty \]

Fortunately, one can introduce Fourier transform of \( L^2 \) functions via a completion, i.e., by representing each function \( f(t) \in L^2 \) as a limit of functions \( f_n(t) \) which are both in \( L^1 \) and \( L^2 \), and defining the Fourier transform of \( f(t) \) as the limit of the Fourier transforms of \( f_n(t) \). Thus, for all practical purposes, we can work as if the limit in

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega = \lim_{a,b \to \infty} \frac{1}{2\pi} \int_{-a}^{b} \hat{f}(\omega)e^{i\omega t} d\omega \]

really existed, i.e., as if this, so called improper integral (because the limits of integration are infinite) converged.

¹The almost everywhere convergence of the Fourier series of functions in \( L^2 [-a, a] \) is a very deep theorem; here is what the Wikipedia has to say about it: “The problem whether the Fourier series of any continuous function converges almost everywhere was posed by Nikolai Lusin in the 1920s and remained open until finally resolved positively in 1966 by Lennart Carleson. Indeed, Carleson showed that the Fourier expansion of any function in \( L^2 \) converges almost everywhere. This result is now known as Carleson’s theorem... Despite a number of attempts at simplifying the proof, it is still one of the most difficult results in analysis. Contrariwise, Andrey Kolmogorov, in his very first paper published [in 1923] when he was 21, constructed an example of a function in \( L^1 \) whose Fourier series diverges almost everywhere (later improved to divergence everywhere).”
One can see $\hat{f}(\omega)$ as providing the amplitude and phase of the “$\omega$-frequency component” of the signal. To see this, recall that for each $\omega$, the value of $\hat{f}(\omega)$ is a complex number that can be represented via its absolute value $\rho(\omega) = |\hat{f}(\omega)|$, $\rho(\omega) \geq 0$, and its argument $\theta(\omega) = \arg\hat{f}(\omega)$, $-\pi < \theta(\omega) \leq \pi$, i.e.,

$$\hat{f}(\omega) = \rho(\omega)e^{i\theta(\omega)}$$

and so

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega)e^{i\theta(\omega)}e^{i\omega t}d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \rho(\omega)e^{i(\omega t + \theta(\omega))}d\omega$$

which shows that $\rho(\omega)$ is the amplitude and $\theta(\omega)$ is the phase shift for $e^{i\omega t}$ harmonic oscillation of frequency $\omega$, and the signal can be seen represented as an “infinitely refined” sum (i.e., an integral) of such harmonic oscillations $\rho(\omega)e^{i(\omega t + \theta(\omega))}$.

Comparing (1.3) and (3.1) we see a remarkable property of the Fourier transform: the transformation from $f(t)$ into $\hat{f}(\omega)$ and back from $\hat{f}(\omega)$ to $f(t)$ are essentially one and the same operation, save the normalization factor $\frac{1}{2\pi}$ and the minus sign in the exponent. The normalization factor disappears if we replace everywhere in our definitions $e^{i\omega t}$ with $e^{2\pi i\omega t}$; thus, if we take the Fourier transform of a signal $f(t)$, thus obtaining its “frequency profile” $\hat{f}$, and then once more take the Fourier transform of $\hat{f}$ obtaining the frequency profile $\hat{\hat{f}}$ of the frequency profile $\hat{f}$ of the original signal, we get $\hat{\hat{f}}(t) = f(-t)$ i.e., we just get back the original signal only with the direction of time reversed!

## 2 Band-limited signals

Most signals of engineering interest, due to physical limitations, cannot contain arbitrarily high frequencies, or we are simply not interested in frequencies above certain limit, treating them as unwanted noise. For example, we cannot hear sounds of frequencies above 20 kHz.

Assume that $f(t)$ has a finite bandwidth; we will chose a unit interval of time so that all frequencies present in the signal are smaller than $\pm \pi$, i.e., that

$$\hat{f}(\omega) = 0 \text{ if } |\omega| \geq \pi.$$ 

It will be clear later how to choose such time measuring unit.

We will also assume that signals have finite energy, i.e., that their usual $L^2$ norm is finite:

$$\int_{-\infty}^{\infty} f(t)^2 dt < \infty$$

The space of such signals, with the usual scalar product and norm of $L^2$ are called the space of band limited signals of finite energy or the Paley-Wiener space and is usually denoted by $BL(\pi)$ or $PW_{\pi}$.

---

1It might look paradoxical that frequencies can be negative. While this is mainly a technical convenience which makes signal representation considerably simpler, one can also remember that a wheel can spin with certain angular velocity in two opposite directions...
Since \( \hat{f}(\omega) = 0 \) outside \((-\pi, \pi)\), we have

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)e^{i\omega t} d\omega \tag{2.1}
\]

We can now represent \( \hat{f}(\omega) \) over the interval \((-\pi, \pi)\) by its Fourier series:

\[
\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{i k \omega},
\]

where

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)e^{-i k \omega} d\omega.
\]

Comparing this equation with (2.1), we see that \( c_k = f(-k) \), i.e., the Fourier coefficients are just the samples of the signal! Thus, the Fourier transform of a \( \pi \)-band limited signal of finite energy is given by

\[
\hat{f}(\omega) = \sum_{k=-\infty}^{\infty} f(-k)e^{i k \omega}. \tag{2.2}
\]

We can now substitute \( \hat{f}(\omega) \) in with its Fourier series we obtain

\[
f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=-\infty}^{\infty} f(-k)e^{i k \omega} \right) e^{i\omega t} d\omega \tag{2.3}
\]

We can now use the substitution \( k = -n \) and, changing the order of summation and integration, get\(^1\)

\[
f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(n) \int_{-\pi}^{\pi} e^{-i \omega n} e^{i \omega t} d\omega
\]

\[
= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(n) \int_{-\pi}^{\pi} e^{i \omega (t-n)} d\omega
\]

\[
= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} f(n) \left[ i \left[ \frac{e^{i \omega (t-n)}}{i(t-n)} \right]_{\omega = -\pi}^{\omega = \pi} \right]
\]

Note that \( \cos(\pi(t-n)) - \cos(-\pi(t-n)) = 0 \) while \( \sin(\pi(t-n)) - \sin(-\pi(t-n)) = 2\sin(\pi(t-n)) \) and we get

\[
f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(t-n))}{\pi(t-n)}. \tag{2.4}
\]

A representation of a band limited signal by formula (2.4) is usually called the Shannon Sampling Theorem, even though he did not discovered it but has only

\(^1\)The mathematicians will complain that we should have first investigated if the series converges uniformly, in order to be able to exchange the order of integration and summation...
popularized it. A more proper name should be “Whittaker - Nyquist - Kotelnikov - Shannon Sampling Theorem”. It asserts that it is possible to obtain a perfect reconstruction of a band limited signal from its discrete samples, sampled at intervals which are half the period of the sinusoid of the maximal allowed frequency, because the sine wave \( \sin \pi t \) of frequency \( \pi \) has period equal to 2, and we sample the signal at all integers. This is usually expressed as “the sampling frequency should be twice the maximal frequency present in the signal”. This, in theory, makes digital signal processing (DSP) possible, because in principle, a continuous time signal can be perfectly captured by its (sufficiently dense uniformly spaced) discrete samples. However, there is a “minor glitch” there: if you look at the sampling expansion (2.4) we see that the interpolation functions decay only as \( 1/n \). Thus, to obtain a good reconstruction of the “analog waveform” of the signal at a non integer point \( t \), we need a very large number of samples around \( t \). Luckily, DSP (Digital Signal Processing) engineers have different, more clever ways of interpolating band limited signals which do not produce unwanted artifacts which would occur if we simply truncated the series (2.4) at any but extremely large number of terms.

The function \( \frac{\sin t}{t} \) is of cardinal importance to signal processing; in fact it is usually called the cardinal sine function or the sinc function and is denoted by \( \text{sinc}(t) \). Thus we have

\[
f(t) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(t-n) \tag{2.5}
\]

Since

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega t} d\omega = \frac{1}{2\pi} \frac{e^{i\omega t}}{it} \bigg|_{\omega=-\pi}^{\pi} = \frac{\sin \pi t}{\pi t} = \text{sinc} \, \pi t; \tag{2.6}
\]

we see that the Fourier transform \( \hat{\text{sinc}}(\omega) \) of \( \text{sinc} \, \pi t \) is given by

\[
\hat{\text{sinc}}(\omega) = \begin{cases} 
1 & \text{if } |\omega| < \pi \\
0 & \text{if } |\omega| > \pi
\end{cases} \tag{2.7}
\]

If we substitute \( t = 0 \) in (2.6) then, since \( \frac{1}{2\pi} \int_{-\pi}^{\pi} d\omega = 1 \), we get \( \text{sinc} \, 0 = 1 \); at all other integers \( n \neq 0 \) we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega n} d\omega = 0
\]

because we have \( n \) many complete periods of \( e^{i\omega n} \) within \([-\pi, \pi]\). Thus, if we sample \( \text{sinc} \, \pi t \) at integers we get 1 at \( t = 0 \) and 0 at all other integer sampling points, as one would expect from (2.5).

### 3 Fourier series in the base \( \{ \text{sinc} \, \pi(t-n) \}_{n \in \mathbb{N}} \)

Every function \( f(x) \) which is square integrable, i.e., such that

\[
\int_{-\infty}^{\infty} f^2(t)dt < \infty
\]
Figure 2.1: sinc $\pi t$ (left) and its Fourier transform (right)

(this is usually denoted by $f \in L^2$) has a Fourier transform $\hat{f}(\omega)$ such that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega;$$  \hfill (3.1)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt;$$  \hfill (3.2)

One can show that the mapping $F : f(t) \mapsto \hat{f}(\omega)$ has the following important property of isometry: for every two functions $f(t), g(t) \in L^2$

$$(f(t), g(t)) = \int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)\overline{g(\omega)}d\omega = \langle \hat{f}(\omega), \hat{g}(\omega) \rangle$$ \hfill (3.3)

and in particular,

$$\|f(t)\|^2 = (f(t), f(t)) = \int_{-\infty}^{\infty} f(t)^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)\overline{\hat{f}(\omega)}d\omega = \|\hat{f}(\omega)\|^2 \hfill (3.4)$$

Thus, $F$ preserves both the “lengths” (i.e., norms) of vectors as well as the angles between vectors, i.e., it preserves the “geometry” of the two spaces of signals, one in the time domain, the other in the “frequency” domain. Note that (2.6) implies

$$\text{sinc} \pi (t - n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(t-n)}d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega t}e^{i\omega n}d\omega; \hfill (3.5)$$

Thus,

$$F[\text{sinc} \pi (t - n)](\omega) = \begin{cases} e^{-i\omega n} & \text{if } |\omega| < \pi \\ 0 & \text{if } |\omega| \geq \pi \end{cases} \hfill (3.6)$$

---

1As we have already mentioned, this is not quite true, because there are functions $f \in L^2$ for which integral (3.1) diverges. This is circumvented by representing $f$ as a limit of functions $f_n$ which are both in $L^1$ and in $L^2$, and the Fourier Transform of such $f$ is defined as the limit of the Fourier Transforms of functions $f_n$. 


\[
\int_{-\infty}^{\infty} \text{sinc}(t-n) \text{sinc}(t-m) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega m} e^{i\omega n} d\omega = \begin{cases} 
1 & \text{if } n = m \\
0 & \text{if } n \neq m 
\end{cases}
\]

This means that functions \(\{\text{sinc}(t-k)\}_{k \in \mathbb{Z}}\) are orthonormal; in fact, they form an orthonormal basis of the vector space of band limited functions of finite energy, and the Shannon expansion (2.5) simply represents a signal \(f(t)\) in that base, i.e., is the Fourier series of \(f(t)\) with respect to the base \(\{\text{sinc}(t-k)\}_{k \in \mathbb{Z}}\).

To see that, using the isometry property, we have that the values of the projections of \(f(t)\) onto the base vectors \(\text{sinc}(t-n)\) are equal to:

\[
\int_{-\infty}^{\infty} f(t) \text{sinc}(t-n) dt = \langle f, \text{sinc}(t-n) \rangle = \langle \hat{f}, \text{sinc}(t-n) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{-i(-n)\omega} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) e^{in\omega} d\omega = f(n)
\]

Since for any orthonormal base \(\{b_n\}\) the corresponding Fourier series expansion is of the form

\[
f = \sum_{n=-\infty}^{\infty} \langle f, b_n \rangle b_n
\]

we get that

\[
f(t) = \sum_{n=-\infty}^{\infty} \langle f, \text{sinc}(t-n) \rangle \text{sinc}(t-n) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(t-n)
\]

Thus, we see that the values of a \(\pi\)-band limited signal are the coefficients of two Fourier series.

1. In the frequency domain (i.e., in the space of the Fourier transforms of the band limited signals) they are the Fourier coefficients of \(\hat{f}(\omega)\) in the usual base of complex exponentials \(\{e^{i\omega} : n \in \mathbb{Z}\}\):

\[
\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(-n) e^{i\omega} = \sum_{n=-\infty}^{\infty} f(n) e^{-i\omega}
\]

2. In the time domain, taking the Inverse Fourier Transform of both sides, we get the Shannon formula which is just the (generalised) Fourier series in the base \(\{\text{sinc}(t-k)\}_{k \in \mathbb{Z}}\):

\[
f(t) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(t-n)
\]

As mentioned, the Fourier transform defines an isometry between the two domains: the “length” of a vector (i.e., a signal) \(f(t)\) is the same if measured either in time or in the frequency domain, in the sense that

\[
\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 d\omega = \|\hat{f}(\omega)\|^2
\]
and the angle between two vectors is the same in both domains, because also

$$\angle(f(t), g(t)) = \frac{\langle f(t), g(t) \rangle}{\|f(t)\| \cdot \|g(t)\|} \quad \text{and} \quad \langle f(t), g(t) \rangle = \langle \hat{f}(\omega), \hat{g}(\omega) \rangle$$

We now establish yet another important isometry. Note that

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t)g(t)dt = \int_{-\infty}^{\infty} \left( \sum_{n=\infty}^{\infty} f(n) \text{sinc} \pi(t - n) \right) \left( \sum_{m=\infty}^{\infty} g(m) \text{sinc} \pi(t - m) \right) dt$$

$$= \sum_{n=\infty}^{\infty} \sum_{m=\infty}^{\infty} f(n)g(m) \int_{-\infty}^{\infty} \text{sinc} \pi(t - n) \text{sinc} \pi(t - m) dt$$

$$= \sum_{n=\infty}^{\infty} f(n)g(n)$$

where in the last step we used the fact that, by orthonormality of the base \{\text{sinc} \pi(t - n)\} we have

$$\int_{-\infty}^{\infty} \text{sinc} \pi(t - n) \text{sinc} \pi(t - m) dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \quad (3.11)$$

Since functions \text{sinc} \pi(t - n) form an orthonormal system, we get

$$\langle f(t), g(t) \rangle = \sum_{n=\infty}^{\infty} f(n)g(n)$$

In particular, we also get

$$\|f(t)\|^2 = \langle f(t), f(t) \rangle = \int_{-\infty}^{\infty} f(t)^2 dt = \sum_{n=\infty}^{\infty} f(n)^2$$

Thus, the space of \pi-band limited signals of finite energy is isometric to the space \text{l}^2 of square summable sequences, \text{l}^2 = \{a_n : \sum_{n=\infty}^{\infty} a_n^2 < \infty\}, where the numbers \(a_n\) can be interpreted as samples of a \pi-band limited signal of finite energy. Note that in such isometry function \text{sinc} \pi(t - n) is mapped to the sequence

$$e_n(m) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \quad (3.12)$$

Again, we have that in the base \{\epsilon_n : n \in \mathbb{Z}\} the Fourier series of the sequence of samples \(\vec{f} = (\langle f(n) \rangle_{n \in \mathbb{Z}}\) which corresponds to the band limited signal \(f(t)\) has the coefficients \(f(n)\), i.e.,

$$\vec{f} = \sum_{n=\infty}^{\infty} f(n)e_n. \quad (3.13)$$
Thus, we get the following isometric spaces:

**Frequency domain:**

\[ \hat{f}(\omega) \text{ base: } \{e^{in\omega} : n \in \mathbb{Z}\} \]

**Fourier series:**

\[ \hat{f}(\omega) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\omega} \]

**Time domain:**

\[ f(t) \text{ base: } \{\text{sinc} \pi(t-n) : n \in \mathbb{Z}\} \]

**Fourier series:**

\[ f(t) = \sum_{n=-\infty}^{\infty} f(n)\text{sinc} \pi(t-n) \]

Sequences from \( l^2 \)

\[ \{f(n)\} \text{ base: } \{e_n = (\ldots 0, 1, 0 \ldots) : n \in \mathbb{Z}\} \]

**Fourier series:**

\[ \vec{f} = \sum_{n=-\infty}^{\infty} f(n)e_n \]

### 4 Filters

So we now know how to represent band-limited signals, and would like to “process” them. For example, we might want to remove certain frequencies, such as the 50 Hz hum which could have come from the mains power supply, or the “hissing” from an old vinyl LPs. Such operations are examples of **filtering**.

In general, a filter \( L \) acts on signals \( f(t) \) producing an output signal \( L[f](t) \). Note that both the input signal \( f(t) \) and the output signal \( L[f](t) \) are taken as “wholes”, in their entire duration; filters do NOT produce outputs \( L[f](t_0) \) at any given instant of time \( t_0 \) from the value of \( f(t_0) \) alone. Thus, filters operate on signals as points of an appropriate inner product space of signals; for that reason they are also called **operators** acting on such spaces; to emphasis this, we place inputs of filters into square brackets: \( L[f] \); since the output is a signal which is a function of time we also write \( L[f](t) \) to denote the output signal.

Filters have to satisfy the following three properties:\(^1\)

1. \( L \) is **continuous**, i.e., if a sequence of signals \( f_n(t) \) converges to a signal \( f(t) \) in the sense of the norm, then the image of the limit signal is the limit of the images. More precisely,

\[
\lim_{n \to \infty} \|f_n(t) - f(t)\|_2 = 0 \quad \text{then also} \quad \lim_{n \to \infty} \|L[f_n](t) - L[f](t)\|_2 = 0.
\]

2. \( L \) is a **linear operator**, i.e., for arbitrary scalars \( c_1, c_2 \) and arbitrary two band limited signals \( f(t), g(t) \),

\[ L[c_1f + c_2g](t) = c_1L[f](t) + c_2L[g](t) \]

3. \( L \) is **time invariant** (also called **shift invariant**) if an image of a time shifted signal is the just an equally shifted image of the original signal. More precisely,

\(^1\)We will restrict ourselves to the space of \( \pi \)-band limited signals with the usual scalar product and the induced norm of \( L^2 \).
\[ L[f(t + \tau)] = L[f(t)](t + \tau) \]

What a filter does to an arbitrary signal in \( BL(\pi) \) is completely determined by what it does to the \( \text{sinc}(\pi t) \)

\[
L(f(t)) = L \left[ \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)} \right]
= \lim_{N \to \infty} L \left[ \sum_{k=-N}^{N} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)} \right] 
= \lim_{N \to \infty} L \left[ \sum_{k=-N}^{N} f(k) \frac{\sin \pi(t - k)}{\pi(t - k)} \right] \quad \text{(by continuity of } L) 
= \lim_{N \to \infty} \sum_{k=-N}^{N} f(k) L \left[ \frac{\sin \pi u}{\pi u} \right]_{u = (t - k)} \quad \text{(by linearity of } L) 
= \sum_{k=-\infty}^{\infty} f(k) L[\text{sinc} \pi u] (t - k)
\]

Here \( L[\text{sinc} \pi u] (t - k) \) stands for first obtaining the image \( l(t) = L[\text{sinc} \pi t] \) and then replacing \( t \) in \( l(t) \) with the shifted variable \( t - k \), i.e., \( L[\text{sinc} \pi u] (t - k) = l(t - k) \). Recall that by (2.7) the Fourier transform \( \hat{\text{sinc}}(\omega) \) of \( \text{sinc} \pi t = \frac{\sin \pi t}{\pi t} \) is given by

\[
\hat{\text{sinc}}(\omega) = \begin{cases} 
1 & -\pi \leq \omega \leq \pi \\
0 & \text{otherwise}
\end{cases}
\]

Consider the response of \( L \) for input \( \text{sinc} \pi t \),

\[ l(t) = L \left[ \frac{\sin \pi t}{\pi t} \right] \]

and let its Fourier transform be \( \hat{L}(\omega) \). Then

\[ l(t) = L \left[ \frac{\sin \pi t}{\pi t} \right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega)e^{i\omega t} d\omega \]

and by the shift-invariance of \( L \),

\[ l(t - k) = L[\text{sinc} \pi u] (t - k) = L[\text{sinc} \pi (t - k)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega)e^{i\omega(t - k)} d\omega. \]

Using the Shannon formula and the linearity of \( L \), for an arbitrary input
Using (2.2) we get

\[
L(f)(t) = \sum_{k=-\infty}^{\infty} f(k) L[\text{sinc}](t - k)
\]

\[
= \sum_{k=-\infty}^{\infty} f(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\omega(t-k)} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k} e^{i\omega t} \hat{L}(\omega) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \hat{L}(\omega) e^{i\omega t} d\omega.
\]

Thus, a linear operator acts on a signal by multiplying the Fourier transform of the signal by the transfer function of the operator, i.e. by the Fourier transform \( \hat{L}(\omega) \) of \( L \left( \frac{\sin \pi t}{\pi t} \right) \). Notice that

\[
L[f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{L}(\omega) e^{i\omega t} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} \hat{L}(\omega) e^{i\omega t} d\omega du
\]

\[
= \int_{-\infty}^{\infty} f(u) \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{L}(\omega) e^{i\omega(t-u)} d\omega du
\]

\[
= \int_{-\infty}^{\infty} f(u) l(t-u) du
\]

where

\[
l(t) = L \left( \frac{\sin \pi t}{\pi t} \right).
\]

The integral

\[
\int_{-\infty}^{\infty} f(u) l(t-u) du
\]

is called a convolution of \( f \) and \( l \), \( f \ast l \). Note that the convolution of two functions corresponds to a product of their Fourier transforms. We also have for every integer \( k \)

\[
L[f](k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{L}(\omega) e^{i\omega k} d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(n) e^{-i\pi n} \sum_{m=-\infty}^{\infty} l(m) e^{-i\pi m} e^{i\pi k} d\omega
\]

\[
= \sum_{n,m=-\infty}^{\infty} f(n) l(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-(n+m))\pi} d\omega
\]

However,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-(n+m))\pi} d\omega = \begin{cases} 
1 & \text{if } k = m + n \\
0 & \text{otherwise}
\end{cases}
\]
Thus, a summand is non zero only for $k = m + n$, i.e., only for $m = k - n$, and we get:

$$L[f](k) = \sum_{n=-\infty}^{\infty} f(n)l(k - n).$$