Advanced Algorithms
COMP4121

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Order Statistic
Problem: Given \( n \) elements, select the \( i^{th} \) smallest element;
- for \( i = 1 \) we get the minimum;
- for \( i = n \) we get the maximum;
- for \( i = \lfloor \frac{n+1}{2} \rfloor \) we get the median.

We can find both the minimum and the maximum in \( O(n) \) many steps (linear time).

Can we find the median also in linear time?

Clearly, we can do it in time \( n \log n \), just MergeSort the array and find the middle element(s) of the sorted array.

Can we do it faster???
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We will show that this can be done in linear time, by both a deterministic and by a randomised algorithm.

Why bother with a randomised algorithm if it can be done in linear time with a deterministic algorithm?

Because in practice the randomised algorithm runs much faster, having much smaller constant $c$ in the bound for the run time $T(n) \leq c \cdot n$.

It turns out that it is easier to solve (both deterministically and with randomisation) the more general problem of finding the $i^{th}$ smallest element for an arbitrary $i$ than to find just the median.
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RAND-SELECT(A, p, r, i)  *choose the $i^{th}$ smallest elt of $A[p..r]>*

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Analysis of **Rand-Select**($A, p, r, i$)

- Clearly, the worst case run time is $\Theta(n^2)$.
- This happens, for example, in a very unlikely event that you always pick either the smallest or the largest element of the array.
- In such a case during each call of **Rand-Select** the size of the array drops only by 1.
- Due to reshuffling of elements around the pivot, each iteration of **Rand-Select** costs the length of the array, and you get

$$T(n) = c(n + (n - 1) + (n - 2) + \ldots + 1) = \Theta(n^2)$$

- This is very unlikely to happen; in fact, as we will now see, most of the time the partitions will be reasonably well balanced.
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- Let us first assume that all the elements in the array are \textbf{distinct}.
- Let us call a partition a \textit{balanced partition} if the ratio between the number of elements in the smaller piece and the number of elements in the larger piece is not worse than 1 to 9 (9 is kind of arbitrary here, any small number > 2 would do).
- What is the probability that we get a balanced partition after choosing the pivot?
- Clearly, this happens if we chose an element which is neither among the smallest $1/10$ nor among the largest $1/10$ of all elements.
- Thus, the probability to end up with a balanced partition is $1 - 2/10 = 8/10$.
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- Thus, the probability to end up with a balanced partition is $1 - 2/10 = 8/10$.
- Let us find the expected number of partitions between two consecutive balanced partitions.
Let us first assume that all the elements in the array are distinct.

Let us call a partition a balanced partition if the ratio between the number of elements in the smaller piece and the number of elements in the larger piece is not worse than 1 to 9 (9 is kind of arbitrary here, any small number > 2 would do).

What is the probability that we get a balanced partition after choosing the pivot?

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Let us find the expected number of partitions between two consecutive balanced partitions.
The probability to get another balanced partition immediately after a balanced partition is $\frac{8}{10}$;

The probability to need two partitions to get another balanced partition is $\frac{2}{10} \cdot \frac{8}{10}$;

In general, the probability that you will need $k$ partitions to end up with another balanced partition is $\left(\frac{2}{10}\right)^{k-1} \cdot \frac{8}{10}$.

Thus, the expected number of partitions between two balanced partitions is

$$E = 1 \cdot \frac{8}{10} + 2 \cdot \frac{2}{10} \cdot \frac{8}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 \cdot \frac{8}{10} + \ldots$$

$$= \frac{8}{10} \cdot \sum_{k=0}^{\infty} (k+1) \left(\frac{2}{10}\right)^k = \frac{8}{10} S$$

where

$$S = 1 + 2 \cdot \frac{2}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 + 4 \cdot \left(\frac{2}{10}\right)^3 + 5 \cdot \left(\frac{2}{10}\right)^4 + \ldots$$
Analysis of \texttt{Rand-Select}(A, p, r, i)

- The probability to get another balanced partition immediately after a balanced partition is \( \frac{8}{10} \);
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Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

- How do we evaluate such a sum \( S \)?
- Trick #1:

\[
S = 1 + 2 \cdot \frac{2}{10} + 3 \cdot \left( \frac{2}{10} \right)^2 + 4 \cdot \left( \frac{2}{10} \right)^3 + 5 \cdot \left( \frac{2}{10} \right)^4 + \ldots
\]

\[
= 1 + \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots
\]

\[
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Evaluating $S = \sum_{k=0}^{\infty} (k + 1)(\frac{2}{10})^k$

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Evaluating $S = \sum_{k=0}^{\infty} (k + 1) \left(\frac{2}{10}\right)^k$

- Summing each row separately we obtain

\[
1 + \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \ldots = \frac{1}{1 - \frac{2}{10}} = \frac{10}{8}
\]

\[
+ \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \ldots = \frac{2}{10} \frac{10}{8}
\]

\[
+ \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \ldots = \left(\frac{2}{10}\right)^2 \frac{10}{8}
\]

\[
+ \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \ldots = \left(\frac{2}{10}\right)^3 \frac{10}{8}
\]

\[\ldots\]

- We can now sum the right hand side column:
Evaluating \( S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \)

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\[
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\[
S = \frac{10}{8} \left( 1 + \frac{2}{10} + \left( \frac{2}{10} \right)^2 + \left( \frac{2}{10} \right)^3 + \left( \frac{2}{10} \right)^4 + \ldots \right) \\
= \frac{10}{8} \frac{1}{1 - \frac{2}{10}} = \left( \frac{10}{8} \right)^2
\]

Thus, we obtain

\[
E = \frac{8}{10} S = \frac{8}{10} \left( \frac{10}{8} \right)^2 = \frac{10}{8} = \frac{5}{4} < 2
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A useful digression (Trick #2): \[ S = \sum_{k=0}^{\infty} (k + 1) \left( \frac{2}{10} \right)^k \]
evaluated another way.

Note that

\[ \sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}. \]

By differentiating both sides with respect to \( q \) we get

\[ \sum_{k=1}^{\infty} k q^{k-1} = \frac{1}{(1 - q)^2}. \]

Substituting \( q = 2/10 \) we get that \( S = (10/8)^2 \).
Performance of **Rand-Select:**

- So, on average, there are only $5/4$ partitions between two balanced partitions.
- Note that before the first balanced partition the size of the array can be bounded by $n$; after the first balanced partition the size of the array is $\leq 9/10 \times n$, after the second balanced partition the size of the array is $\leq (9/10)^2 \times n$ and so on...
- Consequently the total **average** (expected) run time satisfies

$$T(n) < \frac{5}{4} n + \frac{5}{4} \frac{9}{10} n + \frac{5}{4} \left(\frac{9}{10}\right)^2 n + \frac{5}{4} \left(\frac{9}{10}\right)^3 n + \ldots$$

$$= \frac{5/4 n}{1 - \frac{9}{10}} = \frac{50}{4} n = 12.5 n$$

- Where did we tacitly assume that all elements are distinct?
- How did we estimate the probability of choosing the pivot which results in a balanced partition?
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  \]

  \[
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  $$= \frac{5/4 \cdot n}{1 - \frac{9}{10}} = \frac{50}{4} \cdot n = 12.5 \cdot n$$

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Performance of **Rand-Select**:

- So, on average, there are only $\frac{5}{4}$ partitions between two balanced partitions.
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Performance of \textbf{Rand-Select}:

- Note that if all elements are the same \texttt{Rand-Select} would run in quadratic time no matter which elements are chosen as pivots - they are all equal.

- **Homework:** Modify \texttt{Rand-Select} so that it runs in linear time even when there are many repetitions.

- You might want to modify slightly the way how the array is reordered.

- In practice \texttt{Rand-Select} runs really fast, for the same reason the Randomised QuickSort runs fast: it can be implemented as a very tight loop.

- In 1972 Blum, Floyd, Pratt, Rivest and Tarjan designed a deterministic Order Statistic Selection which runs in linear time in the worst case.

- **Main idea:** Use a recursive call of the very same algorithm to choose a good pivot!
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Algorithm Select \((n, i)\):

- Split the numbers in groups of five (the last group might contain less than 5 elements);
- Order each group by brute force in an increasing order.
- Take the collection of all \(\lfloor \frac{n}{5} \rfloor\) middle elements of each group (i.e., the medians of each group of five).
- Apply recursively SELECT algorithm to find the median \(p\) of this collection.
Deterministic Linear Time Algorithm for Order Statistic

- **Algorithm Select** $(n, i)$:
  - Split the numbers in groups of five (the last group might contain less than 5 elements);
  - Order each group by brute force in an increasing order.
  
  ![Diagram of sorting elements in groups]

  - Take the collection of all $\lfloor \frac{n}{5} \rfloor$ middle elements of each group (i.e., the medians of each group of five).

  ![Diagram of selecting medians]

  - Apply recursively SELECT algorithm to find the median $p$ of this collection.
Algorithm Select\((n, i)\) continued:

- partition all elements using \(p\) as a pivot;
- Let \(k\) be the number of elements in the subset of all elements smaller than the pivot \(p\).
- if \(i = k\) then return \(p\)
- else if \(i < k\) then recursively SELECT the \(i^{th}\) smallest element of the set of elements smaller than the pivot.
- else recursively SELECT the \((i - k)^{th}\) smallest element of the set of elements larger than the pivot.

Note: This algorithm is the same as RAND-SELECT except for the way how we chose the pivot.
- instead of choosing pivot randomly we called recursively the very same algorithm to pick the pivot as the median of the middle elements of the groups of five elements.
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Deterministic Linear Time Algorithm for Order Statistic

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- **Note:** This algorithm is the same as **RAND-SELECT** except for the way how we chose the pivot.
  - instead of choosing pivot randomly we called recursively the very same algorithm to pick the pivot as the median of the middle elements of the groups of five elements.
What have we accomplished by such a choice of the pivot?

Note that at least \( \lceil (n/5)/2 \rceil = \lceil n/10 \rceil \) group medians are smaller or equal to the pivot; and at least that many larger than the pivot.

But this implies that at least \( \lceil 3n/10 \rceil \) of the total number of elements are smaller than the pivot, and that many elements...
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Deterministic Linear Time Algorithm for Order Statistic

- What is the run time of our algorithm?

\[ T(n) \leq T(n/5) + T(7n/10) + Cn. \]

- Let us show that \( T(n) < 11Cn \) for all \( n \). Assume that this is true for all \( k < n \) and let us prove it is true for \( n \) as well.
  - Note: this is a proof using the following form of induction:
    \( \varphi(0) \land (\forall n)((\forall k < n)\varphi(k) \rightarrow \varphi(n)) \rightarrow (\forall n)\varphi(n). \)
  - Thus, assume \( T(n/5) < 11C \cdot n/5 \) and \( T(7n/10) < 11C \cdot 7n/10 \); then

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\[
= 109 \frac{Cn}{10} < 11C \cdot n
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which proves out statement that \( T(n) < 11C \cdot n \).
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Deterministic Linear Time Algorithm for Order Statistic

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Note that this algorithm is a genuine recursion (rather than just an iteration) so its execution involves lots of traffic on the stack, which makes this algorithm slow in practice; the randomised version of it, RAND-SELECT, significantly outperforms it.

Similarly RAND-QUICKSORT in practice outperforms MERGESORT, which, unlike RAND-QUICKSORT, is guaranteed to run in time $O(n \log n)$. 
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