Skip Lists
A recent data structure - introduced in 1989 by William Pugh.

Yes, I know, it is recent for people of my age, not at all recent for you ...

A randomised data structure with benefits of balanced trees (e.g., AVL or Red - Black trees):
- $O(\log n)$ expected time for INSERT and SEARCH;
- $O(1)$ time for MIN, MAX, SUCC, PRED;
- Can be enhanced so that finding the $k^{th}$ element in the list also runs in $O(\log n)$ time.

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- Consider a doubly linked list:

- \( \text{MIN, MAX, SUCC, PRED} \) run in time \( O(1) \).
- However, \( \text{SEARCH, INSERT, DELETE} \) run in time \( O(n) \).
- The culprit is searching.
- Can we modify doubly linked links to make search \( O(\log n) \) expected time?
- **Idea:** make shortcuts on several levels:

This is something like the express elevators in skyscrapers which do not stop on every floor.
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  ![Diagram of a doubly linked list]

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Searching for $k$:

Start from the head $H$ and go as far right as you can, without exceeding $k$, using the highest possible level of links;
then drop one level down and repeat the procedure using lower level links.

How can we ensure such a search procedure runs in time $O(\log n)$?
Can we link every other link on the second level, every fourth link on the third level, every eight on the fourth level and so on...
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Problem: insertions and deletions will destroy such a structure...
We need dynamically self balancing structure.
This is where randomisation comes into play, ensuring that in the long run the structure remains (essentially) balanced.

To Insert $k$: first search to find the right place. Then toss a coin until you get a head, and count the number of tails $t$ that you got. Insert $k$ and link it at levels $0 - t$ from the bottom up.
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Deleting an element is just like in a standard doubly linked list, but taking care of all pointers affected.

How fast can we search for an element?

The probability of getting $i$ consecutive tails when flipping a coin $i$ times is $1/2^i$.

Thus, an element has links on levels $0 - i$ (and possibly also on higher levels) with probability $1/2^i$.

If $n$ elements belong to a set with a probability $p$ each, then the expected size of that set is $np$.

Thus, an $n$ element Skip List has on average $n/2^i$ elements with links on level $i$.

Since an element has links only on levels $0 - i$ with probability $1/2^{i+1}$, the total expected number of link levels per element is

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\sum_{i=0}^{\infty} \frac{i + 1}{2^{i+1}} = \sum_{i=1}^{\infty} \frac{i}{2^i} = 2
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Let \( \#(i) \) denote the number of elements on level \( i \).

Since the expected number of elements having a link at level \( i \) is \( E[\#(i)] = \frac{n}{2^i} \), by the Markov inequality the probability of having at least one element at level \( i \) satisfies

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P(\#(i) \geq 1) \leq \frac{E[\#(i)]}{1} = \frac{n}{2^i}
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Thus, the probability to have an element on level \( 2 \log n \) is smaller than \( \frac{n}{2^{2 \log n}} = \frac{n}{2^{\log n^2}} = \frac{n}{n^2} = 1/n \).

More generally, the probability to have an element on level \( k \log n \) is smaller than \( \frac{n}{2^{k \log n}} = \frac{n}{2^{\log n^k}} = \frac{n}{n^k} = 1/n^{k-1} \).

Thus, the probability that level \( k \log n \) is nonempty is smaller than \( 1/n^{k-1} \).

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\[ E \leq \sum_{k=1}^{\infty} \frac{k}{n^k-1} = \left( \frac{n}{n-1} \right)^2 \]

(using the same tricks to evaluate such a sum)

- Thus, the expected number of levels is barely larger than \( \log n \).
- If an element has a link at a level \( i \) then with probability \( 1/2 \) it also has a link at level \( i + 1 \).
- Thus, the expected number of elements between any two consecutive elements with a link on level \( i + 1 \) which have links only up to level \( i \) is smaller than

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\frac{0}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \ldots = 1
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So once on level \( i \), on average we will have to inspect only two elements on that level before going to a lower level.
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- If an element has a link at a level \( i \) then with probability \( 1/2 \) it also has a link at level \( i + 1 \).
- Thus, the expected number of elements between any two consecutive elements with a link on level \( i + 1 \) which have links only up to level \( i \) is smaller than

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\frac{0}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \ldots = 1
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So once on level \( i \), on average we will have to inspect only two elements on that level before going to a lower level.
\[
E \leq \sum_{k=1}^{\infty} \frac{k}{n^{k-1}} = \left( \frac{n}{n-1} \right)^2
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To summarise, on average, there will be fewer than $2 \log n$ levels to go down, with visiting on average only two elements per each level.

Consequently, on average, the search will be in time $O(\log n)$.

For an element with links on levels $0 - i$ we have to store $2(i + 1)$ pointers to other elements and the expected number of elements with highest link on level $i$ is $O(n/2^{i+1})$. Thus, total expected space for all pointers does not exceed

$$O \left( \sum_{i=0}^{\infty} 2(i + 1) \frac{n}{2^{i+1}} \right) = O \left( 2n \sum_{i=0}^{\infty} \frac{i + 1}{2^{i+1}} \right) = O(4n) = O(n)$$

Unless we must ensure that the worst case performance of search is $O(\log n)$, Skip Lists are a better option than BST.
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Homework:

- Note that accessing the $k^{th}$ largest element is still $O(n)$.
- Add something to the structure so that accessing the $k^{th}$ largest element is also $O(\log n)$ expected time.
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