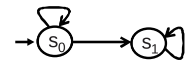
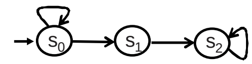
Algorithmic Verification

Comp4151
Lecture 5-A
Ansgar Fehnker

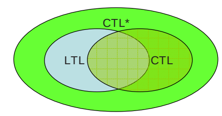
Overview

Modelling
- Finite automata
- Büchi automata
- Kripke structures





Specification
- Linear Time Logic
- Computation Tree Logic
- CTL*



Overview

Explicit state CTL
- Bottom-up recursive labelling algorithm
- Linear in the size of Kripke structure and formula

However
- Suffers from state explosion problem

Fixpoint characterization of CTL
- Leads to set based algorithm

Overview

Semantic of CTL with fixpoints

Given Kripke structure $M=(S, s_0, \rightarrow, \mu)$, with n states, and ϕ in ENF over atomic propositions AP.

- $[|true|] = S$
- $[|false|] = \emptyset$
- $[|\ p\ |] = \{ s \mid p \in \mu(s) \}$
- $[|\neg \phi\ |] = S \setminus [|\ \phi\ |]$
- $[|\ \phi \wedge \psi\ |] = [|\ \phi\ |] \cap [|\ \psi\ |]$
- $[|\ EX\ \phi\ |] = \{ s \mid \exists (s,s') \in \rightarrow s$
- $[|\ E\ \phi\ U\ \psi\ |] = f_{EU}{}^{n}(\emptyset)$
- $[|\ EG\ \phi\ |] = f_{EG}{}^{n}(S)$

- Translates to an algorithm based on sets
- Sets of states as unordered list lists are inefficient
- We need
  - Compact set representation
  - Efficient operations on sets

Binary Decision Diagrams

A longer diversion

With examples and illustrations by Andreas Jacobs and Massimo Benerecetti

Boolean Functions

Definition

A *boolean variable* is variable over values {0,1}.

A *boolean function* over n arguments is a function
f: $\{0,1\}^n \rightarrow \{0,1\}$

Two common representations of boolean functions
- propositional formula
- truth table

Boolean Function

Example:

$f(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge x_3$

compact, but satisfiablity and comparison hard

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	0	0	0
1	0	1	1

exponential size: 2^n lines for n variables

Boolean Function

Binary Decision Tree

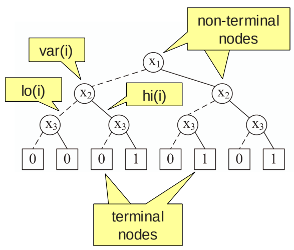
- Represents boolean function as a decision tree
- Non-terminal node i labeled with
  - variable var(i)
  - successors lo(i) and hi(i)
- Terminal nodes are labeled 0 or 1
- Fast lookup of $f(x_1, \ldots, x_n)$ given $x_1, \ldots, x_n$

Boolean Functions

Binary Decision Tree

$f(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge x_3$

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	0	0	0
1	0	1	1

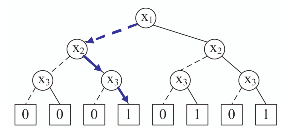


Boolean Functions

Binary Decision Tree

$f(0, 1, 1) = (0 \vee 1) \wedge 1$

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	0	0	0
1	0	1	1

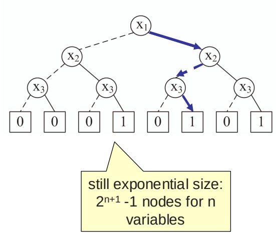


Boolean Functions

Binary Decision Tree

$f(1, 0, 1) = (1 \vee 0) \wedge 1$

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	0	0	0
1	0	1	1



Boolean Functions

Binary Decision Diagrams

Represents boolean functions as *directed acyclic graph* with a single *initial node*

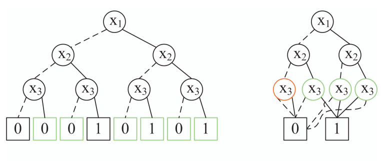
More compact representation than decision trees

Omit duplicated nodes and unnecessary tests.

Binary Decision Diagrams

Reduction

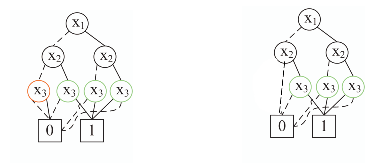
Rule 1: Share identical terminal nodes.



Binary Decision Diagrams

Reduction

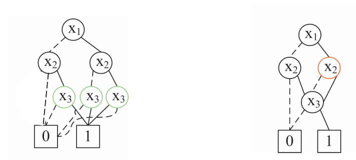
Rule 2: Remove redundant tests



Binary Decision Diagrams

Reduction

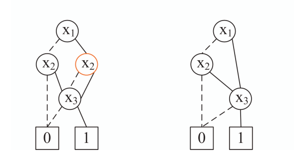
Rule 3: Share identical non-terminal nodes.



Binary Decision Diagrams

Reduction

Rules applied iteratively



Binary Decision Diagrams

Definitions

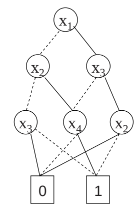
A BDD is *reduced* if none of reduction rules can be applied

Given a variable order $x_1 < \ldots < x_n$ a BDD is *ordered* if for each node i holds

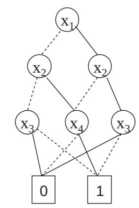
$$j \in \{ lo(i), hi(i) \} \text{ implies } x_i < x_j$$

Binary Decision Diagrams

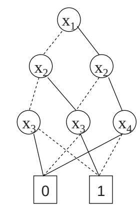
Examples



BDD



OBDD



ROBDD

OBDDs

Theorem

Two reduced ordered BDDs with respect to the same order $x_1 < \ldots < x_n$ represent the same boolean function iff they have the same structure.

Furthermore
- A function is a *tautology* if its ROBDD *u* is *equal* to 1.
- A function is a *satisfiable* if its ROBDD *u* is *not equal* to 0

ROBDDs provide a canonical for boolean functions

OBDDs

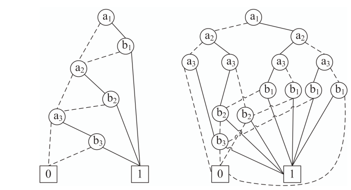
Variable ordering

- Size of an (reduced) OBDD depends on the order

OBDDs

Variable ordering

Example: $(a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee (a_3 \wedge b_3)$



$a_1 < b_1 < a_2 < b_2 < a_3 < b_3$ $\quad$ $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$

OBDDs

Variable ordering

- Size of an (reduced) OBDD depends on the order
- Checking a variable order for optimality is NP-hard
- For some formulas size is exponential for any order
- In practice these cases rarely occur
- There are good heuristics to find good orders

OBDDs

Intermediate Summary

OBDDs provide a compact representation for boolean functions (most of the times)

Satisfiability and comparison are easy to check

However:
- What about other operations on OBDDs?
- What is the relation with model checking?

Operations on OBDDs

Reduce

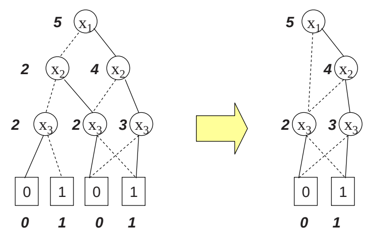
Bottom-up labelling based on the reduction rules

Label nodes with integer numbers as follows
- two terminal nodes with the same value get the same label
- if lo(i)=hi(i) replace set label(i) to label(lo(i))
- if var(i)=var(j) and lo(i)=lo(j) and hi(i)=hi(i) for some j set label(i) to label(j)
- otherwise label(i) with next unused integer

After labelling redirect edges bottom-up

Operations on OBDDs

Reduce



Operations on OBDDs

Restrict

Restrict variable x of function f to constant value b,
i.e. replace all occurrences of x by b.

Example

$f(x_1, x_2, x_3) \quad = (x_1 \vee x_2) \wedge x_3$
$f[0/x_2]\ (x_1, x_2, x_3) \quad = (x_1 \vee 0\ ) \wedge x_3 \ = x_1 \wedge x_3$
$f[1/x_2]\ (x_1, x_2, x_3) \quad = (x_1 \vee 1\ ) \wedge x_3 \ = x_3$

Operations on OBDDs

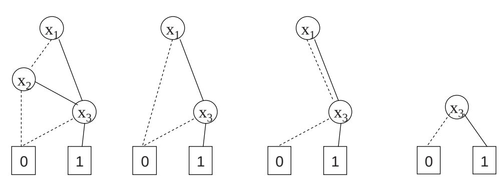
Restrict

Restrict variable x of function f to constant value b,
i.e. replace all occurrences of x by b.

For any node i pointing to a node j with var(j)=x
- repalce pointer of i by lo(j) if b=0
- replace pointer of j by hi(j) if b=1

Operations on OBDDs

Restrict



f $\quad$ $f[0/x_2]$ $\quad$ $f[1/x_2]$ $\quad$ reduce($f[1/x_2]$)

Operations on OBDDs

Apply

$\circ$ can be any boolean operation: and, or, xor, …

Given OBDDs B_f and B_g for boolean formulas f and g.
Apply($\circ$, B_f, B_g) computes the OBDD of $f \circ g$

Basic Idea

- Let x be smallest variable of B_f and B_g (i.e root in B_f or B_g)
- Split into two smaller sub-problem for x=0 and x=1.
- Repeat until leaves are reached. Apply $\circ$ to the leaves.

Recursive application of *Shannon Expansion*

$$f = (\neg x_i \wedge f[0/x_i]) \vee (x_i \wedge f[1/x_i])$$

Operations on OBDDs

Apply

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Recursive application of *Shannon Expansion*

$$f \circ g = (\neg x_i \wedge (f[0/x_i] \circ g[0/x_i])) \vee (x_i \wedge (f[1/x_i] \circ g[1/x_i]))$$

Operations on OBDDs

Apply

Given OBDDs B_f and B_g for boolean formulas f and g.
Let i_f and i_g be the root nodes of B_f and B_g.

Rule 1:

if i_f and i_g are non-terminal nodes and $var(i_f) = var(i_g)$
$\Rightarrow$ label current node with $var(i_f)$
create a low edge to apply($\circ$,lo(i_f),lo(i_g)) — $f[0/x_i] \circ g[0/x_i]$
create a hi edge to apply($\circ$,hi(i_f),hi(i_g)) — $f[1/x_i] \circ g[1/x_i]$

Operations on OBDDs

Apply

Given OBDDs B_f and B_g for boolean formulas f and g.
Let i_f and i_g be the root nodes of B_f and B_g.

Rule 2:

if i_f is a non-terminal and $var(i_f) < var(i_g)$ or i_g a terminal
$\Rightarrow$ label current node with $var(i_f)$
create a low edge to apply($\circ$,lo(i_f),i_g) — $f[0/x] \circ g$
create a hi edge to apply($\circ$,hi(i_f),i_g) — $f[1/x] \circ g$

$var(i_f) < var(i_g)$ implies $x = var(i_f)$ is not in g, thus $g[0/x] = g$

Operations on OBDDs

Apply

Given OBDDs B_f and B_g for boolean formulas f and g.
Let i_f and i_g be the root nodes of B_f and B_g.

Rule 3:

- if i_f is a non-terminal and $var(i_f) > var(i_g)$ or i_g a terminal
- $\Rightarrow$ Similar to Rule 2

Operations on OBDDs

Apply

Given OBDDs B_f and B_g for boolean formulas f and g.
Let i_f and i_g be the root nodes of B_f and B_g.

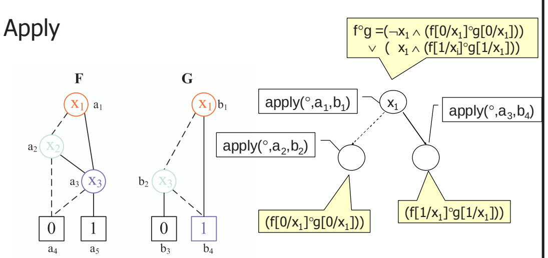
Rule 4:

- if i_f and i_g are terminal nodes labelled b_f and b_g
- $\Rightarrow$ label current node with $b_f \circ b_g$

0 or 1

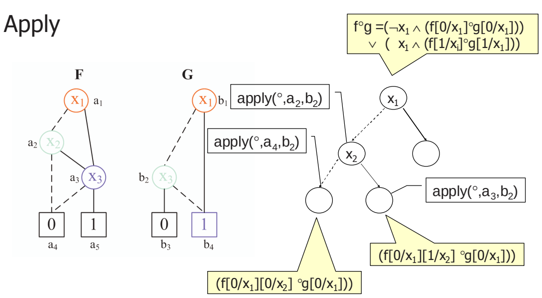
Operations on OBDDs

Apply



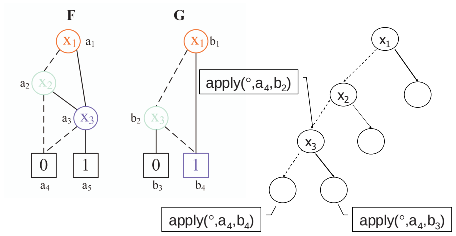
Operations on OBDDs

Apply



Operations on OBDDs

Apply



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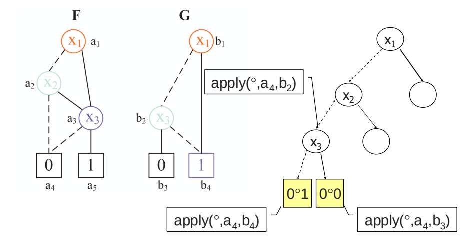
Operations on OBDDs

Apply



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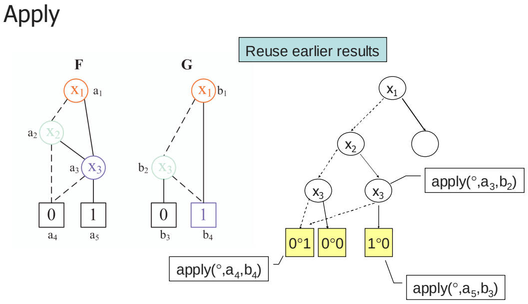
Operations on OBDDs

Apply



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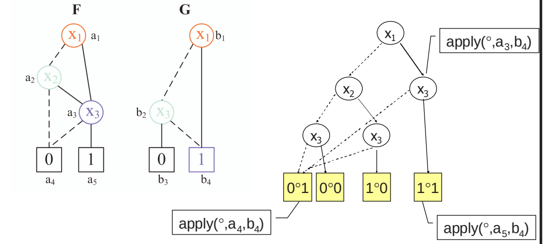
Operations on OBDDs

Apply



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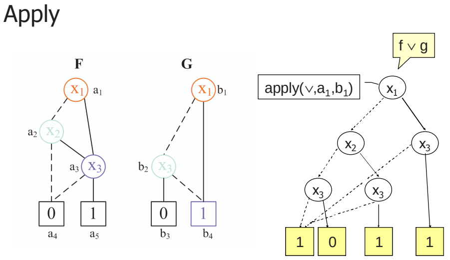
Operations on OBDDs

Apply

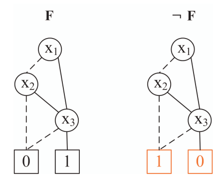


Reduce to obtain ROBDD

Operations on OBDDs

Negate

Given boolean functions f and reduced OBDD B_f, compute ¬f, by swapping the leaves.



Operations on OBDDs

Exist

Given boolean functions f variable x and reduced OBDD B_f, compute ∃x.f.

We have

$$\exists x.f = f[0/x] \vee f[1/x]$$

⇒ We can use an combination of *Restrict* and *Apply*

More efficiency by exploiting common structure in f[0/x] and f[1/x]

Operations on OBDDs

Some complexities

Operation	Complexity
Reduce	O(F)
Negate	constant
Apply	O(|F|*|G|)
Exists	$O(\|F\|*\|G\|*2^{2n})$

Model Checking and OBDDs

Another Intermediate Summary

- OBDDs provide a compact representation for boolean functions (most of the times)
- Satisfiability and comparison are easy to check
- Efficient algorithms for boolean operations

- However:
  - What is the relation with model checking?

Symbolic CTL Model Checking

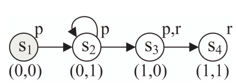
Observations

- *Boolean function* f: $\{0,1\}^n \rightarrow \{0,1\}$ can be used to represent *subsets* of $\{0,1\}^n$
- Each finite set (of states) can be mapped to $\{0,1\}^n$
- Boolean function can be presented as OBDD.
- Given Kripke structure $M = (S, S_0, R, L)$
  - Set of states S
  - Set of intitial states S_0
  - Transition relation R is set of pairs of states
  - Labelling function L, L^{-1} maps labels to sets of states
- States, intial states, transition relation and labeling can be represented as OBDDs

Symbolic CTL Model Checking

State space

- Each state can be represented as vector of boolean values
- The set of initial states can be represented by a boolean function

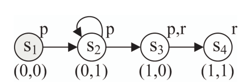


Example
- $S_0 = \{(x_1,x_2) | \neg x_1 \wedge \neg x_2\}$

Symbolic CTL Model Checking

Transition relation

- Each transition can be represented pairs of states
- The transition relation can be represented by a boolean function over $x_1,\ldots, x_n, x_1{}',\ldots, x_n{}'$



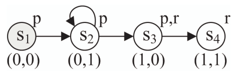
Example
- (s_1,s_2) as ((0,0),(0,1))
- ((0,0), (0,1)) satsifies $(\neg x_1 \wedge \neg x_2 \wedge \neg x'_1 \wedge x'_2)$
- R is the disjunction of boolean functions for all transitions

Symbolic CTL Model Checking

Labelling

- L^{-1} maps labels to sets of states
- This set can be represented as boolean function

Example



- $L^{-1}(p)=\{(0,0),(0,1),(1,0)\}$
- $L^{-1}(p)$ is represented by $(\neg x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2) = \neg x_2 \vee (\neg x_1 \wedge x_2)$

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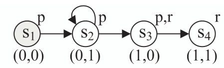
Symbolic CTL Model Checking

Symbolic Representation

Given M = (S, S0, R, L) over set of atomic proposition AP



Explicit representation

$AP = \{p, r\}$
$S = \{s_1, s_3, s_2, s_4\}$
$S_0 = \{s_1\}$
$R = \{(s_1, s_2), (s_2, s_2), (s_2, s_3), (s_3, s_4)\}$
$L(s_1) = \{p\}$ $L(s_2) = \{p\}$
$L(s_3) = \{p,r\}$ $L(s_4) = \{p\}$

Symbolic representation

$s_1 \triangleq (0,0)$ $s_2 \triangleq (0,1)$
$s_3 \triangleq (1,0)$ $s_4 \triangleq (1,1)$
$R = \{ ((0,0), (0,1)), ((0,1), (0,1)), ((0,1), (1,0)), ((1,0), (1,1)) \}$
$L^{-1}(p) = \{(0,0), (0,1), (1,0)\}$
$L^{-1}(r) = \{(1,0), (1,1)\}$

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Symbolic CTL Model Checking

OBDD operations

Booelan operators

$[\|true\|] = S$	OBDD with leaf 1
$[\|false\|] = \emptyset$	OBDD with leaf 0
$[\| p \|] = \{ s \mid p \in \mu(s) \}$	OBDD for $L^{-1}(p)$
$[\| \neg \phi \|] = S \setminus [\| \phi \|]$	Negate(B_ϕ)
$[\| \phi \wedge \psi \|] = [\| \phi \|] \cap [\| \psi \|]$	Apply($\wedge$, B_ϕ, B_ψ)

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Symbolic CTL Model Checking

OBDD operations

Temporal Operators

- $[\| EX\, \phi \|] = \{ s \mid \exists (s,s') \in \rightarrow$ such that $s' \in [\| \phi \|] \}$
- $\exists x'_1,\ldots,x'_n (f_\phi(x'_1,\ldots,x'_n) \wedge f_R(x_1,\ldots,x_n,x'_1,\ldots,x'_n))$
- Impleme[OBDD for ϕ] combi[OBDD for transition relation R] Exists

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Symbolic CTL Model Checking

OBDD operations

Temporal Operators

- [| E ϕ U ψ |] is least fixpont of
  $f_{EU}(P)=[|\psi|] \cup ([|\phi|] \cap [|EX\ P|])$

- Implemented by combination of Apply and Exists

Symbolic CTL Model Checking

OBDD operations

Temporal Operators

- [| EG ϕ |] is the greatest fixpoint of
  $f_{EG}(P) = [|\phi|] \cap [|EX\ P|]$

- Implemented by combination of Apply and Exists

Symbolic CTL Model Checking

Summary

- Fixpoint characterization of CTL translates to an algorithm based on sets
- Set in general and Kripke structures in particular can be modelled with boolean function
- OBDDs provide a canonical representation of boolean functions that provides
  - Compact representation of sets
  - Efficient operations on sets
- Next lecture: SMV