Temporal Logic

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Outline

- Why not "standard" logic?
- What is temporal logic?
- LTL
- CTL*
- CTL
- Fairness

Model Checking Problem

$$M \overset{?}{\models} \phi$$

model, program	satisfies, Implements, refines	property, specification

How to formalize the different components?

for this lecture we assume is M given as a Kripke structure, what about $\models$ and ϕ ?

Kripke Structure

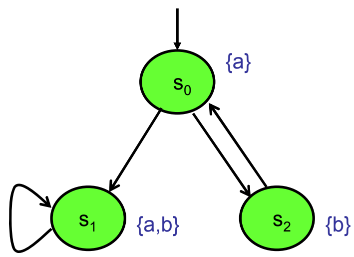
Given a set of atomic propositions AP.

Kripke structure $M=(S, s_0, \rightarrow, \mu)$ is defined by

- S set of states
- s_0 initial state
- $\rightarrow \subseteq S \times S$ transition relation (total)
- $\mu: S \rightarrow 2^{AP}$ labeling function

Any infinite run is accepting, i.e., like Buchi automaton where every state is a final state. Product as for NFA.

Example



How to define properties formally?

- Kripke structure
- automata
- ω regular expression
- logics

Logic can provide succinct notation, "close" to natural language.

Propositional Logic

Syntax

$$\phi ::= p \mid \neg\phi \mid \phi_1 \vee \phi_2$$

Other connectivities ($\wedge$, $\Leftrightarrow$, $\Rightarrow$, …) can be derived (see next slide)

Propositional Logic

Semantics

Given a state s in a Kipke structure M
we define $M,s \models \phi$ inductively by:

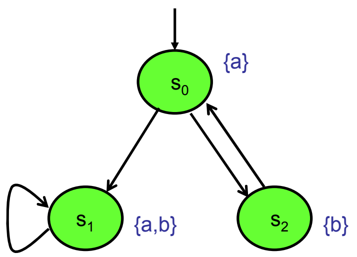
$M,s \models p$	$\Leftrightarrow$	$p \in \mu(s)$
$M,s \models \neg\phi$	$\Leftrightarrow$	not $M,s \models \phi$
$M,s \models \phi_1 \vee \phi_2$	$\Leftrightarrow$	$M,s \models \phi_1$ or $M,s \models \phi_2$

($\wedge$, $\Leftrightarrow$, $\Rightarrow$, true,false…)

Propositional logic is good at describing "static" situations.

Example

- M,s_0 ⊨ a∨b
- M,s_1 ⊨ a
- M,s_2 ⊨ ¬a



Propositional logic is good for describing “static” situations.

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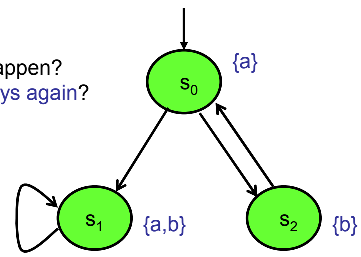
Example

How to describe:
- eventually b will happen?
- a will happen always again?



Propositional logic is unsuitable for describing “dynamic” behavior.

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Dynamic Behavior

Important for reactive systems

- security protocols
- hardware
- operating systems
- embedded systems
- ...

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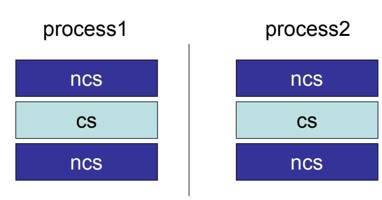
Mutual Exclusion



always only one process in cs
eventually process1 in cs
always eventually process2 in cs

cs = critical section
ncs = none critical section

LTL

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Temporal Logics

- originate from philosophy
- how to express statements including time?
- what is an appropriate model?
  - real-time vs discrete time
  - linear time vs branching time
    (deterministic vs non-deterministic)
  - ...

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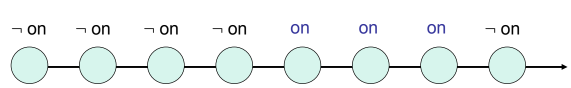
(P)LTL

- Propositional Linear time Temporal Logic
- discrete time
- linear (deterministic) progression



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LTL Syntax

- PLTL formula are inductively defined by:

$$\phi ::= p \mid \neg \phi \mid \phi_1 \vee \phi_2 \mid X\phi \mid F\phi \mid G\phi \mid \phi_1 U \phi_2$$

- p denotes atomic proposition
- X denotes next-state operator
- F denotes eventually/finally
- G denotes always/globally
- U denotes until

LTL semantics

- LTL formula ϕ interpreted over infinite paths of states $\pi = s_0 s_1 s_2 \ldots$
- we define LTL wrt Kripke structure M
- $M,\pi \models \phi$ denotes ϕ holds in a path π of Kripke structure M.
- $M \models \phi$ iff all paths of M satisfy ϕ, i.e., for all π in M we have $M,\pi \models \phi$

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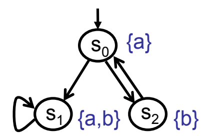
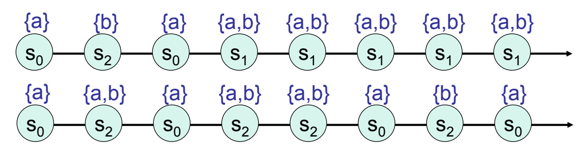
Paths in Kripke Structures



remember transition relation is total



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Semantics of LTL Operators

let π^k denote suffix $s_k s_{k+1} s_{k+2} \ldots$ where $k \geq 0$

$M,\pi \models p$ iff $s_0 \models p$, i.e., $p \in \mu(s_0)$
$M,\pi \models \neg\phi$ iff not $M,\pi \models \phi$
$M,\pi \models \phi_1 \vee \phi_2$ iff $M,\pi \models \phi_1$ or $M,\pi \models \phi_2$
$M,\pi \models X\phi$ iff $M,\pi^1 \models \phi$, i.e., $s_k s_{k+1} s_{k+2} \ldots$ satisfies ϕ
$M,\pi \models F\phi$ iff $\exists\, k \geq 0$ s.t. $M,\pi^k \models \phi$
$M,\pi \models G\phi$ iff $\forall k \geq 0$ $M,\pi^k \models \phi$
$M,\pi \models \phi_1 U \phi_2$ iff $\exists \geq 0$ s.t. $M,\pi^k \models \phi_2$ and $\forall 0 \leq j < k$ we have $M,\pi^j \models \phi_1$

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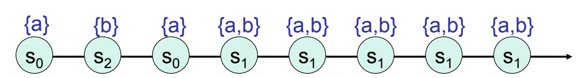
Example



path π satisfies:

- $M,\pi \models a$
- $M,\pi \models \neg b$
- $M,\pi \models Xb$
- $M,\pi \models Fa$
- $M,\pi \models FG(a \wedge b)$
- $M,\pi \models F(bUa)$
- $M,\pi \models FG(aUb)$
- what else?

Exercise

Which temporal operators can be expressed through one or more of the others?

Which cannot?

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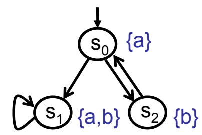
$M \models \phi$

$M \models \phi$ iff $M, \pi \models \phi$ for all paths π



Which properties satisfy this Kripke structure?

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CTL*

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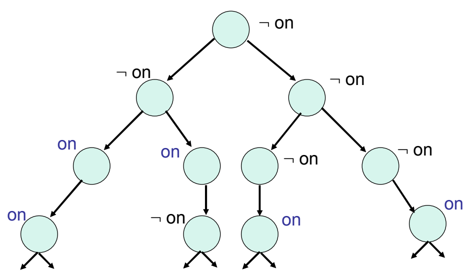
CTL*

- Computational Tree Logic (star)
- discrete time
- branching (non-deterministic) progression
- describes properties of computation trees
- more powerful than LTL

Computation Trees



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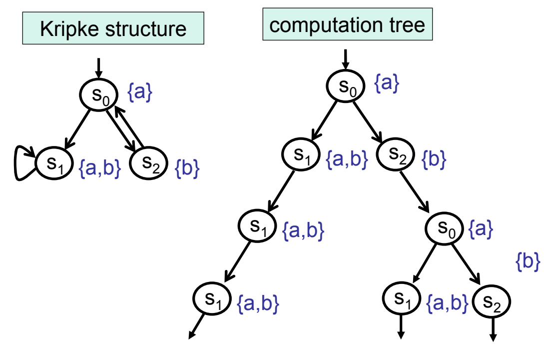
Unwinding Kripke Structure



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Operators in CTL*

- temporal operators
  - same as LTL, describe properties of paths
- path quantifiers
  - A: for all paths (∀) ...
  - E: there exists a path (∃) ...
- more formally ...

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CTL* Formulae

- propositional logic as underlying "static" logic
- two different types of formulae
  - state formula: properties of a state
  - path formulas: property of a path
- all LTL formulae are path formulae
- state formulae ≠ "static" propositional formulae

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State Formulae

$$\phi_s ::= p \mid \neg\phi_s \mid \phi_{s1} \vee \phi_{s2} \mid A\phi_\pi \mid E\phi_\pi$$

- ϕ_s denotes state formula
- ϕ_π path formula
- p atomic proposition
- $A\phi_\pi$ and $E\phi_\pi$ are state formulas
- set of all state formulae = set of all legal CTL* formulae

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Path Quantifiers in State Formulae

- A and E are path quantifiers
- denote universal and existential quantification over paths starting in a certain state
- $A\phi_\pi$ holds in a state s
  iff for all paths starting in s, ϕ_π holds
- $E\phi_\pi$ holds in a state s
  iff there exisits a path starting in s,
  s.t. ϕ_π holds

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Path Formulae

$$\phi_\pi ::= \phi_s \mid \neg\phi_\pi \mid \phi_{\pi1} \vee \phi_{\pi2} \mid X\phi_\pi \mid F\phi_\pi \mid G\phi_\pi \mid \phi_{\pi1} U \phi_{\pi2}$$

- every LTL formula is path formula
- all state formulae are also path formulae
- nesting: $A(GF(A\ a \vee b))$ (example tree?)

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Semantics of CTL*

- define semantics w.r.t. Kripke structure M
- $M,s \models \phi$ denotes
  state formula ϕ holds in a state s of M
- $M,\pi \models \phi$ denotes
  path formula ϕ holds for path π in M
- $\models$ defined inductively, as before

Semantics of CTL* State Formulae

- $M,s \models p$ iff $p \in \mu(s)$
- $M,s \models \neg\phi$ iff not $M,s \models \phi$
- $M,s \models \phi_1 \wedge \phi_2$ iff $M,s \models \phi_1$ and $M,s \models \phi_2$
- $M,s \models A\phi$ iff for all paths π starting in s, $M,\pi \models \phi$
- $M,s \models E\phi$ iff there exists a paths π starting in s, such that $M,\pi \models \phi$

Semantics of CTL* Path Formulae

- $M,\pi \models \phi_s$ iff $s_0 \models \phi_s$
- $M,\pi \models \neg\phi$ iff not $M,\pi \models \phi$
- $M,\pi \models \phi_1 \vee \phi_2$ iff $M,\pi \models \phi_1$ or $M,\pi \models \phi_2$
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basically, same as before

LTL vs. CTL*

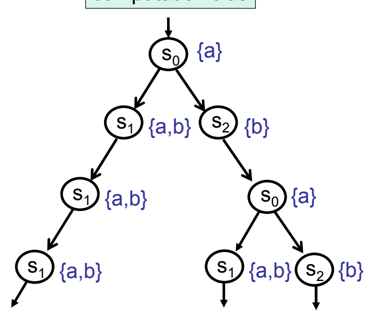
$\phi \in$ LTL implies $A\phi \in$ CTL*

$EF\phi \in$ CTL* but not expressible in LTL (other examples?)

LTL strictly less expressive than CTL*

Examples

computation tree

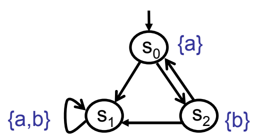


root state s_o satisfies:

- $M,s_0 \models Aa$
- $M,s_0 \models A(\neg b)$
- $M,s_0 \models AXb$
- $M,s_0 \models EFa$
- $M,s_0 \models EFAa$
- what else?

$M \models \phi$

$M \models \phi$ ($\phi \in$ CTL*) iff $M,s_0 \models \phi$ for **initial** state s_0



Which properties satisfies this Kripke structure?

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CTL

a fragment of CTL*

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CTL

- CTL is restricted version of CTL*
- temporal operators as in CTL*
- path quantifier as in CTL*
- However, no unrestricted nesting and Boolean combinations of path formulae
  (next slide)

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CTL Syntax

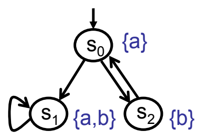
$$\phi ::= p \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid AX\phi \mid EX\phi \mid AF\phi \mid EF\phi \mid AG\phi \mid EG\phi \mid A(\phi_1 U \phi_2) \mid E(\phi_1\ U\ \phi 2)$$

- no arbitrary nesting
- path qualifiers and temporal operators alternate
- no Boolean combinations of path formulae

Not allowed: $a \wedge Fq$, $E(A(F(a \vee b))$, ..., what else?

Example

Which CTL properties hold for this Kripke structure?



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LTL vs. CTL vs. CTL*

- LTL sublogic of CTL*
  - (EF $\phi \in$ CTL*, not expressible in LTL)
- CTL sublogic of CTL*
  - (FG$\phi \in$ CTL*, not expressible in CTL)
- LTL and CTL not comparable
  - FG$\phi \in$ LTL, not expressible in CTL
  - EF$\phi \in$ CTL, not expressible in LTL

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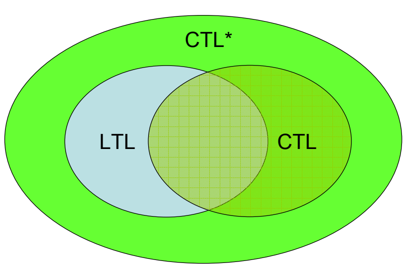
LTL vs. CTL vs. CTL*



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Safety and Liveness

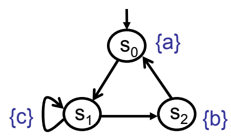
safety: "never something bad will happen"
AG $\neg$(in1 $\wedge$ in2)

liveness: "eventually something good will happen"
EF safe

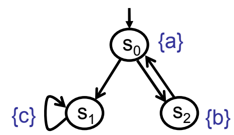
rule of thumb: liveness properties iff counter example requires an infinite trace/infinite deep tree

Fairness

Often liveness properties cannot be proven without certain assumptions, i.e., **fairness**.



M⊨Fb ?



M⊨Fc ?

Weak/Strong Fairness

weak fairness

if an event is continuously enabled, it will occur infinitely often

- in LTL: GF (¬enabled ∨ occurs)

strong fairness

if a event is infinitely often enabled it will occur infinitely often

- in LTL: GF enabled ⇒ GF occurs

Weak/Strong Fairness

weak fairness

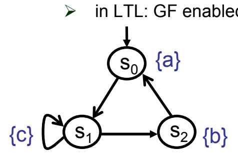
if an event is continuously enabled, it will occur infinitely often

- in LTL: GF (¬enabled ∨ occurs)

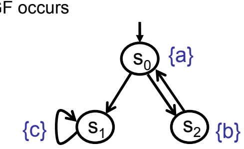
strong fairness

if a event is infinitely often enabled it will occur infinitely often

- in LTL: GF enabled ⇒ GF occurs



under weak fairness: M⊨Fb



under strong fairness: M⊨Fc

Fairness and Model Checking

Weak/strong fairness can be expressed in LTL, however, not in CTL

in **LTL model checking** fairness can be added directly as an assumption

in **CTL model checking** fairness has to be build into the model checking algorithm

Summary

This Lecture

- temporal logic to specify behavior over time
- LTL: linear structure (for all paths)
- CTL(*): branching structure (selective paths)
- LTL, CTL sublogics of CTL*
- CTL, LTL not comparable
- different classes of properties (safety/liveness, fairness)

Next Lecture

- CTL model checking
- how does it work