COMP 4161
NICTA Advanced Course

Advanced Topics in Software Verification

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HOL

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Content

Rough timeline

[a] a1 out; [b] a1 due; [c] a2 out; [d] a2 due; [e] session break; [f] a3 out; [g] a3 due

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Quantifiers

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Scope

- Scope of parameters: whole subgoal
- Scope of $\forall, \exists, \ldots$: ends with ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ ]\!]\ \Longrightarrow\ \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\ \ Q\ x\ y\ ]\!]\ \Longrightarrow\ (\exists x_1.\ Q\ x_1\ y)$$

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Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

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Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice	exploration
apply (rule_tac x = "x" in exI)	**apply** (rule exI)
1. $\bigwedge x.\ x = x$	1. $\bigwedge x.\ x = ?y\ x$
apply (rule refl)	**apply** (rule refl)
	$?y \mapsto \lambda u.u$
simpler & clearer	**shorter & trickier**

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Instantiating Rules

apply (rule_tac x = "$term$" in $rule$)

Like **rule**, but $?x$ in $rule$ is instantiated by $term$ before application.

Similar: **erule_tac**

! x **is in** $rule$**, not in goal** **!**

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Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ??? in exI)

apply (rule exI)

1. $\forall x.\ x = ?y$

apply (rule allI)

1. $\bigwedge x.\ x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.x' = x$

Principle:

$?f\ x_1 \ldots x_n$ **can only be replaced by term** t
if $params(t) \subseteq x_1, \ldots, x_n$

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Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

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Demo: Quantifier Proofs

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Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

apply (rule allI)
1. $\bigwedge x.\ \exists y.\ x = y$

apply (rule_tac x = "x" in exI)

Brittle!

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Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)
1. $\bigwedge x.\ \exists y.\ x = y$

apply (rename_tac N)
1. $\bigwedge N.\ \exists y.\ N = y$

apply (rule_tac x = "N" in exI)

In general:
(rename_tac $x_1 \ldots x_n$) renames the rightmost (inner) n parameters to $x_1 \ldots x_n$

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Forward Proof: frule and drule

apply (frule $< rule >$)

Rule:	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Subgoal:	1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$
Substitution:	$\sigma(B_i) \equiv \sigma(A_1)$
New subgoals:	1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$ $\vdots$ m-1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$ m. $\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $< rule >$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}\ \text{conjunct1} \qquad \frac{P \wedge Q}{Q}\ \text{conjunct2}$$

$$\frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{\forall x.\ P\ x}{P\ ?x}\ \text{spec}$$

Forward Proof: OF

r [**OF** $r_1 \ldots r_n$]

Prove assumption 1 of theorem r with theorem r_1, and assumption 2 with theorem r_2, and ...

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

Forward proofs: THEN

r_1 [THEN r_2] means r_2 [OF r_1]

DEMO: FORWARD PROOFS

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Hilbert's Epsilon Operator

(David Hilbert, 1862-1943)

$\varepsilon\, x.\ Px$ is a value that satisfies P (if such a value exists)

ε also known as **description operator**.
In Isabelle the ε-operator is written SOME $x.\ P\ x$

$$\frac{P\ ?x}{P\ (\text{SOME}\ x.\ P\ x)}\ \text{someI}$$

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More Epsilon

ε implies Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\frac{}{(\text{THE}\ x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

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Some Automation

More Proof Methods:

apply (intro <intro-rules>)	repeatedly applies intro rules
apply (elim <elim-rules>)	repeatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules
apply blast	an automatic tableaux prover (works well on predicate logic)
apply fast	another automatic search tactic

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Epsilon and Automation Demo

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We have learned so far...

- Proof rules for predicate calculus
- Safe and unsafe rules
- Forward Proof
- The Epsilon Operator
- Some automation

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Exercises

- We said that $\mathcal{E}$ implies the Axiom of Choice:

  $\forall x.\, \exists y.\, Q\ x\ y \Longrightarrow \exists f.\, \forall x.\, Q\ x\ (f\ x)$
- Prove the axiom of choice as a lemma, using only the introduction and elimination rules for $\forall$ and $\exists$, namely `allI`, `exI`, `allE`, `exE`, and the introduction rule for $\mathcal{E}$, `someI`, using only the proof methods `rule`, `rule_tac`, `erule`, `erule_tac` and `assumption`.

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