COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

Gerwin Klein, June Andronick, Toby Murray

$$\longrightarrow$$

Content

	Rough timeline
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➜ Foundations & Principles	
• Lambda Calculus, natural deduction	[2,3,4[a]]
• Higher Order Logic	[5,6[b],7]
• Term rewriting	[8,9,10[c]]
➜ Proof & Specification Techniques	
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• Inductively defined sets, rule induction	[13[e],15]
• Datatypes, recursion, induction	[16,17[f],18,19]
• Calculational reasoning, mathematics style proofs	[20]
• Hoare logic, proofs about programs	[21[g],22,23]

[a] a1 out; [b] a1 due; [c] a2 out; [d] a2 due; [e] session break; [f] a3 out; [g] a3 due

Last Time on HOL

- ➜ Defining HOL
- ➜ Higher Order Abstract Syntax
- ➜ Deriving proof rules
- ➜ More automation

The Three Basic Ways of Introducing Theorems

- **Axioms**:

  Expample: **axioms** refl: ”$t = t$”

  Do not use. Evil. Can make your logic inconsistent.

- **Definitions:**

  Example: **definition** inj **where** ”inj $f \equiv \forall x\ y.\ f\ x = f\ y \longrightarrow x = y$”
  Introduces a new lemma called inj_def.

- **Proofs:**

  Example: **lemma** ”inj $(\lambda x.\ x + 1)$”

  The harder, but safe choice.

The Three Basic Ways of Introducing Types

- **typedecl**: by name only

  Example: **typedecl** names

  Introduces new type *names* without any further assumptions

- **types**: by abbreviation

  Example: **types** α rel = "$\alpha \Rightarrow \alpha \Rightarrow bool$"

  Introduces abbreviation *rel* for existing type $\alpha \Rightarrow \alpha \Rightarrow bool$

  Type abbreviations are immediately expanded internally

- **typedef**: by definiton as a set

  Example: **typedef** new_type = "$\{$some set$\}$" $<$proof$>$

  Introduces a new type as a subset of an existing type.

  The proof shows that the set on the rhs in non-empty.

  More on **typedef** in later lectures.

TERM REWRITING

The Problem

Given a set of equations

$$l_1 = r_1$$

$$l_2 = r_2$$

$$\vdots$$

$$l_n = r_n$$

does equation $l = r$ hold?

Applications in:

- ➜ **Mathematics** (algebra, group theory, etc)
- ➜ **Functional Programming** (model of execution)
- ➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$\begin{aligned} l_1 &\longrightarrow r_1 \\ l_2 &\longrightarrow r_2 \\ &\vdots \\ l_n &\longrightarrow r_n \end{aligned}$$

decide $l = r$ **by deciding** $l \overset{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

$\xrightarrow{0} \quad = \quad \{(x,y) | x = y\}$ identity
$\xrightarrow{n+1} \quad = \quad \xrightarrow{n} \circ \longrightarrow$ n+1 fold composition

$\xrightarrow{+} \quad = \quad \bigcup_{i>0} \xrightarrow{i}$ transitive closure
$\xrightarrow{*} \quad = \quad \xrightarrow{+} \cup \xrightarrow{0}$ reflexive transitive closure
$\xrightarrow{=} \quad = \quad \longrightarrow \cup \xrightarrow{0}$ reflexive closure

$\xrightarrow{-1} \quad = \quad \{(y,x) | x \longrightarrow y\}$ inverse
$\longleftarrow \quad = \quad \xrightarrow{-1}$ inverse
$\longleftrightarrow \quad = \quad \longleftarrow \cup \longrightarrow$ symmetric closure

$\overset{+}{\longleftrightarrow} \quad = \quad \bigcup_{i>0} \overset{i}{\longleftrightarrow}$ transitive symmetric closure
$\overset{*}{\longleftrightarrow} \quad = \quad \overset{+}{\longleftrightarrow} \cup \overset{0}{\longleftrightarrow}$ reflexive transitive symmetric closure

How to Decide $l \overset{*}{\longleftrightarrow} r$

Same idea as for β: look for n such that $l \overset{*}{\longrightarrow} n$ and $r \overset{*}{\longrightarrow} n$

Does this always work?

If $l \overset{*}{\longrightarrow} n$ and $r \overset{*}{\longrightarrow} n$ then $l \overset{*}{\longleftrightarrow} r$. Ok.

If $l \overset{*}{\longleftrightarrow} r$, will there always be a suitable n? **No!**

Example:

Rules: $f\ x \longrightarrow a$, $g\ x \longrightarrow b$, $f\ (g\ x) \longrightarrow b$

$f\ x \overset{*}{\longleftrightarrow} g\ x$ because $f\ x \longrightarrow a \longleftarrow f\ (g\ x) \longrightarrow b \longleftarrow g\ x$

But: $f\ x \longrightarrow a$ and $g\ x \longrightarrow b$ and a, b in normal form

Works only for systems with **Church-Rosser** property:

$$l \overset{*}{\longleftrightarrow} r \Longrightarrow \exists n.\ l \overset{*}{\longrightarrow} n \wedge r \overset{*}{\longrightarrow} n$$

Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

Confluence

$$\begin{tikzcd}
 & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
 & t &
\end{tikzcd}$$

Problem:

is a given set of reduction rules confluent?

undecidable

Local Confluence

$$\begin{tikzcd}
 & s \arrow[dl] \arrow[dr] & \\
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
 & t &
\end{tikzcd}$$

Fact: local confluence and termination $\Longrightarrow$ confluence

Termination

$\longrightarrow$ is **terminating** if there are no infinite reduction chains

$\longrightarrow$ is **normalizing** if each element has a normal form

$\longrightarrow$ is **convergent** if it is terminating and confluent

Example:

$\longrightarrow_\beta$ in λ is not terminating, but confluent

$\longrightarrow_\beta$ in $\lambda^\rightarrow$ is terminating and confluent, i.e. convergent

Problem: is a given set of reduction rules terminating?

undecidable

When is $\longrightarrow$ Terminating?

Basic idea: when each rule application makes terms simpler in some way.

More formally: $\longrightarrow$ is terminating when
there is a well founded order $<$ on terms for which $s < t$ whenever $t \longrightarrow s$
(well founded = no infinite decreasing chains $a_1 > a_2 > \ldots$)

Example: $f\ (g\ x) \longrightarrow g\ x$, $g\ (f\ x) \longrightarrow f\ x$

This system always terminates. Reduction order:

$s <_r t$ iff $size(s) < size(t)$ with
$size(s) =$ number of function symbols in s

① Both rules always decrease $size$ by 1 when applied to any term t
② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

Termination in Practice

In practice: often easier to consider just the rewrite rules by themselves, rather than their application to an arbitrary term t.

Show for each rule $l_i = r_i$, that $r_i < l_i$.

Example:

$g\ x <_r f\ (g\ x)$ and $f\ x < g\ (f\ x)$

Requires t to become smaller whenever any subterm of t is made smaller.

Formally:

Requires $<$ to be **monotonic** with respect to the structure of terms:

$$s < t \longrightarrow u[s] < u[t].$$

True for most orders that don't treat certain parts of terms as special cases.

Example Termination Proof

Problem: Rewrite formulae containing $\neg$, $\wedge$, $\vee$ and $\longrightarrow$, so that they don't contain any implications and $\neg$ is applied only to variables and constants.

Rewrite Rules:

- ➜ Remove implications:

  imp: $(A \longrightarrow B) = (\neg A \vee B)$

- ➜ Push $\neg$s down past other operators:

  notnot: $(\neg\neg P) = P$

  notand: $(\neg(A \wedge B)) = (\neg A \vee \neg B)$

  notor: $(\neg(A \vee B)) = (\neg A \wedge \neg B)$

We show that the rewrite system defined by these rules is terminating.

Order on Terms

Each time one of our rules is applied, either:

- an implication is removed, or
- something that is not a $\neg$ is hoisted upwards in the term.

This suggests a 2-part order, $<_r$: $s <_r t$ iff:

- $\mathsf{num\_imps}\ s < \mathsf{num\_imps}\ t$, or
- $\mathsf{num\_imps}\ s = \mathsf{num\_imps}\ t \wedge \mathsf{osize}\ s < \mathsf{osize}\ t$.

Let:

- $s <_i t \equiv \mathsf{num\_imps}\ s < \mathsf{num\_imps}\ t$ and
- $s <_n t \equiv \mathsf{osize}\ s < \mathsf{osize}\ t$

Then $<_i$ and $<_n$ are both well-founded orders (since both functions return nats).

$<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

Order Decreasing

imp clearly decreases $\mathsf{num\_imps}$.

osize adds up all non-$\neg$ operators and variables/constants, weights each one according to its depth within the term.

$$
\begin{array}{lll}
\mathsf{osize}'\ c & \mathsf{acm} & = 2^{\mathsf{acm}} \\
\mathsf{osize}'\ (\neg P) & \mathsf{acm} & = \mathsf{osize}'\ P\ (\mathsf{acm}+1) \\
\mathsf{osize}'\ (P \wedge Q) & \mathsf{acm} & = 2^{\mathsf{acm}} + (\mathsf{osize}'\ P\ (\mathsf{acm}+1)) + (\mathsf{osize}'\ Q\ (\mathsf{acm}+1)) \\
\mathsf{osize}'\ (P \vee Q) & \mathsf{acm} & = 2^{\mathsf{acm}} + (\mathsf{osize}'\ P\ (\mathsf{acm}+1)) + (\mathsf{osize}'\ Q\ (\mathsf{acm}+1)) \\
\mathsf{osize}'\ (P \longrightarrow Q) & \mathsf{acm} & = 2^{\mathsf{acm}} + (\mathsf{osize}'\ P\ (\mathsf{acm}+1)) + (\mathsf{osize}'\ Q\ (\mathsf{acm}+1)) \\
\\
\mathsf{osize}\ \ P & & = \mathsf{osize}'\ P\ 0
\end{array}
$$

The other rules decrease the depth of the things osize counts, so decrease osize.

Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination:	not guaranteed (may loop)
confluence:	not guaranteed (result may depend on which rule is used first)

Control

- ➜ Equations turned into simplification rules with **[simp]** attribute
- ➜ Adding/deleting equations locally:
  apply (simp add: <rules>) and **apply** (simp del: <rules>)
- ➜ Using only the specified set of equations:
  apply (simp only: <rules>)

DEMO

We have seen today...

- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Exercises

- Show, via a pen-and-paper proof, that the `osize` function is monotonic with respect to the structure of terms from that example.