COMP 4161

NICTA Advanced Course

Advanced Topics in Software Verification

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fun

Content

	Rough timeline
➜ Intro & motivation, getting started	[1]
➜ Foundations & Principles	
• Lambda Calculus, natural deduction	[2,3,4[a]]
• Higher Order Logic	[5,6[b],7]
• Term rewriting	[8,9,10[c]]
➜ Proof & Specification Techniques	
• Isar	[11,12[d]]
• Inductively defined sets, rule induction	[13[e],15]
• Datatypes, recursion, induction	[16,17[f],18,19]
• Calculational reasoning, mathematics style proofs	[20]
• Hoare logic, proofs about programs	[21[g],22,23]

[a] a1 out; [b] a1 due; [c] a2 out; [d] a2 due; [e] session break; [f] a3 out; [g] a3 due

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The Choice

- ➜ Limited expressiveness, automatic termination
  - primrec
- ➜ High expressiveness, termination proof may fail
  - fun
- ➜ High expressiveness, tweakable, termination proof manual
  - function

fun — examples

fun sep :: "'a ⇒ 'a list ⇒ 'a list"

where

"sep a (x # y # zs) = x # a # sep a (y # zs)" |

"sep a xs = xs"

fun ack :: "nat ⇒ nat ⇒ nat"

where

"ack 0 n = Suc n" |

"ack (Suc m) 0 = ack m 1" |

"ack (Suc m) (Suc n) = ack m (ack (Suc m) n)"

fun

- The definiton:
  - pattern matching in all parameters
  - arbitrary, linear constructor patterns
  - reads equations sequentially like in Haskell (top to bottom)
  - proves termination automatically in many cases
    (tries lexicographic order)
- Generates own induction principle
- May fail to prove termination:
  - use **function (sequential)** instead
  - allows you to prove termination manually

fun — induction principle

- Each **fun** definition induces an induction principle

- For each equation:

  show P holds for lhs, provided P holds for each recursive call on rhs

- Example **sep.induct**:

$$\llbracket\ \bigwedge a.\ P\ a\ [];$$
$$\bigwedge a\ w.\ P\ a\ [w]$$
$$\bigwedge a\ x\ y\ zs.\ P\ a\ (y\#zs) \Longrightarrow P\ a\ (x\#y\#zs);$$
$$\rrbracket \Longrightarrow P\ a\ xs$$

Termination

Isabelle tries to prove termination automatically

- ➜ For most functions this works with a lexicographic termination relation.
- ➜ Sometimes not $\Rightarrow$ error message with unsolved subgoal
- ➜ You can prove automation separately.

function (sequential) quicksort **where**

quicksort [] = [] $|$

quicksort $(x\#xs)$ = quicksort $[y \leftarrow xs. y \le x]@[x]@$ quicksort $[y \leftarrow xs. x < y]$

by pat_completeness auto

termination

by (relation “measure length”) (auto simp: less_Suc_eq_le)

function is the fully tweakable, manual version of **fun**

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DEMO

How does fun/function work?

Recall **primrec**:

- ➜ defined one recursion operator per datatype
- ➜ inductive definition of its graph $(x, f\ x) \in G$
- ➜ prove totality: $\forall x.\ \exists y.\ (x, y) \in G$
- ➜ prove uniqueness: $(x, y) \in G \Rightarrow (x, z) \in G \Rightarrow y = z$
- ➜ recursion operator: $rec\ x = THE\ y.\ (x, y) \in rec$

How does fun/function work?

Similar strategy for **fun**:

- ➜ a new inductive definition for each **fun** f

- ➜ extract *recursion scheme* for equations in f
- ➜ define graph f_rel inductively, encoding recursion scheme

- ➜ prove totality (= termination)
- ➜ prove uniqueness (automatic)

- ➜ derive original equations from f_rel
- ➜ export induction scheme from f_rel

How does fun/function work?

Can separate and defer termination proof:

- ➜ skip proof of totality

- ➜ instead derive equations of the form: $x \in f\_dom \Rightarrow f\ x = \ldots$
- ➜ similarly, conditional induction principle

- ➜ $f\_dom = acc\ f\_rel$
- ➜ acc = accessible part of f_rel
- ➜ the part that can be reached in finitely many steps

- ➜ termination = $\forall x.\ x \in f\_dom$
- ➜ still have conditional equations for partial functions

Proving Termination

Command **termination fun_name** sets up termination goal

$\forall x.\ x \in fun\_name\_dom$

Three main proof methods:

- ➜ **lexicographic_order** (default tried by **fun**)
- ➜ **size_change** (different automated technique)
- ➜ **relation R** (manual proof via well-founded relation)

Well Founded Orders

Definition

$<_r$ is well founded if well founded induction holds

wf $r \equiv \forall P.\ (\forall x.\ (\forall y <_r x. P\ y) \longrightarrow P\ x) \longrightarrow (\forall x.\ P\ x)$

Well founded induction rule:

$$\frac{\text{wf } r \quad \bigwedge x.\ (\forall y <_r x.\ P\ y) \Longrightarrow P\ x}{P\ a}$$

Alternative definition (equivalent):

there are no infinite descending chains, or (equivalent):

every nonempty set has a minimal element wrt $<_r$

$$\begin{array}{lcl} \text{min } r\ Q\ x & \equiv & \forall y \in Q.\ y \not<_r x \\ \text{wf } r & = & (\forall Q \neq \{\}.\ \exists m \in Q.\ \text{min } r\ Q\ m) \end{array}$$

Well Founded Orders: Examples

- $<$ on $\mathbb{N}$ is well founded
  well founded induction = complete induction
- $>$ and $\leq$ on $\mathbb{N}$ are **not** well founded
- $x <_r y = x \text{ dvd } y \wedge x \neq 1$ on $\mathbb{N}$ is well founded
  the minimal elements are the prime numbers
- $(a, b) <_r (x, y) = a <_1 x \vee a = x \wedge b <_2 y$ is well founded
  if $<_1$ and $<_2$ are
- $A <_r B = A \subset B \wedge \text{finite } B$ is well founded
- $\subseteq$ and $\subset$ in general are **not** well founded

More about well founded relations: *Term Rewriting and All That*

Extracting the Recursion Scheme

So far for termination. What about the recursion scheme?

Not fixed anymore as in primrec.

Examples:

➜ **fun** fib **where**

fib 0 = 1 $|$

fib (Suc 0) = 1 $|$

fib (Suc (Suc n)) = fib n + fib (Suc n)

Recursion: Suc (Suc n) $\leadsto$ n, Suc (Suc n) $\leadsto$ Suc n

➜ **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

Recursion: x $\neq$ 0 $\Longrightarrow$ x $\leadsto$ x - 1

Extracting the Recursion Scheme

Higher Oder:

➜ **datatype** 'a tree = Leaf 'a $|$ Branch 'a tree list

fun treemap :: ('a $\Rightarrow$ 'a) $\Rightarrow$ 'a tree $\Rightarrow$ 'a tree **where**
treemap fn (Leaf n) = Leaf (fn n) $|$
treemap fn (Branch l) = Branch (map (treemap fn) l)

Recursion: x $\in$ set l $\Longrightarrow$ (fn, Branch l) $\leadsto$ (fn, x)

How to extract the context information for the call?

Extracting the Recursion Scheme

Extracting context for equations

$\Rightarrow$

Congruence Rules!

Recall rule **if_cong**:

$$[\![\, \text{b = c; c} \Longrightarrow \text{x = u;} \neg \text{ c} \Longrightarrow \text{y = v} \,]\!] \Longrightarrow$$
$$\text{(if b then x else y) = (if c then u else v)}$$

Read: for transforming x, use b as context information, for y use $\neg b$.

In fun_def: for recursion in x, use b as context, for y use $\neg b$.

Congruence Rules for fun_defs

The same works for function definitions.

declare my_rule[fundef_cong]

(if_cong already added by default)

Another example (higher-order):

$[\![$ xs = ys; $\bigwedge$x. x $\in$ set ys $\Longrightarrow$ f x = g x $]\!]$ $\Longrightarrow$ map f xs = map g ys

Read: for recursive calls in f, f is called with elements of xs

DEMO

Further Reading

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Alexander Krauss,

Automating Recursive Definitions and Termination Proofs in Higher-Order Logic.

PhD thesis, TU Munich, 2009.

`http://www4.in.tum.de/~krauss/diss/krauss_phd.pdf`

We have seen today ...

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- ➜ General recursion with **fun**/**function**
- ➜ Induction over recursive functions
- ➜ How **fun** works
- ➜ Termination, partial functions, congruence rules