DATA 61

COMP4161: Advanced Topics in Software Verification

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Content

- Intro & motivation, getting started [1]

- Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [3[a]]
  - Term rewriting [4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction, Isar (part 2) [6, 7[b]]
  - Hoare logic, proofs about programs, invariants [8]
  - C verification [9]
  - Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

Last Time on HOL

➜ Defining HOL

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➜ Higher Order Abstract Syntax

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➜ Higher Order Abstract Syntax
➜ Deriving proof rules
➜ More automation

Term Rewriting

The Problem

Given a set of equations

$$\begin{gathered} l_1 = r_1 \\ l_2 = r_2 \\ \vdots \\ l_n = r_n \end{gathered}$$

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Applications in:

➜ **Mathematics** (algebra, group theory, etc)

➜ **Functional Programming** (model of execution)

➜ **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$\begin{array}{c} l_1 \longrightarrow r_1 \\ l_2 \longrightarrow r_2 \\ \vdots \\ l_n \longrightarrow r_n \end{array}$$

decide $l = r$ **by deciding** $l \stackrel{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

$$\xrightarrow{0} \quad = \quad \{(x, y) | x = y\} \qquad \text{identity}$$

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$$\begin{array}{lcll} \xrightarrow{0} & = & \{(x,y) \mid x = y\} & \text{identity} \\ \xrightarrow{n+1} & = & \xrightarrow{n} \circ \longrightarrow & \text{n+1 fold composition} \end{array}$$

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How to Decide $l \stackrel{*}{\longleftrightarrow} r$

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Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

Confluence

$$\begin{tikzcd} & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x & & y \end{tikzcd}$$

Problem:
is a given set of reduction rules confluent?

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$$\begin{tikzcd}[column sep=small]
 & s \arrow[dl, "*"'] \arrow[dr, "*"] & \\
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 & t &
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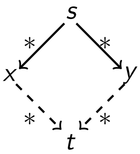
Confluence



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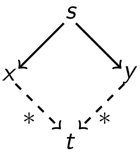

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Fact: local confluence and termination $\Longrightarrow$ confluence

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$\longrightarrow$ is **terminating** if there are no infinite reduction chains
$\longrightarrow$ is **normalizing** if each element has a normal form
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② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

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True for most orders that don't treat certain parts of terms as special cases.

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We show that the rewrite system defined by these rules is terminating.

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Let:

- $s <_i t \equiv$ num_imps $s <$ num_imps t and
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Then $<_i$ and $<_n$ are both well-founded orders (since both return nats). $<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

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\text{osize}' \ (\neg P) & x & = \text{osize}' \ P \ (x+1) \\
\text{osize}' \ (P \wedge Q) & x & = 2^x + (\text{osize}' \ P \ (x+1)) + (\text{osize}' \ Q \ (x+1)) \\
\text{osize}' \ (P \vee Q) & x & = 2^x + (\text{osize}' \ P \ (x+1)) + (\text{osize}' \ Q \ (x+1)) \\
\text{osize}' \ (P \longrightarrow Q) & x & = 2^x + (\text{osize}' \ P \ (x+1)) + (\text{osize}' \ Q \ (x+1)) \\
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The other rules decrease the depth of the things osize counts, so decrease osize.

Term Rewriting in Isabelle

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apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

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(may loop)

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apply simp

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(may loop)

confluence: not guaranteed
(result may depend on which rule is used first)

Control

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- ➜ Adding/deleting equations locally:

  apply (simp add: <rules>)   and   **apply** (simp del: <rules>)
- ➜ Using only the specified set of equations:

  apply (simp only: <rules>)

Demo

We have seen today...

- ➜ Equations and Term Rewriting

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- ➜ Equations and Term Rewriting
- ➜ Confluence and Termination of reduction systems
- ➜ Term Rewriting in Isabelle

Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.

