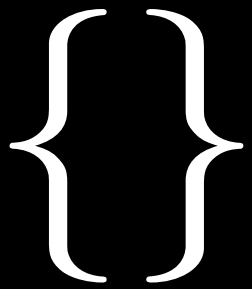
COMP4161: Advanced Topics in Software Verification



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T3/2019

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  - Higher Order Logic, Isar (part 1) [3[a]]
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[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- Sets

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- Type Definitions

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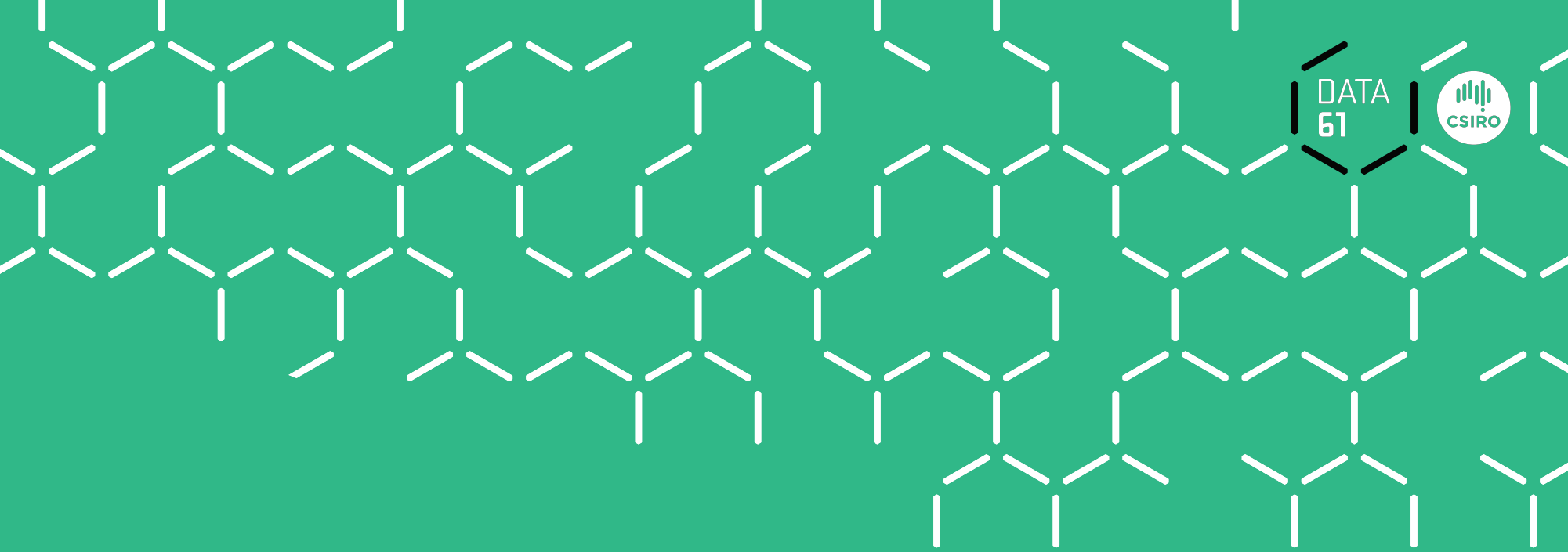
- ➜ Sets
- ➜ Type Definitions
- ➜ Inductive Definitions



Inductive Definitions

How They Work

The Nat Example

$$\frac{}{0 \in N} \qquad \frac{n \in N}{n + 1 \in N}$$

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- Objective: **no junk**. Only what must be in X shall be in X.
- Gives rise to a nice proof principle (rule induction)

Formally

$$\text{Rules } \frac{a_1 \in X \quad \ldots \quad a_n \in X}{a \in X} \text{ with } a_1, \ldots, a_n, a \in A$$

$$\text{define set } X \subseteq A$$

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Fact: $X = \bigcap \{B \subseteq A.\ B\ R\text{−closed}\}$

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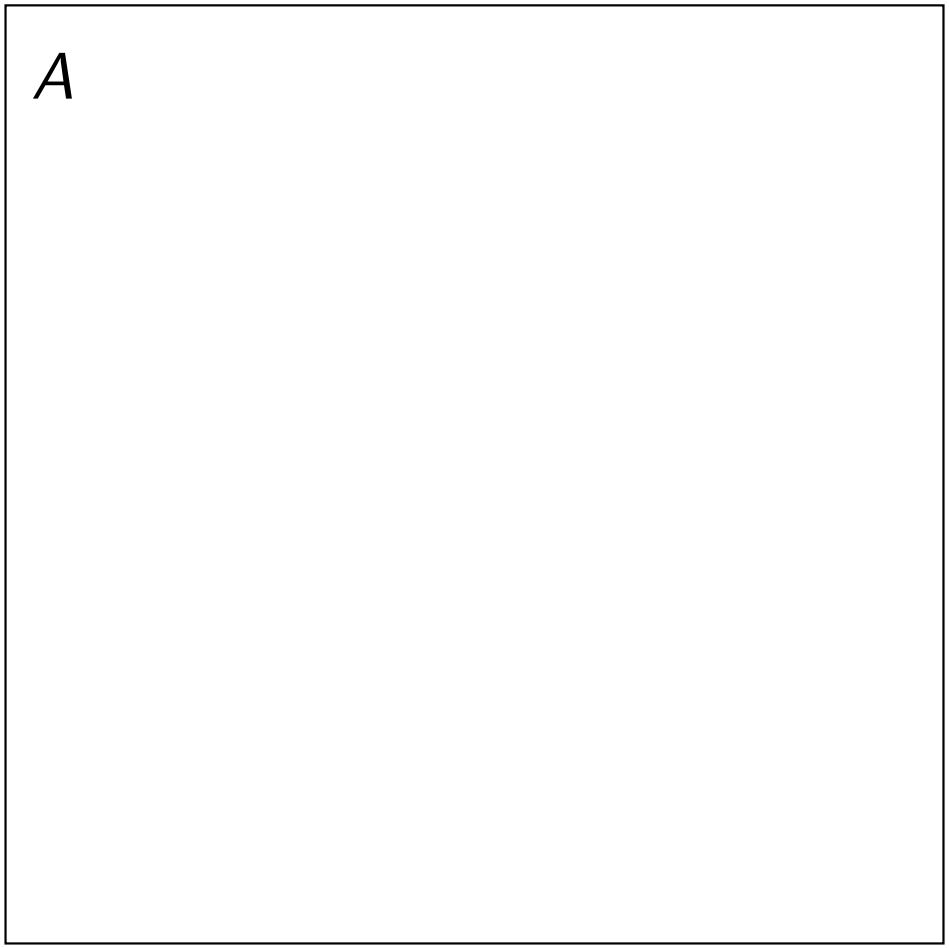

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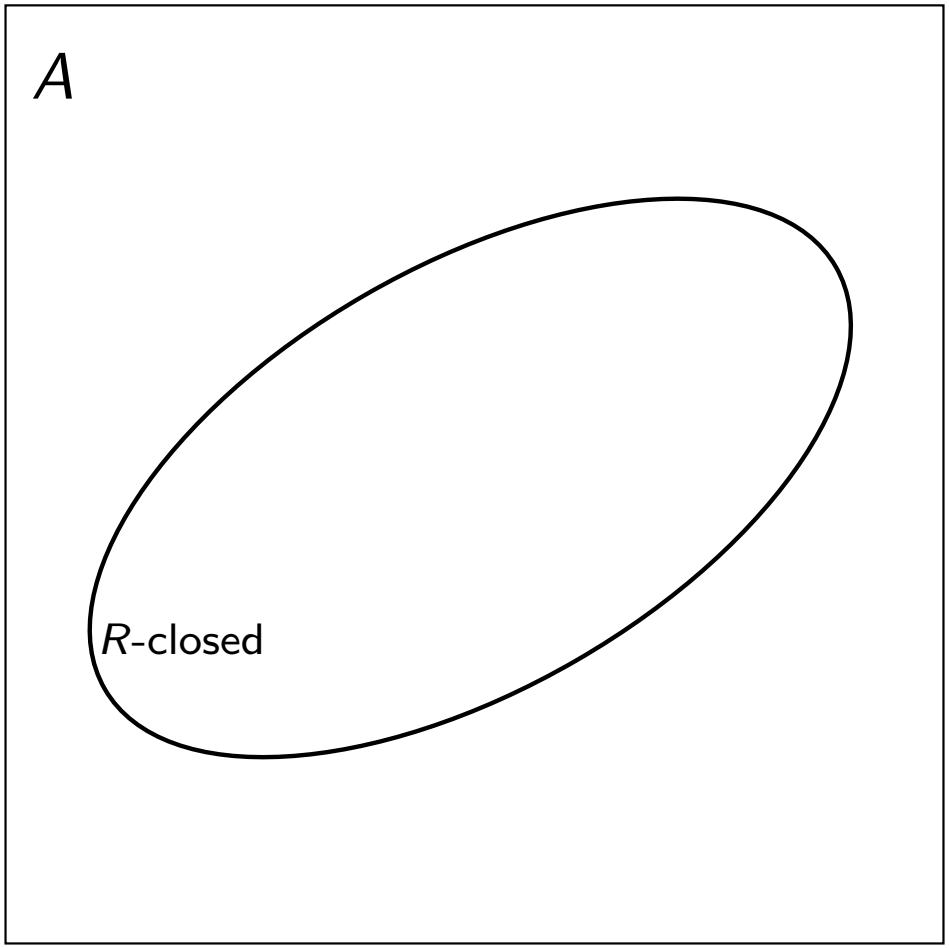

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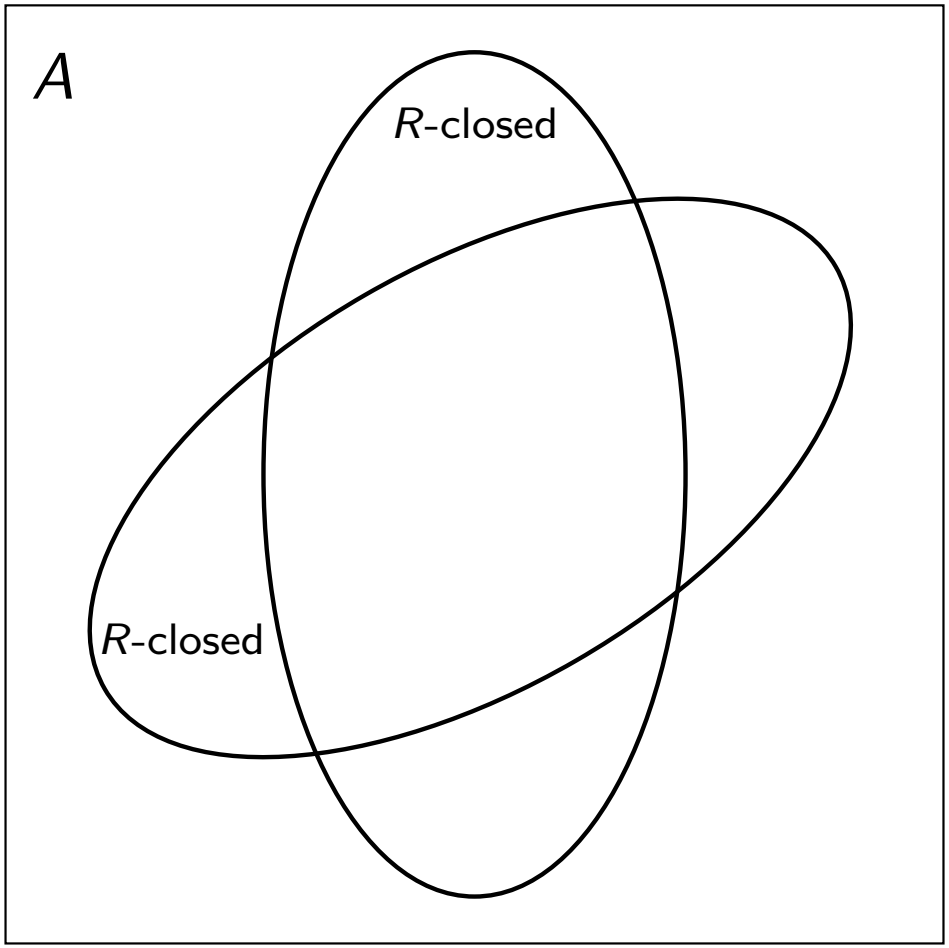

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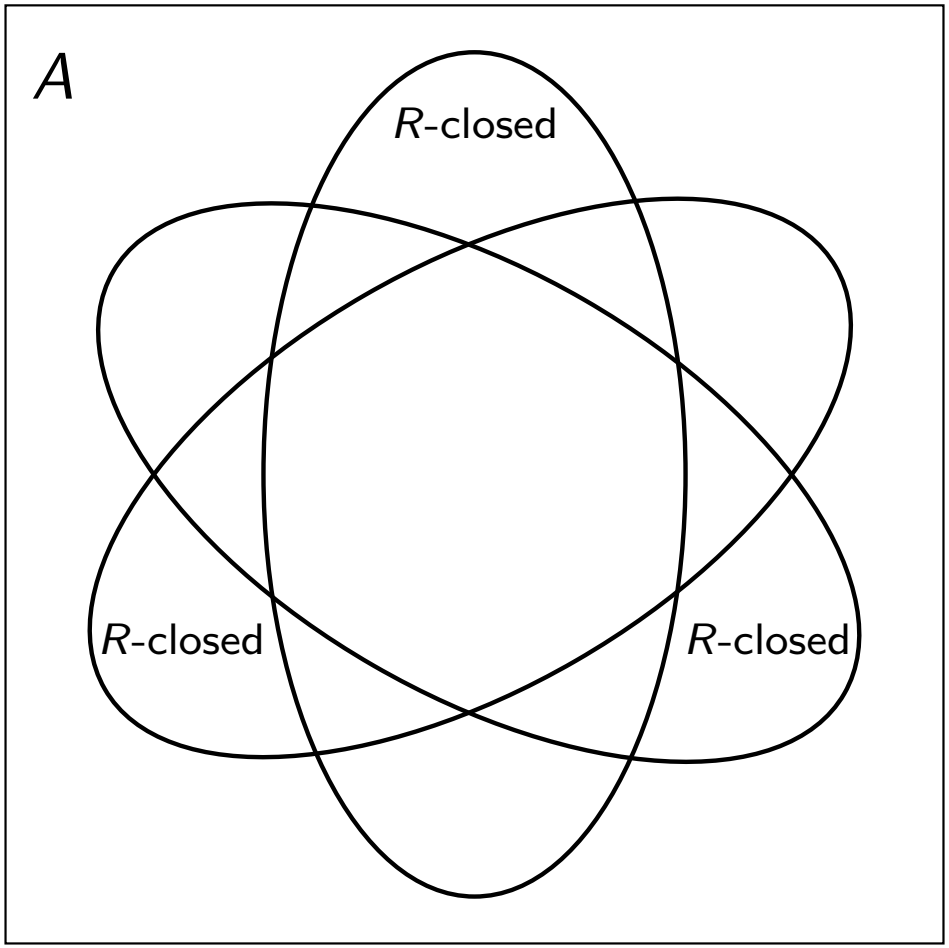

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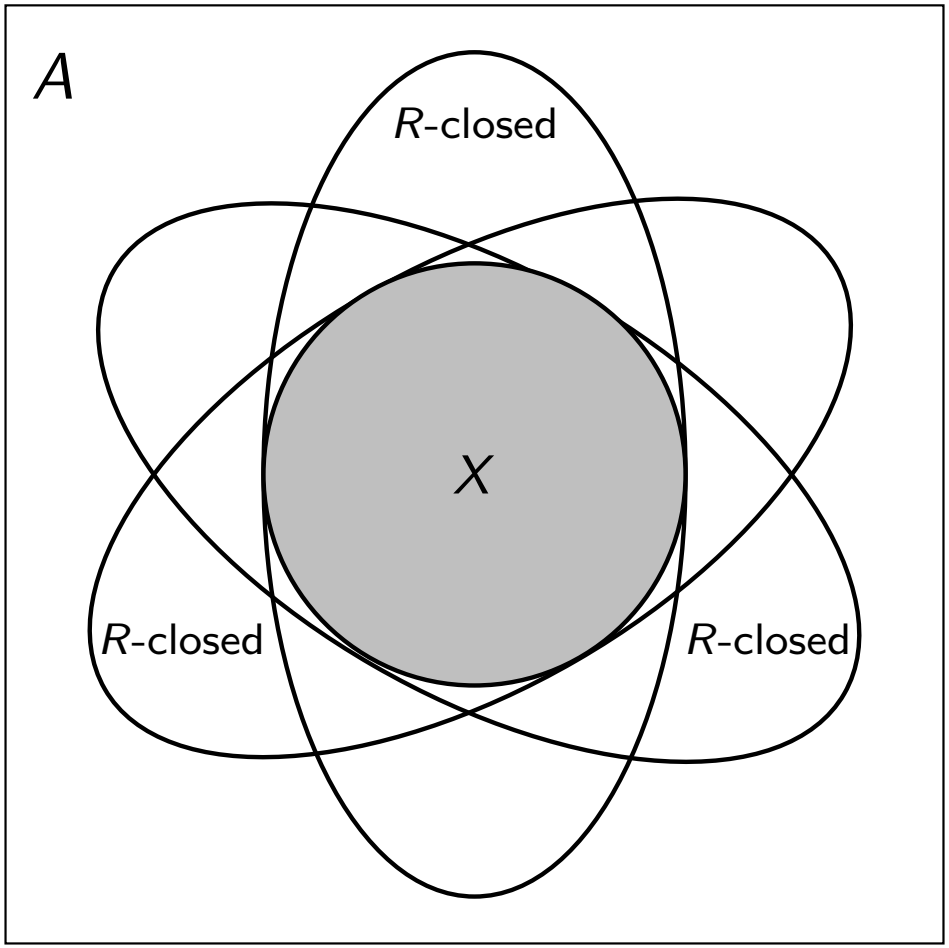

Rule Induction

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induces induction principle

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In general:

$$\frac{\forall(\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

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Rules with side conditions

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induction scheme:

$$\begin{aligned}(\forall(\{a_1, \ldots a_n\}, a) \in R.\ & P\ a_1 \wedge \ldots \wedge P\ a_n \wedge \\ & C_1 \wedge \ldots \wedge C_m \wedge \\ & \{a_1, \ldots, a_n\} \subseteq X \Longrightarrow P\ a) \\ & \Longrightarrow \\ & \forall x \in X.\ P\ x\end{aligned}$$

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$$X_\omega = \bigcup_{n \in \mathbb{N}} (R^n\ \{\}) = X$$

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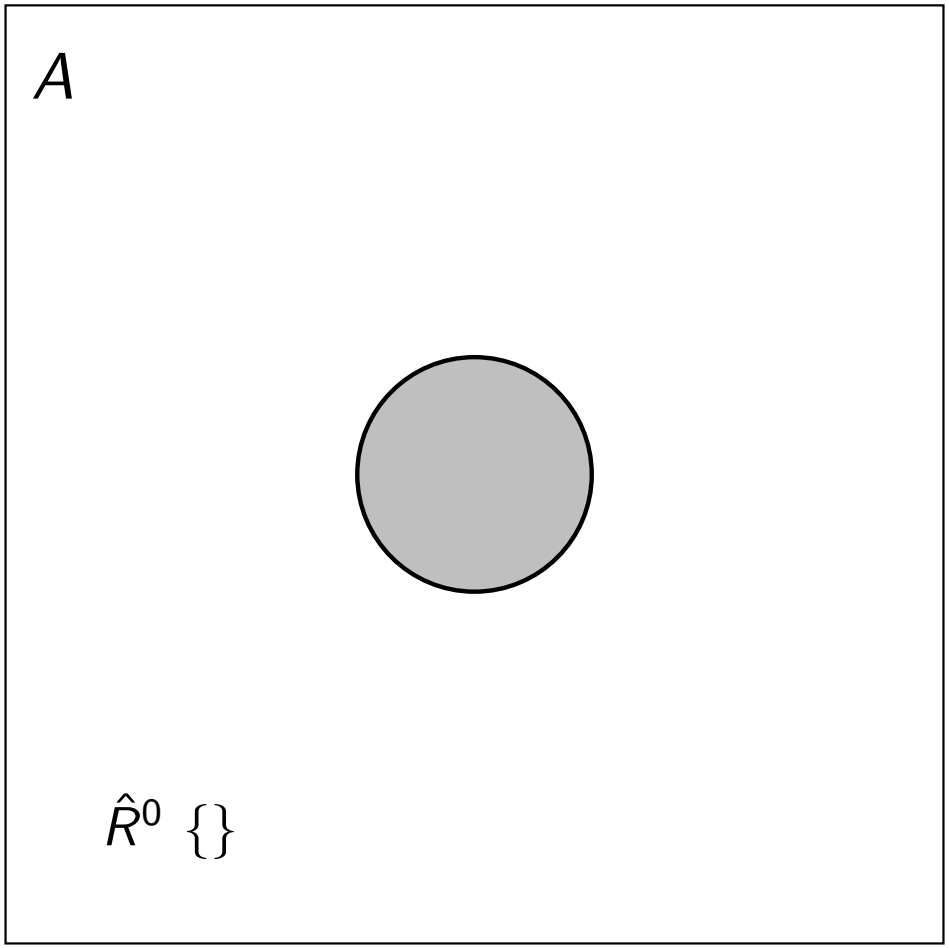

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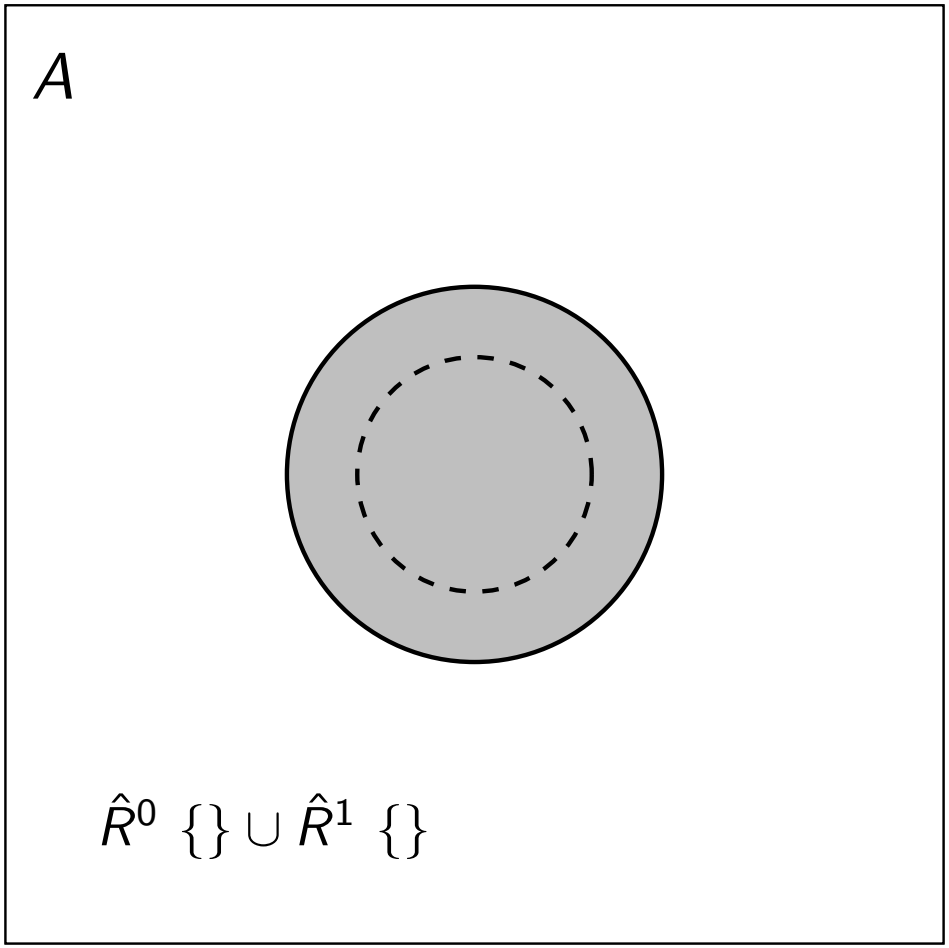

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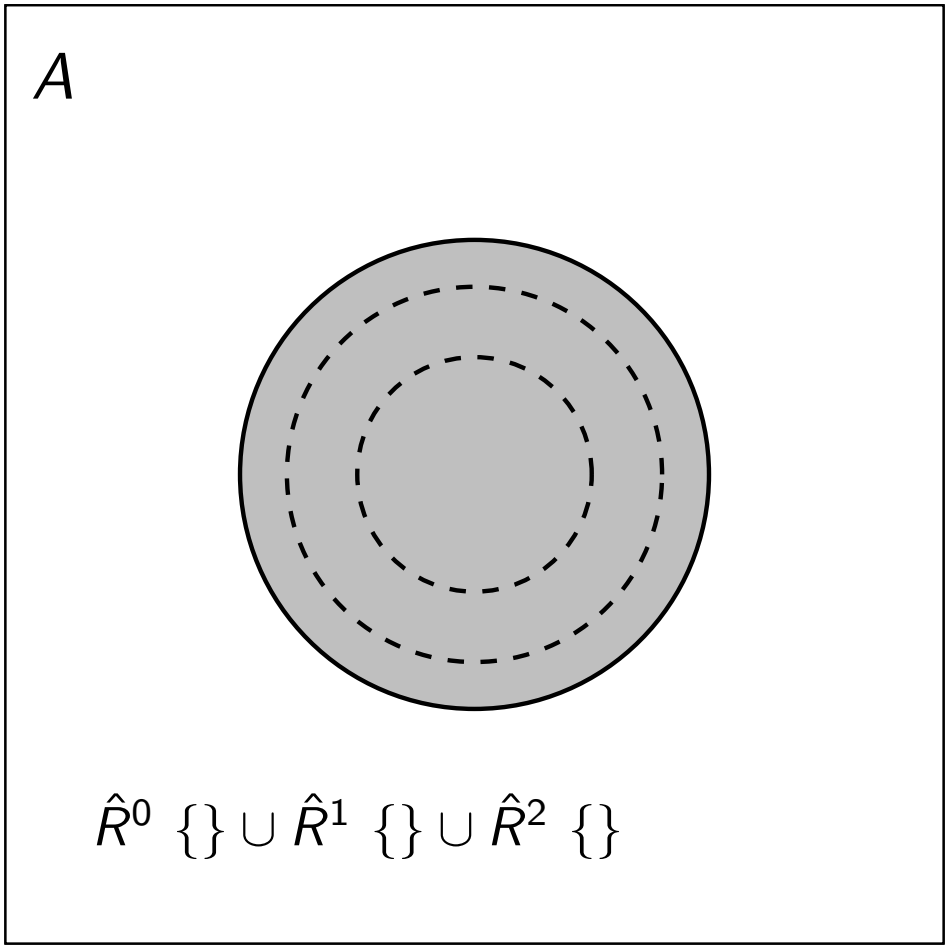

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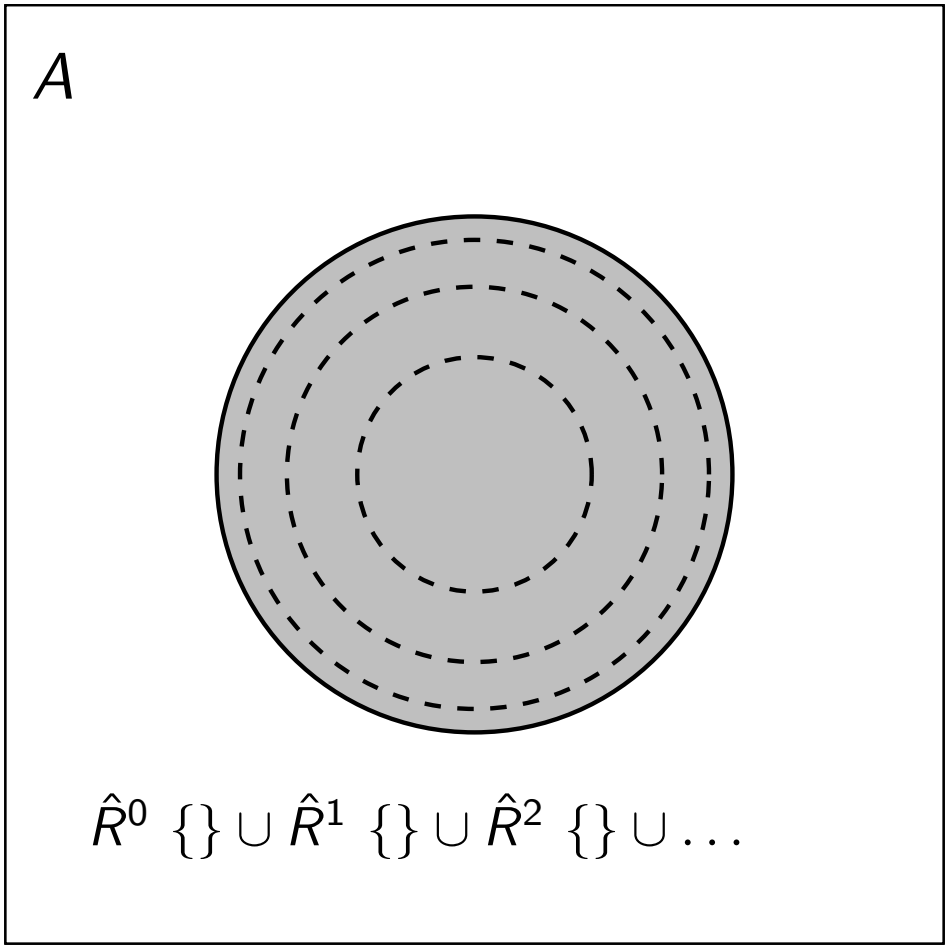

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Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize this lecture in Isabelle:

- ➜ Define **closed** $f\ A :: (\alpha\ \text{set} \Rightarrow \alpha\ \text{set}) \Rightarrow \alpha\ \text{set} \Rightarrow \text{bool}$
- ➜ Show $\text{closed}\ f\ A \wedge \text{closed}\ f\ B \Longrightarrow \text{closed}\ f\ (A \cap B)$ if f is monotone (**mono** is predefined)
- ➜ Define **lfpt** f as the intersection of all f-closed sets
- ➜ Show that lfpt f is a fixpoint of f if f is monotone
- ➜ Show that lfpt f is the least fixpoint of f
- ➜ Declare a constant $R :: (\alpha\ \text{set} \times \alpha)\ \text{set}$
- ➜ Define $\hat{R} :: \alpha\ \text{set} \Rightarrow \alpha\ \text{set}$ in terms of R
- ➜ Show soundness of rule induction using R and lfpt $\hat{R}$

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- ➜ Formalisation in Isabelle

