COMP4161: Advanced Topics in Software Verification

fun

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Content

- ➜ Intro & motivation, getting started [1]
- ➜ Foundations & Principles
  - Lambda Calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [3[a]]
  - Term rewriting [4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction [5]
  - Datatypes, recursion, induction, Isar (part 2) [6, 7[b]]
  - Hoare logic, proofs about programs, invariants [8]
  - C verification [9]
  - Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

General Recursion

The Choice

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- ➜ Limited expressiveness, automatic termination
  - primrec
- ➜ High expressiveness, termination proof may fail
  - fun
- ➜ High expressiveness, tweakable, termination proof manual
  - function

fun — examples


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fun sep :: "'a ⇒ 'a list ⇒ 'a list"
where
    "sep a (x # y # zs) = x # a # sep a (y # zs)" |
    "sep a xs = xs"
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fun ack :: "nat ⇒ nat ⇒ nat"
where
    "ack 0 n = Suc n" |
    "ack (Suc m) 0 = ack m 1" |
    "ack (Suc m) (Suc n) = ack m (ack (Suc m) n)"
```


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- pattern matching in all parameters
- nested, linear constructor patterns
- reads equations sequentially like in Haskell (top to bottom)
- proves termination automatically in many cases
  (tries lexicographic order)

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➜ May fail to prove termination:

- use **function (sequential)** instead
- allows you to prove termination manually

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- Example **sep.induct**:

$$\begin{array}{l}\llbracket \bigwedge a.\ P\ a\ [];\\ \quad \bigwedge a\ w.\ P\ a\ [w]\\ \quad \bigwedge a\ x\ y\ zs.\ P\ a\ (y\#zs) \Longrightarrow P\ a\ (x\#y\#zs);\\ \rrbracket \Longrightarrow P\ a\ xs\end{array}$$

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- ➜ For most functions this works with a lexicographic termination relation.
- ➜ Sometimes not ⇒ error message with unsolved subgoal
- ➜ You can prove termination separately.

function (sequential) quicksort **where**
quicksort [] = [] |
quicksort (x#xs) = quicksort [$y \leftarrow xs . y \leq x$]@[$x$]@ quicksort [$y \leftarrow xs . x < y$]
by pat_completeness auto

termination
by (relation "measure length") (auto simp: less_Suc_eq_le)

Demo

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- ➜ recursion operator for datatype D_rec, defined via THE.
- ➜ primrec: apply datatype recursion operator

How does fun/function work?

Similar strategy for **fun**:

- ➜ a new inductive definition for each **fun** *f*
- ➜ extract *recursion scheme* for equations in *f*
- ➜ define graph *f_rel* inductively, encoding recursion scheme
- ➜ prove totality (= termination)
- ➜ prove uniqueness (automatic)
- ➜ derive original equations from *f_rel*
- ➜ export induction scheme from *f_rel*

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- ➜ termination = $\forall x.\ x \in f\_dom$
- ➜ still have conditional equations for partial functions

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The Size-change Principle for Program Termination, POPL 2001.

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Three main proof methods:

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- ➜ **size_change** (automated translation to simpler size-change graph[1])
- ➜ **relation R** (manual proof via well-founded relation)

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Well Founded Orders

Definition

$<_r$ is well founded if well founded induction holds

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Alternative definition (equivalent):

there are no infinite descending chains, or (equivalent):

every nonempty set has a minimal element wrt $<_r$

$\text{min}\ (<_r)\ Q\ x \quad \equiv \quad \forall y \in Q.\ y \not<_r x$

$\text{wf}\ (<_r) \quad = \quad (\forall Q \neq \{\}.\ \exists m \in Q.\ \text{min}\ r\ Q\ m)$

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  if $<_1$ and $<_2$ are well founded
- $A <_r B = A \subset B \wedge$ finite B is well founded
- $\subseteq$ and $\subset$ in general are **not** well founded

More about well founded relations: *Term Rewriting and All That*

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- **fun** f **where** f x = (if x = 0 then 0 else f (x - 1) * 2)

  Recursion: x $\neq$ 0 $\Longrightarrow$ x $\leadsto$ x - 1

Extracting the Recursion Scheme

Higher Order:

➜ **datatype** 'a tree = Leaf 'a | Branch 'a tree list

fun treemap :: ('a ⇒ 'a) ⇒ 'a tree ⇒ 'a tree **where**
treemap fn (Leaf n) = Leaf (fn n) |
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How does Isabelle extract context information for the call?

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Recall rule **if_cong**:

$$[\!|\ b = c;\ c \Longrightarrow x = u;\ \neg\ c \Longrightarrow y = v\ |\!] \Longrightarrow \\ (\text{if } b \text{ then } x \text{ else } y) = (\text{if } c \text{ then } u \text{ else } v)$$

Read: for transforming x, use b as context information, for y use $\neg b$.

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In fun_def: for recursion in x, use b as context, for y use $\neg b$.

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Read: for recursive calls in f, f is called with elements of xs

Demo

Further Reading

Alexander Krauss,
Automating Recursive Definitions and Termination Proofs in Higher-Order Logic.
PhD thesis, TU Munich, 2009.

`https://www21.in.tum.de/~krauss/papers/krauss-thesis.pdf`

We have seen today ...

- ➜ General recursion with **fun**/**function**
- ➜ Induction over recursive functions
- ➜ How **fun** works
- ➜ Termination, partial functions, congruence rules

