COMP4161: Advanced Topics in Software Verification

INV

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[a] a1 due; [b] a2 due; [c] a3 due

Today

Practice with invariants!

Recall:

- ➜ it needs to be an invariant
- ➜ it needs to be enough

Example 1

$A := 0;$
$B := 0;$

WHILE $A \neq a$
DO
  $B := B + b;$
  $A := A + 1$
OD

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$\{\ B = b * a\ \}$

Example 1

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$
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Example 1

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$A := 0;$
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INV $\{$ B = b * A$\}$
WHILE $A \neq a$
DO
  $B := B + b;$
  $A := A + 1$
OD
$\{\ B = b * a\ \}$

Example 1

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$

$A := 0;$

$B := 0;$ $\qquad 0 = b * 0$

INV $\{$ B = b * A$\}$

WHILE $A \neq a$ $\qquad B = b * A \wedge A \neq a \ \longrightarrow\ B + b = b * (A + 1)$

DO

$\quad B := B + b;$

$\quad A := A + 1$

OD $\qquad B = b * A \wedge A = a \ \longrightarrow\ B = b * a$

$\{\ B = b * a\ \}$

Example 2

$A := 0;$
$B := 0;$

WHILE $A < a$
DO
  $B := B + b;$
  $A := A + 1$
OD

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Example 2

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$
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Example 2

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$
$A := 0;$
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INV $\{$ B = b * A$\}$
WHILE $A < a$
DO
    $B := B + b;$
    $A := A + 1$
OD
$\{\ B = b * a\ \}$

Example 2

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$

$A := 0;$

$B := 0;$ $\qquad 0 = b * 0$

INV $\{$ B = b * A$\}$

WHILE $A < a$ $\qquad B = b * A \wedge A < a \longrightarrow B + b = b * (A + 1)$

DO

$\quad B := B + b;$

$\quad A := A + 1$

OD $\qquad B = b * A \wedge A \geq a \longrightarrow B = b * a$

$\{\ B = b * a\ \}$

Example 2

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$

$A := 0;$

$B := 0;$ $\qquad 0 = b * 0 \wedge 0 \leq a$

INV $\{\ B = b * A\}\ \wedge A \leq a$

WHILE $A < a$ $\qquad B = b * A \wedge A < a \wedge A \leq a \longrightarrow B + b = b * (A + 1) \wedge A + 1 \leq a$

DO

$\quad B := B + b;$

$\quad A := A + 1$

OD $\qquad B = b * A \wedge A \geq a \wedge A \leq a \longrightarrow B = b * a$

$\{\ B = b * a\ \}$

Example 3

$A := a;$
$B := 1;$

WHILE $A \neq 0$
DO
  $B := B * b;$
  $A := A - 1$
OD

Example 3

$A := a;$
$B := 1;$

WHILE $A \neq 0$
DO
  $B := B * b;$
  $A := A - 1$
OD
$\{ \ B = b^a \ \}$

Example 3

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$
$A := a;$
$B := 1;$

WHILE $A \neq 0$
DO
  $B := B * b;$
  $A := A - 1$
OD
$\{\ B = b^a\ \}$

Example 3

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$
$A := a;$
$B := 1;$
INV $\{\ B = b^{a-A}\}$
WHILE $A \neq 0$
DO
  $B := B * b;$
  $A := A - 1$
OD
$\{\ B = b^a\ \}$

Example 3

$\{\ a \geq 0\ \wedge\ b \geq 0\ \}$

$A := a;$

$B := 1;$ $\qquad 1 = b^{a-a}$

INV $\{\ B = b^{a-A}\}$

WHILE $A \neq 0$ $\qquad B = b^{a-A} \wedge A \neq 0 \longrightarrow\ B * b = b^{a-(A-1)}$

DO

$\quad B := B * b;$

$\quad A := A - 1$

OD $\qquad B = b^{a-A} \wedge A = 0\ \longrightarrow\ B = b^{a}$

$\{\ B = b^{a}\ \}$

Example 4

$X := x;$
$Y := [];$

WHILE $X \neq []$

DO
  $Y := (hd\ X \# Y);$
  $X := tl\ X$
OD

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$X := x;$
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WHILE $X \neq []$

DO
  $Y := (hd\ X \# Y);$
  $X := tl\ X$
OD
$\{\ Y = rev\ x\ \}$

Example 4

$\{\ True\ \}$
$X := x;$
$Y := [];$

WHILE $X \neq []$

DO
  $Y := (hd\ X \# Y);$
  $X := tl\ X$
OD
$\{\ Y = rev\ x\ \}$

Example 4

$\{\ True\ \}$
$X := x;$
$Y := [];$
INV $\{\ (rev\ X)@Y = rev\ x\}$
WHILE $X \neq []$

DO
  $Y := (hd\ X \# Y);$
  $X := tl\ X$
OD
$\{\ Y = rev\ x\ \}$

Example 4

$\{\ True\ \}$

$X := x;$

$Y := [];$ $\qquad (rev\ x)@[] = rev\ x$

INV $\{\ (rev\ X)@Y = rev\ x\}$

WHILE $X \neq []$

$\qquad (rev\ X)@Y = rev\ x\ \wedge X \neq [] \longrightarrow$

$\qquad (rev\ (tl\ X))@((hd\ X)\#Y) = rev\ x$

DO

$\quad Y := (hd\ X\#Y);$

$\quad X := tl\ X$

OD $\qquad (rev\ X)@Y = rev\ x\ \wedge X = [] \longrightarrow Y = rev\ x$

$\{\ Y = rev\ x\ \}$

Example 5

$A := a;\ B := b;\ C := 1;$

WHILE $B \neq 0$
DO

  WHILE (B mod 2 = 0)

    DO
    $A := A * A;$
    $B := B\ div\ 2;$
    OD
  $C := C * A;$
  $B := B - 1$
OD

Example 5

Try with $b = 10 = 2^1 + 2^3$ or $b = 12 = 2^2 + 2^3$ (and e.g. a=3)

$A := a;\ B := b;\ C := 1;$

WHILE $B \neq 0$
DO

  WHILE (B mod 2 = 0)

    DO
    $A := A * A;$
    $B := B\ div\ 2;$
    OD
  $C := C * A;$
  $B := B - 1$
OD

Example 5

Try with $b = 10 = 2^1 + 2^3$ or $b = 12 = 2^2 + 2^3$ (and e.g. a=3)

$\{\ a \geq 0 \wedge b \geq 0\ \}$

$A := a;\ B := b;\ C := 1;$

WHILE $B \neq 0$

DO

  WHILE (B mod 2 = 0)

    DO

    $A := A * A;$

    $B := B\ div\ 2;$

    OD

  $C := C * A;$

  $B := B - 1$

OD

$\{\ C = a^b\ \}$

Example 5

Try with $b = 10 = 2^1 + 2^3$ or $b = 12 = 2^2 + 2^3$ (and e.g. a=3)

$\{\ a \geq 0 \wedge b \geq 0\ \}$
$A := a;\ B := b;\ C := 1;$
INV $\{\ a^b = C * A^B\}$
WHILE $B \neq 0$
DO
INV $\{\ a^b = C * A^B\}$
  WHILE (B mod 2 = 0)

    DO
    $A := A * A;$
    $B := B\ div\ 2;$
    OD
  $C := C * A;$
  $B := B - 1$
OD
$\{\ C = a^b\ \}$

Example 5

Try with $b = 10 = 2^1 + 2^3$ or $b = 12 = 2^2 + 2^3$ (and e.g. a=3)

$\{\ a \geq 0 \wedge b \geq 0\ \}$

$A := a;\ B := b;\ C := 1;$ $\qquad a^b = 1 * a^b$

INV $\{\ a^b = C * A^B\}$

WHILE $B \neq 0$ $\qquad a^b = C * A^B \wedge B \neq 0 \longrightarrow\ a^b = (C * A) * A^{B-1}$

DO

INV $\{\ a^b = C * A^B\}$

WHILE (B mod 2 = 0)

$a^b = C * A^B \wedge B\ mod\ 2 = 0 \longrightarrow\ a^b = C * (A * A)^{B\ div\ 2}$

DO

$A := A * A;$

$B := B\ div\ 2;$

OD

$C := C * A;$

$B := B - 1$

OD $\qquad a^b = C * A^B \wedge B = 0 \longrightarrow\ C = a^b$

$\{\ C = a^b\ \}$

Example 6

$l := 0; u := \mathit{length}\ A - 1; A := a$

WHILE $l \leq u$
DO

WHILE $l < \mathit{length}\ A \wedge A!l \leq \mathit{piv}$ DO $l := l + 1$ OD;

WHILE $0 < u \wedge \mathit{piv} \leq A!u$ DO $u := u - 1$ OD;

IF $l \leq u$ THEN $A := A[l := A!u, u := A!l]$ ELSE SKIP FI
OD

Example 6

$l := 0; u := \mathit{length}\ A - 1; A := a$

WHILE $l \leq u$
DO

WHILE $l < \mathit{length}\ A \wedge A!l \leq \mathit{piv}$ DO $l := l + 1$ OD;

WHILE $0 < u \wedge \mathit{piv} \leq A!u$ DO $u := u - 1$ OD;

IF $l \leq u$ THEN $A := A[l := A!u, u := A!l]$ ELSE SKIP FI
OD
$\{\ \mathit{LEQ}\ A\ u \wedge \mathit{EQ}\ A\ u\ l \wedge \mathit{GEQ}\ A\ l \wedge A$ permutes $a\ \}$

Example 6

$LEQ\ A\ n = \forall k.\ k < n \longrightarrow\ A!k \leq piv$
$GEQ\ A\ n = \forall k.\ n < k < length\ A \longrightarrow A!k \geq piv$
$EQ\ A\ n\ m = \forall k.\ n \leq k \leq m \longrightarrow A!k = piv$

$\{\ 0 < length\ A\ \}$
$l := 0; u := length\ A - 1; A := a$

WHILE $l \leq u$
DO

WHILE $l < length\ A \wedge A!l \leq piv$ DO $l := l + 1$ OD;

WHILE $0 < u \wedge piv \leq A!u$ DO $u := u - 1$ OD;

IF $l \leq u$ THEN $A := A[l := A!u, u := A!l]$ ELSE SKIP FI
OD
$\{\ LEQ\ A\ u \wedge EQ\ A\ u\ l \wedge GEQ\ A\ l \wedge A$ permutes $a\ \}$

Example 6

$LEQ\ A\ n = \forall k.\ k < n \longrightarrow\ A!k \leq piv$
$GEQ\ A\ n = \forall k.\ n < k < length\ A \longrightarrow A!k \geq piv$
$EQ\ A\ n\ m = \forall k.\ n \leq k \leq m \longrightarrow A!k = piv$

$\{\ 0 < length\ A\ \}$
$l := 0; u := length\ A - 1; A := a$
INV $\{\ LEQ\ A\ l \wedge GEQ\ A\ u \wedge u < length\ A \wedge l \leq length\ A \wedge A$ permutes $a\}$
WHILE $l \leq u$
DO
  INV $\{\ LEQ\ A\ l \wedge GEQ\ A\ u \wedge u < length\ A \wedge l \leq length\ A \wedge A$ permutes $a\}$
  WHILE $l < length\ A \wedge A!l \leq piv$ DO $l := l + 1$ OD;

  INV $\{\ LEQ\ A\ l \wedge GEQ\ A\ u \wedge u < length\ A \wedge l \leq length\ A \wedge A$ permutes $a\}$
  WHILE $0 < u \wedge piv \leq A!u$ DO $u := u - 1$ OD;

  IF $l \leq u$ THEN $A := A[l := A!u, u := A!l]$ ELSE SKIP FI
OD
$\{\ LEQ\ A\ u \wedge EQ\ A\ u\ l \wedge GEQ\ A\ l \wedge A$ permutes $a\ \}$

Example 7

Reminder:

datatype ref = Ref int | Null

Pointer access: p→field

Pointer update: p→field :== v

Definition:

"*List nxt p Ps*" is a linked list, starting at pointer *p* following the next pointer through the function *nxt*, and where *Ps* contains the list of the pointers of the linked list.

$\{ \; List \; nxt \; p \; Ps \wedge X \in Ps \; \}$

WHILE $p \neq Null \wedge p \neq Ref \; X$

DO

$\quad p := p \rightarrow nxt;$

OD

Example 7

Reminder:

datatype ref = Ref int | Null

Pointer access: p→field

Pointer update: p→field :== v

Definition:

"*List nxt p Ps*" is a linked list, starting at pointer *p* following the next pointer through the function *nxt*, and where *Ps* contains the list of the pointers of the linked list.

$\{\ List\ nxt\ p\ Ps \wedge X \in Ps\ \}$

WHILE $p \neq Null \wedge p \neq Ref\ X$

DO

$p := p \rightarrow nxt;$

OD

$\{\ p = Ref\ X\ \}$

Example 7

Reminder:

datatype ref = Ref int | Null

Pointer access: p→field

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Definition:

"*List nxt p Ps*" is a linked list, starting at pointer *p* following the next pointer through the function *nxt*, and where *Ps* contains the list of the pointers of the linked list.

$\{\ List\ nxt\ p\ Ps \wedge X \in Ps\ \}$

WHILE $p \neq Null \wedge p \neq Ref\ X$

DO

$p := p \rightarrow nxt;$

OD

$\{\ p = Ref\ X\ \}$

Example 7

Reminder:

datatype ref = Ref int | Null

Pointer access: p→field

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Definition:

"*List nxt p Ps*" is a linked list, starting at pointer *p* following the next pointer through the function *nxt*, and where *Ps* contains the list of the pointers of the linked list.

$\{\ List\ nxt\ p\ Ps \wedge X \in Ps\ \}$

INV $\{\ \exists Qs.\ List\ nxt\ p\ Qs \wedge X \in Qs\}$

WHILE $p \neq Null \wedge p \neq Ref\ X$

DO

$\quad p := p \rightarrow nxt;$

OD

$\{\ p = Ref\ X\ \}$

Example 7

Reminder:

datatype ref = Ref int | Null

Pointer access: p→field

Pointer update: p→field :== v

Definition:

"*List nxt p Ps*" is a linked list, starting at pointer *p* following the next pointer through the function *nxt*, and where *Ps* contains the list of the pointers of the linked list.

$\{\ List\ nxt\ p\ Ps \wedge X \in Ps\ \}$ $\quad \exists Qs.\ List\ nxt\ p\ Qs \wedge X \in Qs$

INV $\{\ \exists Qs.\ List\ nxt\ p\ Qs \wedge X \in Qs\}$

WHILE $p \neq Null \wedge p \neq Ref\ X$ $\quad \exists Qs.\ List\ nxt\ p\ Qs \wedge X \in Qs$

$\wedge p \neq Null \wedge p \neq Ref\ X \longrightarrow$

$\exists Qs.\ List\ nxt\ (p \rightarrow nxt)\ Qs \wedge X \in Qs$

DO

$p := p \rightarrow nxt;$

OD $\quad \exists Qs.\ List\ nxt\ p\ Qs \wedge X \in Qs$

$\wedge (p = Null \vee p = Ref\ X) \longrightarrow\ p = Ref\ X$

$\{\ p = Ref\ X\ \}$

Example 8

What is is Isabelle function doing?

$$
\begin{array}{l}
\textit{fun } f \ :: \ 'a \ \textit{list} \Rightarrow' a \ \textit{list} \Rightarrow' a \ \textit{list} \ \textit{where} \\
\quad f \ [] \ ys = ys | \\
\quad f \ xs \ [] = xs | \\
\quad f \ (x\#xs) \ (y\#ys) = x\#y\# \ f \ xs \ ys
\end{array}
$$

Example 8

What is is Isabelle function doing?

$$
\begin{array}{l}
\textit{fun splice} :: {}'a\ \textit{list} \Rightarrow' a\ \textit{list} \Rightarrow' a\ \textit{list}\ \textit{where} \\
\quad \textit{splice}\ []\ ys = ys \mid \\
\quad \textit{splice}\ xs\ [] = xs \mid \\
\quad \textit{splice}\ (x\#xs)\ (y\#ys) = x\#y\#\ f\ xs\ ys
\end{array}
$$

Example 8

What is is Isabelle function doing?

$$
\begin{array}{l}
\textit{fun splice} :: {}'a\ \textit{list} \Rightarrow' a\ \textit{list} \Rightarrow' a\ \textit{list}\ \textit{where} \\
\quad \textit{splice}\ []\ ys = ys| \\
\quad \textit{splice}\ xs\ [] = xs| \\
\quad \textit{splice}\ (x\#xs)\ (y\#ys) = x\#y\#\ f\ xs\ ys
\end{array}
$$

Let's write it with linked lists!

Example 8

$$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$$

$$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$$

Example 8

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

WHILE $q \neq Null$

DO

$\quad qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

$List\ nxt\ p\ Ps = Path\ nxt\ p\ Ps\ Null$

$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{$

$\}$

WHILE $q \neq Null$

DO

$qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

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$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs$

$List\ nxt\ pp\ PPs$

$\}$

WHILE $q \neq Null$

DO

$qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

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$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs\ QQs$

$List\ nxt\ pp\ PPs\ \wedge List\ nxt\ q\ QQs$

$\}$

WHILE $q \neq Null$

DO

$qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

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$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs\ QQs\ PPPs.$

$List\ nxt\ pp\ PPs\ \wedge List\ nxt\ q\ QQs\ \wedge Path\ nxt\ p\ PPPs\ pp$

$\}$

WHILE $q \neq Null$

DO

$qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

$List\ nxt\ p\ Ps = Path\ nxt\ p\ Ps\ Null$

$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs\ QQs\ PPPs.$

$List\ nxt\ pp\ PPs\ \wedge List\ nxt\ q\ QQs\ \wedge Path\ nxt\ p\ PPPs\ pp$

$\wedge\ PPPs@splice\ PPs\ QQs\ =\ splice\ Ps\ Qs$

$\}$

WHILE $q \neq Null$

DO

$qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

$List\ nxt\ p\ Ps = Path\ nxt\ p\ Ps\ Null$

$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs\ QQs\ PPPs.\quad size\ QQs \leq size\ PPs \wedge$

$\quad List\ nxt\ pp\ PPs\ \wedge List\ nxt\ q\ QQs\ \wedge Path\ nxt\ p\ PPPs\ pp$

$\quad \wedge\ PPPs@splice\ PPs\ QQs\ =\ splice\ Ps\ Qs$

$\}$

WHILE $q \neq Null$

DO

$\quad qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Example 8

$List\ nxt\ p\ Ps = Path\ nxt\ p\ Ps\ Null$

$Path\ nxt\ p\ Ps\ Null$ is a linked list from p to q following function nxt and containing list of pointers Ps

$\{\ List\ nxt\ p\ Ps \wedge List\ nxt\ q\ Qs \wedge (set\ Ps \cap set\ Qs) = \{\} \wedge size\ Qs \leq size\ Ps\ \}$

$pp := p;$

INV $\{\ \exists PPs\ QQs\ PPPs.\quad size\ QQs \leq size\ PPs\ \wedge$

$\quad List\ nxt\ pp\ PPs\ \wedge List\ nxt\ q\ QQs\ \wedge Path\ nxt\ p\ PPPs\ pp$

$\quad \wedge\ PPPs@splice\ PPs\ QQs\ =\ splice\ Ps\ Qs\ \wedge$

$\quad set\ PPs \cap set\ QQs = \{\} \wedge distinct\ PPPs \wedge set\ PPPs \cap (set\ PPs \cup set\ QQs) = \{\}$

$\}$

WHILE $q \neq Null$

DO

$\quad qq := q \rightarrow nxt; q \rightarrow nxt := pp \rightarrow nxt; pp \rightarrow nxt = q; pp := q \rightarrow nxt; q := qq;$

OD

$\{\ List\ nxt\ p\ (splice\ Ps\ Qs)\ \}$

Demo