



COMP4161: Advanced Topics in Software Verification



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Content

→ Foundations & Principles

- Intro, Lambda calculus, natural deduction [1,2]
- Higher Order Logic, Isar (part 1) [2,3^a]
- Term rewriting [3,4]

→ Proof & Specification Techniques

- Inductively defined sets, rule induction, datatype induction, primitive recursion [4,5]
- General recursive functions, termination proofs [7^b]
- Proof automation, Hoare logic, proofs about programs, invariants [8]
- C verification [9,10]
- Practice, questions, examp prep [10^c]

^aa1 due; ^ba2 due; ^ca3 due

Last Time on HOL

- Defining HOL
- Higher Order Abstract Syntax
- Deriving proof rules
- More automation

Term Rewriting

The Problem

Given a set of equations

$$l_1 = r_1$$

$$l_2 = r_2$$

\vdots

$$l_n = r_n$$

does equation $l = r$ hold?

Applications in:

- **Mathematics** (algebra, group theory, etc)
- **Functional Programming** (model of execution)
- **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use equations as reduction rules

$$l_1 \longrightarrow r_1$$

$$l_2 \longrightarrow r_2$$

\vdots

$$l_n \longrightarrow r_n$$

decide $l = r$ by deciding $l \xleftrightarrow{*} r$

Arrow Cheat Sheet

$\xrightarrow{0}$	$= \{(x, y) x = y\}$	identity
$\xrightarrow{n+1}$	$= \xrightarrow{n} \circ \longrightarrow$	n+1 fold composition
$\xrightarrow{+}$	$= \bigcup_{i>0} \xrightarrow{i}$	transitive closure
$\xrightarrow{*}$	$= \xrightarrow{+} \cup \xrightarrow{0}$	reflexive transitive closure
\rightrightarrows	$= \longrightarrow \cup \xrightarrow{0}$	reflexive closure
$\xrightarrow{-1}$	$= \{(y, x) x \longrightarrow y\}$	inverse
\longleftarrow	$= \xrightarrow{-1}$	inverse
\longleftrightarrow	$= \longleftarrow \cup \longrightarrow$	symmetric closure
$\xleftrightarrow{+}$	$= \bigcup_{i>0} \xleftrightarrow{i}$	transitive symmetric closure
$\xleftrightarrow{*}$	$= \xleftrightarrow{+} \cup \xleftrightarrow{0}$	reflexive transitive symmetric closure

How to Decide $l \overset{*}{\longleftrightarrow} r$

Same idea as for β : look for n such that $l \overset{*}{\longrightarrow} n$ and $r \overset{*}{\longrightarrow} n$

Does this always work?

If $l \overset{*}{\longrightarrow} n$ and $r \overset{*}{\longrightarrow} n$ then $l \overset{*}{\longleftrightarrow} r$. Ok.

If $l \overset{*}{\longleftrightarrow} r$, will there always be a suitable n ? **No!**

Example:

Rules: $f x \longrightarrow a$, $g x \longrightarrow b$, $f (g x) \longrightarrow b$

$f x \overset{*}{\longleftrightarrow} g x$ because $f x \longrightarrow a \longleftarrow f (g x) \longrightarrow b \longleftarrow g x$

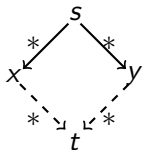
But: $f x \longrightarrow a$ and $g x \longrightarrow b$ and a, b in normal form

Works only for systems with **Church-Rosser** property:

$$l \overset{*}{\longleftrightarrow} r \implies \exists n. l \overset{*}{\longrightarrow} n \wedge r \overset{*}{\longrightarrow} n$$

Fact: \longrightarrow is Church-Rosser iff it is confluent.

Confluence

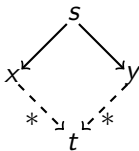


Problem:

is a given set of reduction rules confluent?

undecidable

Local Confluence



Fact: local confluence and termination \implies confluence

Termination

- is **terminating** if there are no infinite reduction chains
- is **normalizing** if each element has a normal form
- is **convergent** if it is terminating and confluent

Example:

- _β in λ is not terminating, but confluent
- _β in λ^{\rightarrow} is terminating and confluent, i.e. convergent

Problem: is a given set of reduction rules terminating?

undecidable

When is \longrightarrow Terminating?

Basic idea: when each rule application makes terms simpler in some way.

More formally: \longrightarrow is terminating when there is a well founded order $<$ on terms for which $s < t$ whenever $t \longrightarrow s$
(well founded = no infinite decreasing chains $a_1 > a_2 > \dots$)

Example: $f(g\ x) \longrightarrow g\ x$, $g(f\ x) \longrightarrow f\ x$

This system always terminates. Reduction order:

$s <_r t$ iff $size(s) < size(t)$ with
 $size(s)$ = number of function symbols in s

- ① Both rules always decrease *size* by 1 when applied to any term t
- ② $<_r$ is well founded, because $<$ is well founded on \mathbb{N}

Termination in Practice

In practice: often easier to consider just the rewrite rules by themselves,

rather than their application to an arbitrary term t .

Show for each rule $l_i = r_i$, that $r_i < l_i$.

Example:

$$g\ x < f\ (g\ x) \text{ and } f\ x < g\ (f\ x)$$

Requires

u to become smaller whenever any subterm of u is made smaller.

Formally:

Requires $<$ to be **monotonic** with respect to the structure of terms:

$$s < t \longrightarrow u[s] < u[t].$$

True for most orders that don't treat certain parts of terms as special cases.

Example Termination Proof

Problem: Rewrite formulae containing \neg , \wedge , \vee and \longrightarrow , so that they don't contain any implications and \neg is applied only to variables and constants.

Rewrite Rules:

→ Remove implications:

$$\text{imp: } (A \longrightarrow B) = (\neg A \vee B)$$

→ Push \neg s down past other operators:

$$\text{notnot: } (\neg\neg P) = P$$

$$\text{notand: } (\neg(A \wedge B)) = (\neg A \vee \neg B)$$

$$\text{notor: } (\neg(A \vee B)) = (\neg A \wedge \neg B)$$

We show that the rewrite system defined by these rules is terminating.

Order on Terms

Each time one of our rules is applied, either:

- an implication is removed, or
- something that is not a \neg is hoisted upwards in the term.

This suggests a 2-part order, $<_r$: $s <_r t$ iff:

- $\text{num_imps } s < \text{num_imps } t$, or
- $\text{num_imps } s = \text{num_imps } t \wedge \text{osize } s < \text{osize } t$.

Let:

- $s <_i t \equiv \text{num_imps } s < \text{num_imps } t$ and
- $s <_n t \equiv \text{osize } s < \text{osize } t$

Then $<_i$ and $<_n$ are both well-founded orders (since both return nats).

$<_r$ is the lexicographic order over $<_i$ and $<_n$. $<_r$ is well-founded since $<_i$ and $<_n$ are both well-founded.

Order Decreasing

imp clearly decreases numimps.

osize adds up all non- \neg operators and variables/constants, weights each one according to its depth within the term.

$$\begin{aligned} \text{osize}' c & \quad x = 2^x \\ \text{osize}' (\neg P) & \quad x = \text{osize}' P (x + 1) \\ \text{osize}' (P \wedge Q) & \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1)) \\ \text{osize}' (P \vee Q) & \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1)) \\ \text{osize}' (P \longrightarrow Q) & \quad x = 2^x + (\text{osize}' P (x + 1)) + (\text{osize}' Q (x + 1)) \\ \text{osize } P & \quad = \text{osize}' P 0 \end{aligned}$$

The other rules decrease the depth of the things osize counts, so decrease osize.

Term Rewriting in Isabelle

Term rewriting engine in Isabelle is called **Simplifier**

apply simp

- uses simplification rules
- (almost) blindly from left to right
- until no rule is applicable.

termination: not guaranteed
(may loop)

confluence: not guaranteed
(result may depend on which rule is used first)

Control

- Equations turned into simplification rules with **[simp]** attribute
- Adding/deleting equations locally:
apply (simp add: <rules>) and **apply** (simp del: <rules>)
- Using only the specified set of equations:
apply (simp only: <rules>)

Demo

We have seen today...

- Equations and Term Rewriting
- Confluence and Termination of reduction systems
- Term Rewriting in Isabelle

Exercises

- Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the structure of terms from that example.