COMP4161: Advanced Topics in Software Verification

{P} ... {Q}

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Content

- ➜ Foundations & Principles
  - Intro, Lambda calculus, natural deduction [1,2]
  - Higher Order Logic, Isar (part 1) [2,3[a]]
  - Term rewriting [3,4]
- ➜ Proof & Specification Techniques
  - Inductively defined sets, rule induction, datatype induction, primitive recursion [4,5]
  - General recursive functions, termination proofs [7[b]]
  - Proof automation, Hoare logic, proofs about programs, invariants [8]
  - C verification [9,10]
  - Practice, questions, examp prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

A Crash Course in Semantics

(For more,
see Concrete Semantics)

IMP - a small Imperative Language

Commands:
datatype com

=	SKIP	
|	Assign vname aexp	(_ := _)
|	Semi com com	(_; _)
|	Cond bexp com com	(IF _ THEN _ ELSE _)
|	While bexp com	(WHILE _ DO _ OD)

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type_synonym aexp	=	state ⇒ nat
type_synonym bexp	=	state ⇒ bool

Example Program

Usual syntax:

$$
\begin{array}{l}
B := 1; \\
\text{WHILE } A \neq 0 \text{ DO} \\
\quad B := B * A; \\
\quad A := A - 1 \\
\text{OD}
\end{array}
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$$

Expressions are functions from state to bool or nat:

$$
\begin{array}{l}
B := (\lambda\sigma.\ 1); \\
\text{WHILE } (\lambda\sigma.\ \sigma\ A \neq 0) \text{ DO} \\
\quad B := (\lambda\sigma.\ \sigma\ B * \sigma\ A); \\
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$$

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Now we need: the meaning (semantics) of programs

How to define execution of a program?

- ➜ A wide field of its own
- ➜ Some choices:
  - Operational (inductive relations, big step, small step)
  - Denotational (programs as functions on states, state transformers)
  - Axiomatic (pre-/post conditions, Hoare logic)

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Demo: The Definitions in Isabelle

Proofs about Programs

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Example:
Show that example program from before implements the factorial function.

$$\textbf{lemma}\ \langle \text{factorial}, \sigma \rangle \to \sigma' \Longrightarrow \sigma' B = \text{fac}\ (\sigma A)$$
$$(\text{where} \quad \text{fac}\ 0 = 1, \quad \text{fac}\ (\text{Suc}\ n) = (\text{Suc}\ n) * \text{fac}\ n)$$

Demo: Example Proof

Too tedious

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Is there something easier?

Floyd/Hoare

Idea: describe meaning of program by pre/post conditions

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Floyd/Hoare

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$\{x = n\} \quad \text{IF } y < 0 \text{ THEN } x := x + y \text{ ELSE } x := x - y \quad \{x = n - |y|\}$

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Proofs: have rules that directly work on such triples

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$$\{P\} \quad c \quad \{Q\}$$

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$$\models \{P\}\ c\ \{Q\} \quad \equiv \quad \forall \sigma\ \sigma'.\ P\ \sigma \wedge \langle c, \sigma \rangle \to \sigma' \longrightarrow Q\ \sigma'$$

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This lecture: partial correctness only (easier)

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$$\frac{P \Longrightarrow P' \quad \{P'\}\ c\ \{Q'\} \quad Q' \Longrightarrow Q}{\{P\} \quad c \quad \{Q\}}$$

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Demo: Hoare Logic in Isabelle