NMR I

May 4, 2005

1 Closed World Assumption

General notation. Δ, Γ are sets of clauses. $A_1, A_2, \ldots$ are ground atoms. $L_1, L_2, \ldots$ are literals. $C_1, C_2, \ldots$ are clauses. Models are Herbrand models.

Definition 1 (CWA) $CWA(\Delta) = Cn(\Delta \cup \{\neg A | \Delta \not\vdash A\})$.

Corollary 1 (Conservative) *If $CWA(\Delta)$ is consistent and $CWA(\Delta) \vdash A$ then $\Delta \vdash A$.*

Definition 2 (Subsume/subsumption) *If $\Delta \vdash A_1 \vee A_2 \vee \ldots \vee A_n$ implies $\Delta \vdash A_k$ for some k, $1 \leq k \leq n$, then the clause $A_1 \vee A_2 \vee \ldots \vee A_n$ is subsumed (by A_k.*

Definition 3 (Minimal/Least model) *A model M of Γ is minimal if there is no other Γ model M' such that $M' \subset M$. M is (the) least model if it is the smallest model (hence it is of course minimal too, and every other model contains it).*

Lemma 1 (Cn property) *If $\Delta = Cn(\Gamma)$ and $\Delta \vdash C$ then $\Gamma \vdash C$.*

Proposition 1 (CWA consistency and subsumption) *$CWA(\Delta)$ is consistent iff every clause of atoms C such that $\Delta \vdash C$ is subsumed.*

LHS implies RHS:
Suppose there is a clause C of atoms $A_1 \vee A_2 \vee \ldots \vee A_n$ such that $\Delta \vdash C$ but is not subsumed. Note that $CWA(\Delta) \vdash C$. By non-subsumption of C, $\Delta \not\vdash A_k$ for every k, $1 \leq k \leq n$. Hence $CWA(\Delta) \vdash \neg A_k$ for every k, $1 \leq k \leq n$, and therefore $CWA(\Delta) \vdash \neg A_1 \wedge \neg A_2 \ldots \neg A_n$. Thus $CWA(\Delta) \vdash \neg(A_1 \vee A_2 \vee \ldots \vee A_n)$, i.e. $CWA(\Delta) \vdash \neg C$, hence $CWA(\Delta)$ is inconsistent.

RHS implies LHS:
Suppose $CWA(\Delta)$ is inconsistent. Then $\Delta \cup \{\neg A | \Delta \not\vdash A\}$ is unsatisfiable. By compactness a finite subset of this is unsatisfiable, say, $\Delta \cup \{\neg A_1, \neg A_2, \ldots, \neg A_k\}$. So $\Delta \cup \{\neg A_1 \wedge \neg A_2 \ldots \wedge \neg A_k\}$ is unsatisfiable, and hence $\Delta \cup \{\neg(A_1 \vee A_2 \ldots \vee A_k)\}$ is unsatisfiable. Therefore $\Delta \vdash A_1 \vee A_2 \ldots \vee A_k$ but by assumption $\Delta \not\vdash A_i$ for every i such that $1 \leq i \leq k$.

Corollary 2 (CWA Consistency and least model) *$CWA(\Delta)$ is consistent iff Δ has a (the) least model.*

Suppose there is an unsubsumed clause $A_1 \vee \ldots \vee A_n$ such that $\Delta \vdash A_1 \vee \ldots \vee A_n$. Then $\Delta \not\vdash A_i$ for each i, $1 \leq i \leq n$, and hence $\Delta \cup \neg A_i$ has a model for each such i. Let the corresponding Herbrand model be M_i, whence $A_i \notin M_i$. If there is a least model M, then $M \subseteq \cap_i M_i$ and therefore $A_i \notin M$ for each i. Thus M cannot be a model of Δ, and so there is no least model.

Converse is an exercise.

If there is an infinite descending chain of models $M_1 \supseteq M_2 \supseteq \ldots$ of Δ then $\cap_i M_i$ is a model of Δ.