Exercise 1.

(a) $R$ is not an equivalence relation since it is neither reflexive nor transitive: $(1, 2) \in R$ and $(2, 1) \in R$, but $(1, 1) \notin R$.

(b) Yes, $R$ is an equivalence relation: Notice that $a + 2b$ is divisible by 3 whenever $a - b$ is divisible by 3, hence $(a, b) \in R$ iff $a \mod 3 = b \mod 3$, which is reflexive, symmetric and transitive.

Exercise 2. If $m = n \pmod{p}$ then $m = k \cdot p + r$ and $n = l \cdot p + r$ for some $k, l \in \mathbb{Z}$ and $r \in \{0, \ldots, p-1\}$. Then,

$$m^2 = k^2 p^2 + 2kpr + r^2$$
$$n^2 = l^2 p^2 + 2lpr + r^2$$

Hence, $m^2 \mod p = r^2 \mod p = n^2 \mod p$, so $m^2 = n^2 \pmod{p}$.

Exercise 3.

• $R$ is reflexive: for every $a, b$ such that $a = b$, by definition $(a, b) \in R$.

• $R$ is antisymmetric: for any $a \neq b$, if $(a, b) \in R$ then it must be that $a \leq b - 0.5$, therefore $b \geq a + 0.5 > a - 0.5$ so $(b, a) \notin R$.

• $R$ is transitive: for any $a, b, c$, this is trivial if $a = b$ or $b = c$, otherwise if $a \neq b$ and $b \neq c$ then $(a, b) \in R \wedge (b, c) \in R \Rightarrow a \leq b - 0.5 \wedge b \leq c - 0.5 \Rightarrow a \leq c - 1 \Rightarrow a \leq c - 0.5 \Rightarrow (a, c) \in R$.

Therefore $R$ is a partial order. It is not a total order since any pair $a, b$ where $a < b < a + 0.5$ (for instance, 1.1 and 1.2) are not related in either direction.

Exercise 4.

(a) Check that $R$ is reflexive, antisymmetric and transitive in the same way as for the relation in Exercise 2.

(b) It is not a lattice; for example, the pair (1,2) do not have a greatest lower bound (or, in fact, any lower bound — both 1 and 2 are minimal elements).

Exercise 5. For the product order the maximum length is 19; for example,

$$(1, 1) \subseteq_P (1, 2) \subseteq_P \ldots \subseteq_P (1, 9) \subseteq_P (1, 10) \subseteq_P (2, 10) \subseteq_P \ldots \subseteq_P (9, 10) \subseteq_P (10, 10)$$

For the lexicographic order, since it is a total order, the longest chain contains all of the 100 elements in the set:

$$(1, 1) \subseteq_L (1, 2) \subseteq_L \ldots \subseteq_L (1, 10) \subseteq_L (2, 1) \subseteq_L \ldots \subseteq_L (9, 10) \subseteq_L (10, 1) \subseteq_L \ldots \subseteq_L (10, 10)$$