

Solutions to COMP9334 Week 5 Sample Problems

Problem 1: A system is being designed. The inter-arrival times of customers are expected to be exponentially distributed with mean $1/\lambda = 50$ msec. Three options are considered as illustrated in Figure 1.

- (a) One single-server queue with infinite buffer space. The service times are exponentially distributed with mean $1/\mu = 20$ msec.
- (b) Two single-server queues, each with infinite buffer space. Customers are randomly dispatched to each queue with an equal probability. The service times are exponentially distributed with mean $1/\mu = 40$ msec at each server.
- (c) One two-server queue with infinite buffer space. The service times are exponentially distributed with mean $1/\mu = 40$ msec at each server.

Find the response time in each option using queueing analysis.

Solution:

- (a) This option is an $M/M/1$ queue with $\lambda = 0.02$ customers per msec and $\mu = 0.05$ customers per msec. Using the $M/M/1$ mean response time formula, we have

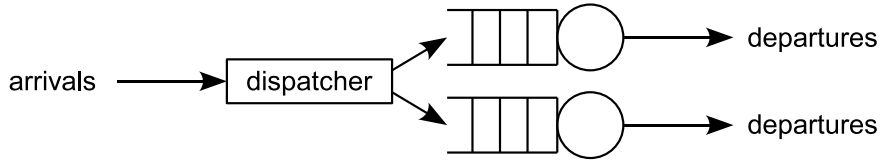
$$T = \frac{1}{\mu - \lambda} = \frac{1}{0.05 - 0.02} = 33.3 \text{ msec}$$

- (b) In this option, the arrival rate at each queue is half of the system arrival rate. Effectively, we have two $M/M/1$ queues, each with $\lambda = 0.01$ customers per msec and $\mu = 0.025$ customers per msec. Using the $M/M/1$ mean response time formula, we have

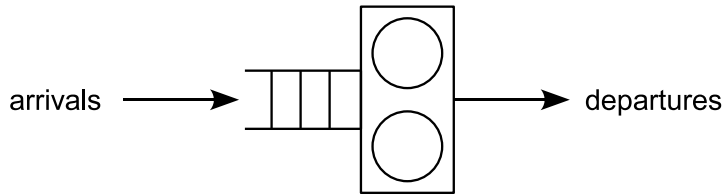
$$T = \frac{1}{\mu - \lambda} = \frac{1}{0.025 - 0.01} = 66.7 \text{ msec}$$



(a) One single-server queue



(b) Two single-server queues



(c) One two-server queue

Figure 1: Three options.

- (c) This option is effectively an $M/M/2$ queue with $\lambda = 0.02$ customers per msec, $\mu = 0.025$ customers per msec, and $\rho = \frac{\lambda}{2\mu} = 0.4$. Using the $M/M/m$ mean response time formula and letting $m = 2$, we have

$$C(\rho, 2) = \frac{\frac{(2\rho)^2}{2!}}{(1 - \rho)(1 + 2\rho) + \frac{(2\rho)^2}{2!}} = \frac{2\rho^2}{1 + \rho}$$

and therefore

$$T = \frac{C(\rho, 2)}{2\mu(1 - \rho)} + \frac{1}{\mu} = \frac{\rho^2}{\mu(1 - \rho^2)} + \frac{1}{\mu} = \frac{1}{\mu(1 - \rho^2)} = \frac{1}{0.025(1 - 0.4^2)} = 47.6 \text{ msec}$$

Problem 2: Consider a single-server queue with infinite buffer space

(a) Consider the situation

- The inter-arrival time is a constant and is given by 1 sec
- The service time required by each customer is always 0.5 sec

What is the mean waiting time per customer?

(b) Consider the situation

- The inter-arrival time is exponentially distributed with mean 1 sec
- The service time required by each customer is exponentially distributed with mean 0.5 sec

What is the mean waiting time per customer?

(c) Compare the answers of (a) and (b), what conclusions can you draw?

Solution:

(a) There is no queueing at all in this case. The mean waiting time W is 0 sec.

(b) The mean arrival rate is $\lambda = 1$ customers per sec. The mean service rate is $\mu = 2$ customers per sec. Using the $M/M/1$ mean response time formula, we have

$$T = \frac{1}{\mu - \lambda} = \frac{1}{2 - 1} = 1 \text{ sec}$$

Therefore, the mean waiting time W is given by

$$W = T - \frac{1}{\mu} = 1 - 0.5 = 0.5 \text{ sec}$$

(c) You may draw a number of conclusions here:

- Queueing is a result of the randomness in arrival and service patterns. In (b), the mean arrival rate is 1 customers per sec, which means that the instantaneous arrival rate can sometimes be greater than 1 customers per sec. The high instantaneous arrival rate can create a backlog in the system and create queueing.
- Waiting time is not just a function of the mean arrival and the mean service rate. It depends on the distributions of inter-arrival times and service times too.

Problem 3: An Internet Service Provider has 4 dial-up ports. Connection requests follow a Poisson arrival process with mean arrival rate of 3 requests per hour. The session duration of each connection request is exponentially distributed with a mean of 1.5 hours. What is the probability that a connection request will be rejected?

Solution: This is effectively an $M/M/4/4$ system with $\lambda = 3$ requests per hour, $1/\mu = 1.5$ hours, and $A = \lambda/\mu = 4.5$. The probability that a connection request is blocked is given by the Erlang B formula

$$\mathbf{E}(A, m) = \frac{\frac{A^m}{m!}}{\sum_{k=0}^m \frac{A^k}{k!}} = \frac{\frac{4.5^4}{4!}}{\sum_{k=0}^4 \frac{4.5^k}{k!}} = 0.3567$$

Alternatively, we can compute $\mathbf{E}(A, m)$ using the following recursive formula

$$\mathbf{E}(A, k) = \frac{A\mathbf{E}(A, k-1)}{k + A\mathbf{E}(A, k-1)}, \quad k = 1, 2, \dots, m$$

with the initial condition $\mathbf{E}(A, 0) = 1$. For $A = 4.5$, $m = 4$, we have:

$$\mathbf{E}(A, 1) = \frac{A\mathbf{E}(A, 0)}{1 + A\mathbf{E}(A, 0)} = \frac{4.5}{1 + 4.5} = 0.8182$$

$$\mathbf{E}(A, 2) = \frac{A\mathbf{E}(A, 1)}{2 + A\mathbf{E}(A, 1)} = \frac{4.5 \times 0.8182}{2 + 4.5 \times 0.8182} = 0.6480$$

$$\mathbf{E}(A, 3) = \frac{A\mathbf{E}(A, 2)}{3 + A\mathbf{E}(A, 2)} = \frac{4.5 \times 0.6480}{3 + 4.5 \times 0.6480} = 0.4929$$

$$\mathbf{E}(A, 4) = \frac{A\mathbf{E}(A, 3)}{4 + A\mathbf{E}(A, 3)} = \frac{4.5 \times 0.4929}{4 + 4.5 \times 0.4929} = 0.3567$$