AXIOM SYSTEMS FOR PROPOSITIONAL LOGIC

Aims

- notions of axiomatic system, deduction
- particular axiom system for propositional logic
- use of this system to establish tautologies, equivalences, consistency, argument validity

Basic Idea (Kelly Ch 4)

- so far ...
  - *truth tables* used to define
    - tautologies, (in)consistency, validity, ...
  - gives the *semantics* of propositional logic
    - semantics = considerations involving “truth”
- now ...
  - a *proof theory* approach (c.f. Prolog!)
    - axioms, inference rules, proofs, theorems, ...
- later – *connections* between the semantic and proof theory approaches

Proofs in Propositional Logic

- In general, a proof consists of a set of hypotheses which are converted via correct reasoning steps to a conclusion (what you were asked to prove).
- In Propositional Logic, we begin with a set of axioms, and convert them to a theorem via correct uses of the rules of inference.
  (Axioms $\approx$ Prolog program; conclusion $\approx$ Prolog query.)

Axioms of Propositional Logic (Kelly Section 4.2.1)

- actually axiom schema – allow all substutions for A, B, C
  - Ax1 $A \to (B \to A)$
  - Ax2 $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
  - Ax3 $(\neg A \to \neg B) \to (B \to A)$
- e.g. $P \to (Q \to P)$ is an *instance* of Ax1.
- e.g. $(D \to E) \to (S \to (D \to E))$ is also an instance of Ax1.
- **Definition of a Well Formed Formula (wff)**:
  - Any propositional letter is a wff
  - If W and V are wff so is $W \to V$
  - If W is a wff so is $\neg W$
  - Nothing else is a wff

Logical Deduction (Kelly Section 4.3)

- there is one rule of deduction or *inference rule*:
  modus ponens: from A and $A \to B$ infer B
- a *deduction* of conclusion C from a set H of hypotheses is a sequence $F_1, F_2, \ldots, F_n$ of wff such that $F_n = C$ and for each $i \in \{1, \ldots, n\}$
  - F_i is an instance of an axiom, or
  - F_i is a hypothesis in H, or
  - F_i is derived by modus ponens from F_j and F_k where $j, k < i$
- if there is a deduction of C from H, write $H \vdash C$ (also say that C is a *consequence* of H).

- say that C is a *theorem* (of Propositional Logic) if H is empty, written $\vdash C$.

Examples

- A boring example with a very short proof

$$\vdash P \to (Q \to P) \text{ Ax1}$$

- Another boring example

$$\vdash (D \to E) \to (S \to (D \to E)) \text{ Ax1}$$

A slightly more interesting example with a 5 line proof (see Kelly pages 69–70)

$$\vdash A \to A$$

1.	$A \to ((B \to A) \to A)$	Ax1
2.	$(A \to ((B \to A) \to A)) \to ((A \to (B \to A)) \to (A \to A))$	Ax2
3.	$(A \to (B \to A)) \to (A \to A)$	MP 1, 2
4.	$A \to (B \to A)$	Ax1
5.	$A \to A$	MP 3, 4

Exercise: Prove the following Simplification MetaTheorems:

$$\vdash (A \land B) \to A$$

$$\vdash (A \land B) \to B$$

Here $A \land B$ is *defined* as $\neg(A \to \neg B)$.

A further example: remember the disappointed student?

- H_1: If she studies the sciences then she prepares to earn a good living.
  H_2: If she studies the humanities, then she prepares for a good life.
  H_3: If for a good living or for a good life then the years are well spent.
  H_4: The years were not well spent.
  C: She didn't study science or humanities.
- Is the argument "H_1, H_2, H_3, H_4, therefore C" valid?

H_1	is	$S \to G$
H_2	is	$H \to L$
H_3	is	$G \lor L \to W$
H_4	is	$\neg W$

- Proof $\{H_1, H_2, H_3, H_4\} \vdash \neg(S \lor H)$

1.	$\neg W \to \neg(G \lor L)$	Contraposition H_3
2.	$\neg(G \lor L)$	MP 1, H_4
3.	$\neg G \land \neg L$	De Morgan 2
4.	$\neg G \to \neg S$	Contraposition H_1
5.	$\neg G$	Simplification 3
6.	$\neg S$	MP 4, 5
7.	$\neg L \to \neg H$	Contraposition H_2
8.	$\neg L$	Simplification 3
9.	$\neg H$	MP 7, 8
10.	$\neg H \land \neg S$	Conjunction* 6, 9
11.	$\neg(S \lor H)$	De Morgan 10

*Exercise: Prove the Conjunction MetaTheorem i.e. $A, B \vdash A \land B$

Discussion – Finding axiomatic proofs

- no firm search strategy
- proofs generally hard to find
  - usually a mixture of seeing what you can do with the hypotheses you have, and seeing what might lead to the desired conclusion.
- worked examples
  (A MetaTheorem is a theorem about a logical system)
  - Kelly MetaTheorem 4.2 $\vdash \neg A \to (A \to B)$ eventually a seven line proof (page 71) with a second proof (page 72)
  - Kelly MetaTheorem 4.3 $\{A \to B, B \to C\} \vdash A \to C$
  - also 4.4, 4.5, …, 4.10

Deduction Theorem (MetaTheorem 4.6)

$$\text{If } H \cup \{A\} \vdash B \text{ then } H \vdash A \to B$$

is particularly useful.

Connecting proof theory and semantics (Kelly Section 4.5)

- in semantics, talk about . . . tautologies
  in proof theory, talk about . . . theorems
- how are these two collections related?
- 4.5.1 Soundness: Are all theorems tautologies?

  $$\text{if } \vdash A \text{ then } \models A$$

  Yes: every theorem is a tautology (Kelly MetaTheorem 4.11), so Propositional Logic is sound (generates *only* valid conclusions).
- 4.5.2 Consistency: is it possible to derive a theorem and its negation?
  No: it is not possible for both A and $\neg A$ to be theorems (Kelly MetaTheorem 4.12), so Propositional Logic is consistent.

- 4.5.3 Completeness: is every tautology a theorem?

  $$\text{if } \models A \text{ then } \vdash A$$

  Yes: every tautology is a theorem (Kelly MetaTheorem 4.19), so Propositional Logic is complete (generates *every* valid conclusion).

Summary

- proof system based on axioms and inference rules defines a computational procedure for verifying deductions . . . though not necessarily practical for *generating* deductions
- correctness of proof system with respect to semantics is guaranteed by soundness, completeness