Justified Representation in Approval-Based Committee Voting

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Voting with Approval Ballots

• A set of alternatives \( C \)

• \( n \) voters \( \{1, \ldots, n\} \)

• Each voter approves a subset of candidates \( A_i \subseteq C \)

• **Goal**: select \( k \) winners
Outline

• Approval-based multiwinner rules

• Justified Representation (JR)

• Which rules satisfy JR?

• Extended Justified Representation (EJR)
• Approval-based multiwinner rules

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Approval Voting (AV)

- Each candidate gets one point from each voter who approves her.
- k candidates with the highest score are selected.
  - Ties broken deterministically.

For k = 3, AV outputs \{c_1, c_2, c_3\}.
Satisfaction Approval Voting (SAV)

- Brams & Kilgour ’14

- Voter $i$ scores committee $W$ as $|A_i \cap W|/|A_i|$

- **Goal**: select a size-$k$ committee with the maximum score

For $k=2$
- AV outputs $\{c_1, c_2\}$,
- SAV outputs $\{c_3, c_4\}$
Minimax Approval Voting (MAV)

- Brams, Kilgour & Sanver ’07

- **Distance** from ballot $A_i$ to a committee $W$:
  \[ d(A_i, W) = |A_i \setminus W| + |W \setminus A_i| \]

- **Goal**: select a size-$k$ committee that minimizes $\max_i d(A_i, W)$
Proportional Approval Voting (PAV)

• Simmons ’01
• Voter $i$ derives utility of 1 from her $1^{st}$ approved candidate, $1/2$ from $2^{nd}$, $1/3$ from $3^{rd}$, etc.
• $u_i(W) = 1 + 1/2 + ... + 1/|W \cap A_i|$ for $k=2$
• Goal: select a size-$k$ committee $W$ that maximizes $u(W) = \sum_i u_i(W)$ for AV outputs $\{c_1, c_2\}$, PAV outputs $\{c_1, c_3\}$ or $\{c_2, c_3\}$
Reweighted Approval Voting (RAV)

- Thiele, early 20th century
- Sequential version of PAV
- Initialize: \( \omega(i) = 1 \) for all \( i \), \( W = \emptyset \)
- Repeat \( k \) times:
  - add to \( W \) a candidate with max approval weight
    \( \omega(c) = \sum_{i \text{ approves } c} \omega(i) \)
  - update the weight of each voter to
    \( \omega(i) = 1/(1+|A_i \cap W|) \)

For \( k=2 \)
- PAV outputs \{\( c_2 \), \( c_3 \)\}
- RAV outputs \{\( c_1 \), \( c_2 \)\} or \{\( c_1 \), \( c_3 \)\}
Generalizing PAV and RAV: Arbitrary Weights

- PAV and RAV both use weight vector \((1, 1/2, 1/3, \ldots)\)
- We can use an arbitrary weight vector \((w_1, w_2, \ldots)\) with \(w_1 = 1\) , \(w_1 \geq w_2 \geq \ldots\) instead:
  \((w_1, w_2, \ldots)\)-PAV and \((w_1, w_2, \ldots)\)-RAV
- \((1, 0, \ldots)\)-RAV: choose candidates one by one to cover as many uncovered voters as possible at each step (Greedy Approval Voting (GAV))
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- Approval-based multiwinner rules
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- Which rules satisfy JR?
- Extended Justified Representation (EJR)
Representation

• 5 voters get 3 representatives, 4 voters get 0 representatives

• Intuition: each cohesive group of voters of size $n/k$ “deserves” at least one representative

for $k=3$

AV outputs $\{c_1, c_2, c_3\}$
Justified Representation

• **Definition**: a committee $W$ provides justified representation (JR) for a list of ballots $(A_1, \ldots, A_n)$ and committee size $k$ if for every set of voters $X$ with $|X| \geq n/k$ and $\bigcap_{i \in X} A_i \neq \emptyset$ it holds that $W$ contains at least one candidate from $\bigcup_{i \in X} A_i$.

• $k=1$: JR is satisfied unless there exists a candidate approved by all, but we pick a candidate not approved by anyone.
Can We Always Satisfy JR?

- **Claim**: GAV (aka $(1, 0, \ldots)$-RAV) always outputs a committee that provides JR.

- **Proof**:
  - Suppose after $k$ steps we have $\frac{n}{k}$ uncovered voters who all approve $a$
  - $a$’s weight is $\geq \frac{n}{k}$
  - then at each step we chose a set that covered $\geq \frac{n}{k}$ uncovered voters
  - thus we should have covered all $n$ voters
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Rules that fail JR

- AV fails JR for $k \geq 3$
- SAV fails JR for $k \geq 2$
- MAV fails JR for $k \geq 2$
  - except if each ballot is of size $k$ and ties are broken in favour of JR

For $k=3$, AV outputs \{c_1, c_2, c_3\}
PAV, RAV and JR

• **Theorem:** PAV satisfies JR
• **Theorem:** \((w_1, w_2, \ldots)\)-PAV satisfies JR iff \(w_j \leq 1/j\) for all \(j\)
• **Theorem:** RAV satisfies JR for \(k = 2\) and fails JR for \(k \geq 10\)
  – \(k = 3, \ldots, 9\) is open!
• **Theorem:** \((w_1, w_2, \ldots)\)-RAV fails JR if \(w_2 > 0\)
  – \((1, 0, \ldots)\)-RAV is GAV and satisfies JR
## Summary: JR

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• Approval-based multiwinner rules

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Is JR Enough?

• Should we choose $c_4$ ???

• Perhaps a very large coherent group of voters “deserves” several representatives?

• **Idea**: if $n/k$ voters who agree on a candidate “deserve” one representative, then maybe $\lfloor n/k \rfloor$ voters who agree on $l$ candidates “deserve” $l$ representatives?
Extended Justified Representation

• **Definition**: a committee \( W \) provides extended justified representation (EJR) for a list of ballots \( (A_1, \ldots, A_n) \) and committee size \( k \) if for every \( l > 0 \), every set of voters \( X \) with \( |X| \geq l \cdot n/k \) and \( \bigcap_{i \in X} A_i \geq l \) it holds that \( |W \cap A_i| \geq l \) for at least one \( i \in X \).

• \( l = 1 \): justified representation
Satisfying EJR

• **Observation**: GAV fails EJR

• **Theorem**: PAV satisfies EJR, but is **NP-hard** to compute
  – do any other rules satisfy EJR?

• **Theorem**: \((w_1, w_2, \ldots)\)-PAV fails EJR if
  \((w_1, w_2, \ldots) \neq (1, 1/2, 1/3, \ldots)\)

• **Theorem**: checking if a committee provides EJR is **coNP-complete**

• **Open**: complexity of finding an EJR committee
## Summary: EJR

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Thank you!