Proportionally Representative Participatory Budgeting with Ordinal Preferences

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Abstract

Participatory budgeting (PB) is a democratic paradigm whereby voters decide on a set of projects to fund with a limited budget. We consider PB in a setting where voters report ordinal preferences over projects and have (possibly) asymmetric weights. We propose proportional representation axioms and clarify how they fit into other preference aggregation settings, such as multi-winner voting and approval-based multi-winner voting. As a result of our study, we also discover a new solution concept for approval-based multi-winner voting, which we call Inclusion PSC (IPSC). IPSC is stronger than proportional justified representation (PJR), incomparable to extended justified representation (EJR), and yet compatible with EJR. The well-studied Proportional Approval Voting (PAV) rule produces a committee that satisfies both EJR and IPSC; however, both these axioms can also be satisfied by an algorithm that runs in polynomial-time.

1 Introduction

Participatory budgeting (PB) provides a grassroots and democratic approach to selecting a set of public projects to fund within a given budget (Aziz and Shah 2020). It has been deployed in several cities all over the globe (Shah 2007). In contrast to standard political elections, PB requires consideration of the (heterogeneous) costs of projects and must respect a budget constraint. When examining PB settings formally, standard voting axioms and methods that ignore budget constraints and differences in each project's cost need to be reconsidered. In particular, it has been discussed in policy circles that the success of PB partly depends on how well it provides representation to minorities (Bhatnaga et al. 2003). We take an axiomatic approach to the issue of proportional representation in PB.

In this paper, we consider PB with *weak ordinal preferences*. Ordinal preferences provide a simple and natural input format whereby participants rank candidate projects and are allowed to express indifference. A special class of ordinal preferences are dichotomous preferences (sometimes referred to as approval ballots); this input format is used in most real-world applications of PB. However, in recent years, some PB applications have shifted to requiring linear order inputs. For example, in the New South Wales state of Australia, participants are asked to provide a partial strict ranking over projects.[1] The PB model we consider encompasses both approval ballots and linear order inputs.

In most of the PB settings considered, the participants are assumed to have the same weight. However, in many scenarios, symmetry may be violated. For example, in liquid democracy or proxy voting settings, a voter could be voting on behalf of several voters so may have much more voting weight. Similarly, asymmetric weights may naturally arise if PB is used in settings where voters have contributed different amounts to a collective budget or voters are affected by the PB outcome to different extents. Therefore, we consider PB where voters may have asymmetric weights.

While there is much discussion on fairness and representation issues in PB, there is a critical need to formalize reasonable axioms to capture these goals. We present two new axioms that relate to the proportional representation axiom, *proportionality for solid coalitions (PSC)*, advocated by Dummett for multi-winner elections (Dummett 1984). PSC has been referred to as "a sine qua non for a fair election rule"(Woodall 1994) and the essential feature of a voting rule that makes it a system of proportional representation (Tideman 1995). We use the key ideas underlying PSC to design new axioms for PB settings. Our axioms provide yardsticks against which existing and new rules and algorithms can be measured. We also provide several justifications for our new axioms.

Contributions We formalize the setting of PB with weak ordinal preferences. Previously, only restricted versions of the setting, such as PB with approval ballots, have been axiomatically studied (Aziz, Lee, and Talmon 2018). We then propose two new axioms *Inclusive PSC (IPSC)* and *Comparative PSC (CPSC)* that are meaningful proportional representation and fairness axioms for PB with ordinal preferences. In contrast to previous fairness axioms for PB with approval ballots (see, e.g., Aziz, Lee, and Talmon 2018), both IPSC and CPSC imply exhaustiveness (i.e., no addi-

[1] https://mycommunityproject.service.nsw.gov.au

	Approval Ballots	Ordinal Prefs
Divisible	(e.g. Bogomolnaia, Moulin, and Stong 2005)	(e.g. Aziz and Stursberg 2014)
Indivisible	(e.g. Goel et al. 2019)	**This paper**

Table 1: Classification of the literature on fair participatory budgeting with ordinal preferences.

tional candidate can be funded without exceeding the budget limit).

We show that an outcome satisfying Inclusive PSC is always guaranteed to exist and can be computed in polynomial time. The concept appears to be the "right" concept for several reasons. First, it is stronger than the local-BPJR-L concept proposed for PB when voters have dichotomous preferences (Aziz, Lee, and Talmon 2018). Second, it is also stronger than *generalised PSC* for multi-winner voting with ordinal preferences (Aziz and Lee 2020). Third, when voters have dichotomous preferences, it implies the well-studied concept PJR for multi-winner voting, is incomparable to the EJR axiom (Aziz et al. 2018), and yet is compatible with EJR. In particular, the well-studied proportional approval voting rule (PAV) computes an outcome that satisfies both IPSC and EJR; however, there also exists polynomial-time algorithms that can achieve this. Even for this restricted setting, it is of independent interest. To show that there exists a polynomial-time algorithm to compute an outcome satisfying IPSC, we present the *PB Expanding Approvals Rule (PB-EAR)* algorithm.

We also show that the CPSC is equivalent to the generalised PSC axiom for multi-winner voting with weak preferences, to Dummett's PSC axiom for multi-winner voting with strict preferences, and to PJR for multi-winner voting with dichotomous preferences.

2 Related Work

PB with ordinal preferences can be classified across different axes. One axis concerns the input format. Voters either express dichotomous preferences or general weak or linear orders. Along another axis, either the projects are divisible or indivisible. When the inputs are dichotomous preferences, there has been work both for divisible (Bogomolnaia, Moulin, and Stong 2005; Aziz, Bogomolnaia, and Moulin 2019) as well as indivisible projects (Aziz, Lee, and Talmon 2018; Faliszewski and Talmon 2019). When the input concerns rankings, then there is work where the projects are divisible (see, e.g., Aziz and Stursberg 2014; Airiau et al. 2019). Some of the work is cast in the context of probabilistic voting but is mathematically equivalent to PB for divisible projects.

To the best of our knowledge, fairness axioms for PB for discrete projects have not been studied deeply when the input preferences are general ordinal preferences. Therefore, this paper addresses an important gap in the literature. Table 1 provides a classification of the literature.

Aziz, Lee, and Talmon (2018), Faliszewski and Talmon (2019), and Baumeister, Boes, and Seeger (2020) focused on PB with discrete projects where the input preference format is approval ballots. We show that our general axioms have connections with proportional representation axioms proposed by Aziz, Lee, and Talmon (2018) for the case of approval-ballots. We will also show how our approach has additional merit even for the case of approval-ballots. For example, in contrast to previously proposed axioms in Aziz, Lee, and Talmon (2018), our axioms imply a natural property called exhaustiveness.

Fluschnik et al. (2017) consider the discrete PB model and study the computational complexity of maximizing various notions of social welfare, including Nash social welfare. Benadè et al. (2017) study issues surrounding preference elicitation in PB with the goal of maximizing utilitarian welfare. In their model, they also consider input formats in which voters express ordinal rankings. However, their focus is not on proportional representation. Fain, Goel, and Munagala (2016) considered PB both for divisible settings as well as discrete settings. However, their focus was on cardinal utilities. In particular, they focus on a demanding but cardinal-utility centric concept of core fairness. Our ordinal approach caters to many settings in which voters only express rankings over projects. Other works on cardinal utilities include Fain, Munagala, and Shah (2018) and Bhaskar, Dani, and Ghosh (2018). In recent work, Rey, Endriss, and de Haan (2020) study an end-to-end model of participatory budgeting and focus primarily on strategic behaviour.

The paper is also related to a rapidly growing literature on multi-winner voting (Aziz et al. 2017a; Faliszewski et al. 2017; Aziz et al. 2017b; Elkind et al. 2017; Janson 2016; Schulze 2002; Tideman 2006). PB is a strict generalization of multi-winner voting. Our axiomatic approach is inspired by the PSC axiom in multi-winner voting. The axiom was advocated by Dummett (1984). PSC has been referred to as the most important requirement for proportional representation in multi-winner voting (Woodall 1994, 1997; Tideman and Richardson 2000; Woodall 1994; Tideman 1995). Figure 1 provides an overview of which model reduces to which other model. We dedicate a separate section to multi-winner voting because one of our axioms gives rise to a new and interesting axiom for the restricted setting of multi-winner voting.

3 Preliminaries

A PB setting is a tuple $(N, C, \succsim, b, w, L)$ where N is the set of n voters, C is the set of candidate projects (candidates), and L is the total *budget limit*. In the context of PB, it makes sense to refer to C as the set of projects. However, we will also refer to them as candidates especially when making connections with multi-winner voting. The function $w : C \rightarrow \mathbb{R}^+$ specifies the cost $w(c)$ of each candidate

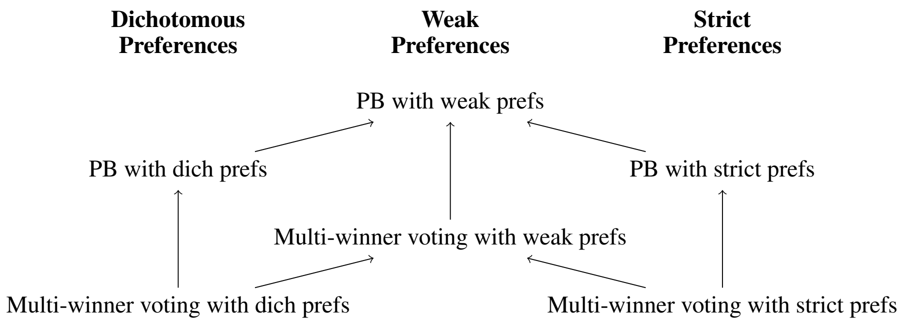

Figure 1: Relations between properties. An arrow from (A) to (B) denotes that (B) is more general than (A).

$c \in C$. We will more generally refer to $w(c)$ as the weight of the candidate project c. The function $b : N \to \mathbb{R}^+$ specifies a *voter weight* b_i for each $i \in N$. We assume that $\sum_{i\in N} b_i$ is $|N|$. For any set of voters $S \subseteq N$, we will denote $\sum_{i\in S} b_i$ by $b(S)$. Therefore $b(N) = n$. Abusing notation slightly, for any set of candidates $C' \subseteq C$, we will denote $\sum_{c\in C'} w(c)$ by $w(C')$. An outcome, denoted by W, is a set of candidates. A set of candidates (or outcome) $W \subseteq C$ is *feasible* with respect to L if $w(W) \leq L$. The preference profile $\succsim$ specifies for each voter $i \in N$, her ordinal preference relation over C. In the terminology of Benadè et al. (2017), the input format can be viewed as '*rank by value*' so that voters rank projects according to how they value them without taking costs into account.

We write $a \succsim_i b$ to denote that voter i values candidate a at least as much as candidate b and use $\succ_i$ for the strict part of $\succsim_i$, i.e., $a \succ_i b$ if and only if $a \succsim_i b$ but not $b \succsim_i a$. Finally, $\sim_i$ denotes i's indifference relation, i.e., $a \sim_i b$ if and only if both $a \succsim_i b$ and $b \succsim_i a$. The relation $\succsim_i$ results in (non-empty) equivalence classes $E_i^1, E_i^2, \ldots, E_i^{m_i}$ for some m_i such that $a \succ_i a'$ if and only if $a \in E_i^l$ and $a' \in E_i^{l'}$ for some $l < l'$. Often, we will use these equivalence classes to represent the preference relation of a voter as a preference list. If each equivalence class is of size 1, then the preference will be a called *strict preference*. If for each voter, the number of equivalence classes is at most two, the preferences are referred to as *dichotomous preferences*. When the preferences of the voters are dichotomous, the voters can be seen as approving a subset of voters. In this case, for each voter $i \in N$, the first equivalence class E_i^1 is also referred to as an *approval ballot* and is denoted by $A_i \subseteq C$. Note that in this special case, where a voter i has dichotomous preferences, the approval set A_i contains all information about voter i's preference. The vector $A = (A_1, \ldots, A_n)$ is referred to as the *approval ballot profile*. If a voter is indifferent between all candidates, then voter i's approval ballot could be interpreted to be either $A_i = \emptyset$ or $A_i = C$; our results and axioms are independent of this interpretation.

Multi-winner voting can be viewed as a special kind of PB setting in which $w(c) = 1$ for all $c \in C$ and $b_i = 1$ for all $i \in N$. The budget limit L is typically denoted by committee size k. Any setting that allows for weak preferences can be viewed as encapsulating the corresponding setting with approval ballots. The reason is that approval ballots can be viewed as dichotomous preferences.

It will be useful to distinguish between two types of PB outcomes: exhaustive and maximal cost outcomes. These concepts do not rely on the preferences of voters and, instead, are defined solely in terms of the cost of candidates, $w(c)$, and the budget, L.

Definition 1 (Exhaustive outcomes)**.** *An outcome W is said to be exhaustive w.r.t. L if $w(W) \leq L$ and $w(W \cup \{c\}) > L$ for all $c \in C \setminus W$.*

Definition 2 (Maximal cost outcomes)**.** *An outcome W is said to be a maximal cost outcome w.r.t. L if $W \in \arg\max_{C'}\{w(C') \ : \ C' \subseteq C$ and $w(C') \leq L\}$.*

Note that a maximal cost outcome is always exhaustive but an exhaustive outcome need not be maximal cost. In multi-winner voting, since we only consider outcomes that use up the budget limit of k, it means that all feasible outcomes are both exhaustive and maximal cost.

4 Proportional Representation in PB with Ordinal Preferences

Before we develop and formally define our concepts, we give some simple examples to provide intuition behind our main ideas.

We first warm up with an example that captures the *proportionality for solid coalitions (PSC)* concept of Dummett (1984). The example concerns a context in which multi-winner voting coincides with PB.

Example 1 (Motivating example I)**.** *Suppose there are 9 voters and 4 projects: a, b, c, d. The budget limit is 3 dollars and each project costs 1 dollar. Hence, three projects are to be selected. Suppose the preferences of the voters are as follows.*

$$\begin{aligned} 1-6:&\quad a \succ b \succ c \succ d \\ 7-8:&\quad d \succ c \succ b \succ a \\ 9:&\quad c \succ a \succ b \succ d \end{aligned}$$

PSC requires that both a and b are selected among the three selected projects. Informally speaking, the rationale is that two-thirds of the voters most prefer a and then b, and if they are assumed to have control over two-thirds of the budget, then they have the ability to afford both a and b.

Following the original PSC axiom for multi-winner elections, our concepts are based on the idea that if a group of voters is large, and cohesively most prefers a certain set of projects,[2] then sufficient funding should be given to projects within the set.

Example 2 (Motivating example II)**.** *Let voter preferences be*

$$\begin{aligned} 1-30: &\quad a \succ b \succ c \succ d \\ 31-100: &\quad d \succ c \succ b \succ a. \end{aligned}$$

Suppose the total budget limit is 100*, and the weights of the projects are* $w(a) = 50$*,* $w(b) = 30$*,* $w(c) = 30$*,* $w(d) = 40$*. The first group of voters (1-30) have* $30/100$ *of the voter population size. Our concepts can be motivated by supposing that all the voters have equal control of the budget. Thus, the first group of voters can be viewed as controlling 30 units of the total budget limit of 100. However, these voters cannot "afford" their most preferred project,* a*, as its weight of 50 is more than 30 units of the budget that they control. Yet, the first group of voters' second most preferred project,* b*, is affordable, having weight of only 30 units. Accordingly, the first group of voters can be thought of as having a justified demand that a project no worse than their second-most preferred project is selected, i.e., either project* a *or* b*. By a similar argument, the second group of voters have a justified demand that both project* d *and* c *are selected, since* $w(c) + w(d) \leq 70$*. However, they do not have a justified demand that projects* d, c *and* b *are selected, since* $w(c) + w(d) + w(b) > 70$*. Notice that a key difference between multi-winner elections and the PB setting is that projects may have heterogeneous weights.*

The concepts become more complicated when ties are considered in the preference lists.

Example 3 (Motivating example III)**.** *Consider a modification of Example 2 such that the first group of voters are indifferent between* b *and* c *as follows.*

$$\begin{aligned} 1-30: &\quad a \succ b \sim c \succ d, \\ 31-100: &\quad d \succ c \succ b \succ a. \end{aligned}$$

Then, the voters in the first group would not care if c *is selected or* b *is selected.*

More generally, our concepts do not require voters in a single group to have perfectly aligned preferences.

Example 4 (Motivating example IV)**.** *Consider a modification of Example 2 such that the first group of voters are split into two subgroups as follows.*

$$\begin{aligned} 1-15: &\quad a \succ b \succ c \succ d \\ 16-30: &\quad b \succ a \succ c \succ d \\ 31-100: &\quad d \succ c \succ b \succ a. \end{aligned}$$

In this case, the first group (1-15) and the second group (16-30) of voters do not agree on which project is most preferred but they are cohesive in the sense that they unanimously agree that the two-most preferred projects are a *and* b*. Since none of the groups can afford their respective most-preferred project with the budget they control, our concepts require that these two groups are allowed to combine their budgets to make a justified demand for either project* a *or* b*.*

Reasoning about proportional representation becomes, yet again, more complicated when a group of voters can be combined with many different groups.

Example 5 (Motivating example V)**.** *Let voter preferences be*

$$\begin{aligned} 1-14: &\quad a \succ b \succ c \succ d \\ 15-30: &\quad a \succ c \succ b \succ d \\ 31-100: &\quad c \succ a \succ b \succ d, \end{aligned}$$

the total budget limit 100*, and the weights of the projects* $w(a) = 90$*,* $w(b) = 30$*,* $w(c) = 80$*,* $w(d) = 40$*. Here, the second group of voters (15-30) share a most-preferred project (project* a*) with the first group of voters (1-14) but also share their two-most preferred projects,* a *and* c*, with the third group of voters (31-100). However, the first and second group combined cannot afford project* a*, which has weight 90. Yet, the second and third group can afford project* c*, which has weight* 80*.*

The last example highlights an additional and key challenge presented by the PB setting that is not present in the multi-winner setting. When groups of voters are combined, their justified demand for projects depends not only on the size of the groups (i.e., the size of the budget that they control), but also the weight of the projects that they prefer. The concepts that we introduce and develop are flexible enough to capture all of the variants of the example described above.

Before presenting our main concepts in the next section, we introduce the notion of a generalised solid coalition and some technical notation. The notion of a generalised solid coalition is central to the PSC of Dummett axioms (Dummett 1984) and the related concepts that we develop. Intuitively, a set of voters N' forms a generalised solid coalition for a set of candidate projects C' if every voter in N' weakly prefers every candidate project in C' to any candidate project outside of C'. Importantly, voters that form a generalised solid coalition for a candidate-project-set C' are not required to have identical preference orderings over candidate projects within C' nor $C \backslash C'$.

Definition 3 (Generalised solid coalition)**.** *Suppose voters have weak preferences. A set of voters* N' *is a* generalised solid coalition *for a set of candidates* C' *if every voter in* N' *weakly prefers each candidate in* C' *to each candidate in* $C \backslash C'$*. That is, for all* $i \in N'$ *and for any* $c' \in C'$*,* $\forall c \in C \backslash C' \quad c' \succsim_i c$*. The candidates in* C' *are said to be* solidly supported *by the voter set* N'*, and conversely the voter set* N' *is said to* solidly support *the candidate set* C'*.*

Like Dummett's PSC axioms (Dummett 1984), our axioms will capture intuitive features of proportional representation by ensuring that minority groups of voters are represented in the PB outcome so long as they share similar preferences over candidates, i.e., they form a generalised solid coalition, and the amount of representation given to a group

[2] I.e., there is a set of projects that all voters of the group unanimously prefer to all other projects; as will be shown below, this does not require voters to have perfectly aligned preferences.

of voters that form a generalised solid coalition is (approximately) in proportion to their size.

Lastly, we introduce some technical notation and terminology that is required for our main concepts. Let $c^{(i,j)}$ denote voter i's j-th most preferred candidate or one such candidate if indifferences are present. To attain such a candidate $c^{(i,j)}$ in the presence of indifferences the following procedure can be used: (1) break all ties in voter i's preferences temporarily to get an artificial strict order and (2) identify the j-th candidate $c^{(i,j)}$ in the artificial strict order. If a set of voters N' supports a set of candidates C', we will refer to $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \setminus C'$ as the *periphery* of the set of candidates C' with respect to voter set N'.

Main New Concepts

We now present our key concepts for proportional representaton. The concepts are inspired by the PSC concept that was proposed by Dummett (1984) for multi-winner voting for strict preferences. The PSC concept requires that if a set of voters N' solidly supports a set of candidates C', then a proportional number of candidates should be selected from C' especially if C' is large enough.

Although the PSC is quite intuitive and natural, extending it for our general PB settings needs to be done with care. In particular, the presence of candidate weights, budget limits, and indifference cause several complications so we need to define the concepts for the general PB setting carefully. The concepts are based on the requirements put forth on the outcome W. Each requirement corresponds to set of voters $N' \subseteq N$ solidly supporting a set of candidates C'. Since these voters solidly support C', the proportional representation concepts require that sufficient amount of weight in W should come from either candidates in C' or candidates in the *periphery* of the set of candidates C' with respect to voter set N'.[3]

When formally defining these requirements of the weight composition of W, we also need to take care that voters in N' do not require very heavy weight candidates to be included in the outcome. Another guiding principle while formalizing the concepts is that the existence of an outcome satisfying the concepts is not ruled out because of previous insights on subdomains of PB such as multi-winner voting. Next, we use the ideas mentioned above to formally introduce our first key solution concept.

Definition 4 (Comparative PSC (CPSC) for PB with general preferences)**.** *A budget W satisfies* Comparative PSC (CPSC) *if there exists no set of voters $N' \subseteq N$ such that N' solidly supports a set of candidates C' and there is a subset of candidates $C'' \subseteq C'$ such that*

$$w(\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W) < w(C'') \leq b(N')L/n.$$

The intuition for CPSC is that if a set of voters N' solidly supports a subset C' then it may *start to think* that at least weight $b(N')L/n$ worth of candidates should be selected from C' or its periphery especially if there is enough weight present. At the very least it should not be the case that there is a feasible subset of $C'' \subseteq C'$ of weight at most $b(N')L/n$ but the weight of $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W$ is strictly less.

Inclusion PSC is defined similarly to Comparative PSC.

Definition 5 (Inclusion PSC for PB with general preferences)**.** *An outcome W satisfies* Inclusion PSC (IPSC) *if there exists no set of voters $N' \subseteq N$ who have a solidly supported set of candidates C' such that there exists some candidate $c^* \in C' \setminus (\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W)$ such that*

$$w(c^* \cup (\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W)) \leq b(N')L/n.$$

The intuition for IPSC is that if a set of voters N' solidly supports a subset C' then it may *start to think* that a weight $b(N')L/n$ should be selected from C' or its periphery especially if there is enough weight present. At the very least it should not be the case that weight of $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W$ does not exceed $b(N')L/n$ even if some unselected candidate in $c^* \in C'$ can be added to $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W$.

For both IPSC and CPSC, we avoid violation if for N' solidly supporting candidates in C', the weight of $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W$ is large enough. That is, we only impose representation requirements for sets of voters who solidly support a set of candidates. If, instead, representation requirements were enforced for all sets of voters, regardless of whether they solidly supported a set of candidates or not, then it may not be possible to satisfy either axiom. This observation has already been made in the context of multi-winner voting (see, e.g., Aziz et al. 2017a). Similarly, both axioms focus on whether the weight $\{c : \exists i \in N' \text{ s.t. } c \succsim_i c^{(i,|C'|)}\} \cap W$ is large enough. If we only care about the weight of $C' \cap W$, then, again, it can be impossible to satisfy the requirements for all solid coalitions (Aziz et al. 2017a).

Next, we show that IPSC and CPSC are independent. The intuition is as follows. CPSC is stronger than IPSC in one respect: it cares about the maximum weight of candidates that are preferred by a coalition of voters whereas IPSC cares about set inclusion. On the other hand, IPSC is stronger in the following respect. For a violation of CPSC, we restrict ourselves to a subset of the solidly supported set of candidates $C'' \subseteq C'$. For a violation of IPSC, we even allow for inclusion of a candidate c that is not in the set of solidly supported set of candidates.

Proposition 1. *For PB with ordinal preferences, IPSC does not imply CPSC and CPSC does not imply IPSC.*

Both IPSC and CPSC imply exhaustiveness as shown in the proposition below.

Proposition 2 (CPSC and IPSC are exhaustive)**.** *Any outcome W that satisfies CPSC or IPSC is exhaustive.*

CPSC implies the stronger maximal cost property. As will be shown within the proof of Proposition 1, an IPSC outcome need not be a maximal cost outcome.

[3] Allowing for the weight representation to come from the periphery is essential because otherwise even for multi-winner voting, an outcome satisfying the requirements may not exist.

Proposition 3 (CPSC implies maximal cost)**.** *Any outcome W that satisfies CPSC is a maximal cost outcome.*

Concepts with Approval Ballots

We revisit our central concepts in the special but well-studied domain of approval ballots. We provide characterizations of both CPSC and IPSC when the voters have dichotomous preferences. At the end of this section, we show that these characterizations highlight connections between our axioms (CPSC, IPSC) and axioms that have previously been established in the PB literature. The following proposition provides a characterization of CPSC in this domain.

Proposition 4 (Comparative PSC (CPSC) for PB with approval preferences)**.** *Suppose voters have dichotomous preferences. An outcome W satisfies* Comparative PSC (CPSC) *if and only if the following two conditions hold:*

(i) *there exists no set of voters $N' \subseteq N$ such that there is a subset of candidates $C'' \subseteq \bigcap_{i\in N'} A_i$ such that $w(C'') \leq b(N')L/n$ but $w(W \cap \bigcup_{i\in N'} A_i) < w(C'')$, and*

(ii) *the outcome W is a maximal cost outcome.*

We also obtain a characterization of IPSC under approval ballots.

Proposition 5 (Inclusion PSC for PB with approval preferences)**.** *Suppose voters have dichotomous preferences. An outcome W satisfies* Inclusion PSC (IPSC) *if and only if the following two conditions hold:*

(i) *there exists no set of voters $N' \subseteq N$ such that $w(\cup_{i\in N'} A_i \cap W) < b(N')L/n$ and there exists some $c \in (\cap_{i\in N'} A_i) \setminus (\cup_{i\in N'} A_i \cap W)$ such that $w(\{c\} \cup (\cup_{i\in N'} A_i \cap W)) \leq b(N')L/n$, and*

(ii) *the outcome W is exhaustive.*

PB with approval ballots has been considered by Aziz, Lee, and Talmon (2018). For example, they proposed the concept BPJR-L. In the restricted setting studied by Aziz, Lee, and Talmon (2018), CPSC for PB with approval preferences is equivalent to the combination of the B-PJR-L and the maximal cost concepts. BPJR-L is weaker than CPSC because BPJR-L does not imply maximal cost.

Remark 1. *In the standard multi-winner setting, outcomes are required to have maximal cost (and hence are exhaustive). Thus, condition (ii) in Proposition 4 and 5 are always satisfied in the multi-winner setting.*

IPSC for PB with approval preferences is stronger than the Local-BPJR-L proposed by Aziz, Lee, and Talmon (2018).

5 Computing Proportional Outcomes

In this section, we focus on the computational aspects of proportionally representative outcomes. Our first observation is that computing a CPSC outcome is computationally hard, even for one voter. The reduction is from the knapsack problem.

Proposition 6. *Computing a CPSC outcome is weakly NP-hard even for the case of one voter.*

Next, we show that even for one voter with strict preferences, a CPSC outcome may not exist.

Example 6. *Consider the following PB instance with one voter and 4 candidate projects. The voters' preferences are as follows. $1 : a \succ b \succ c \succ d$. The limit L is 4 and the weights are: $w(a) = 3, w(b) = w(c) = w(d) = 2$. CPSC requires that project a must be selected. It also requires that $\{b, c\}$ should be selected. Therefore, a CPSC outcome does not exist.*

Later, we will show that in a more restrictive setting (multi-winner approval voting) a CPSC outcome always exists, can be computed in polynomial-time, and coincides with a well-established proportional representation axiom, called PJR.

Algorithm 1 PB Expanding Approvals Rule (PB-EAR)

Input: $(N, C, \succsim, b, L, w)$ $\{\succsim$ can contain weak preferences; if a voter i expresses her preferences over a subset $C' \subset C$, then $C \setminus C'$ is considered the last equivalence class of the voter.$\}$
Output: $W \subseteq C$ such that $w(W) \leq L$.
1: $j \longleftarrow 1$; $W \longleftarrow \emptyset$
2: **while** $w(W) < L$ and some candidate can be added to W without exceeding budget limit L **do**
3:   **for** $i \in N$ **do**
4:     $A_i^{(j)} \longleftarrow \{c \in C \,:\, c \succsim_i c^{(i,j)}\}$
5:   **end for**
6:   $C^* \longleftarrow \{c \in C \backslash W \,:\, \sum_{\{i\in N \,:\, c\in A_i^{(j)}\}} b_i \geq n\frac{w(c)}{L}\}$
7:   **if** $C^* = \emptyset$ **then**
8:     $j \longleftarrow j + 1$
9:   **else**
10:     Select a candidate c^* from C^* and add it to W
11:     $N' \longleftarrow \{i \,:\, c^* \in A_i^{(j)}\}$
12:     Modify the weights of voters in N' so the total weight of voters in N', i.e., $\sum_{i\in N'} b_i$, decreases by exactly $n\frac{w(c)}{L}$.
13:   **end if**
14: **end while**
15: **return** W

In contrast to CPSC, we show that an IPSC outcome is not only guaranteed to exist but it can be computed in polynomial time via Algorithm 1, which we refer to as PB-EAR. The algorithm is a careful generalization of the EAR algorithm of Aziz and Lee (2020). In the algorithm, W is initially empty. Some most preferred candidate c is selected (i.e., added into the set W) if it has sufficient support $n \cdot (w(c))/L$ from the voters. If c is selected, then $n \cdot (w(c))/L$ voting weight of the voters who most prefer c is decreased; it does not matter which of these voters' weight is decreased nor by how much — so long as a total of $n \cdot (w(c))/L$ voting weight is reduced. If no such candidate exists, candidates further down in the preference lists of all voters are considered. It is clear that PB-EAR runs in polynomial time. The argument for PB-EAR satisfying IPSC does not depend on what way candidate c^* is selected is Step 10.

Proposition 7. *PB-EAR satisfies Inclusion PSC for PB.*

We note here that not all IPSC outcomes are possible outcomes of PB-EAR even for the restricted setting of multi-winner voting.

6 Special Focus on Multi-winner Voting

In this section, we dive into the well-studied setting of multi-winner voting, which is also referred to as committee voting. In this setting, k candidates are to be selected from the set of candidates. Note that PB reduces to multi-winner voting if the weight of each candidate is 1 and the budget limit is set to k.

We uncover some unexpected relations between fairness concepts for this particular setting. We also show that whereas CPCS does not give rise to a new fairness concept, IPSC gives rise to a new fairness concept even for the setting concerning approval ballots. When discussing concepts for PB, we will assume that voters have equal voter weight of 1. This will make it possible to form connections with concepts for multi-winner voting in which all the voters are typically treated equally.

Let us first introduce generalised PSC, which was proposed by Aziz and Lee (2020) and applies to multi-winner settings with ordinal preferences. Aziz and Lee (2020) showed that generalised PSC extends the PJR concept for multi-winner voting with approval ballots.

Definition 6 (Generalised PSC (Aziz and Lee 2020))**.** *A committee W satisfies* generalised PSC *if for every positive integer ℓ, and for all generalised solid coalitions N' supporting candidate subset C' with size $|N'| \geq \ell n/k$, there exists a set $C^* \subseteq W$ with size at least $\min\{\ell, |C'|\}$ such that for all $c'' \in C^*$, $\exists i \in N'$: $c'' \succsim_i c^{(i,|C'|)}$.*

In the multi-winner setting, our axioms have connections with previously studied axioms related to PSC. In particular, we show that CPSC is equivalent to generalised PSC, and IPSC implies generalised PSC. The latter result implies that IPSC is a stronger concept than CPSC. This is, perhaps, surprising given that in more general settings CPSC appears to be a more demanding concept than IPSC because computing a CPSC outcome is NP-hard and a CPSC outcome may may not exist.

Proposition 8. *For multi-winner voting,*

(i) *CPSC is equivalent to Generalised PSC, and*

(ii) *IPSC implies Generalised PSC (or CPSC).*

As another corollary, we note that since testing PJR is coNP-complete (Aziz et al. 2018), testing CPSC is coNP-complete.

Approval-based multi-winner voting

In this subsection, we explore our axioms in the well-studied setting of approval-based multi-winner elections. We begin by stating two established PR axioms: *Proportional Justified Representation (PJR)* (Sánchez-Fernández et al. 2017) and *Extended Justified Representation (EJR)* (Aziz et al. 2017a).

Definition 7 (PJR)**.** *Suppose all voters have dichotomous preferences. A committee W with $|W| = k$ satisfies PJR for an approval ballot profile $\boldsymbol{A} = (A_1, \dots, A_n)$ over a candidate set C if for every positive integer $\ell \leq k$ there does not exists a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \frac{n}{k}$ such that the following two conditions hold:*

(i) $\left|\bigcap_{i \in N^*} A_i\right| \geq \ell$*, and*

(ii) $\left|\left(\bigcup_{i \in N^*} A_i\right) \cap W\right| < \ell$*.*

Definition 8 (EJR)**.** *Suppose all voters have dichotomous preferences. A committee W with $|W| = k$ satisfies EJR for an approval ballot profile $\boldsymbol{A} = (A_1, \dots, A_n)$ over a candidate set C if for every positive integer $\ell \leq k$ there does not exists a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \frac{n}{k}$ such that the following two conditions hold:*

(i) $\left|\bigcap_{i \in N^*} A_i\right| \geq \ell$*, and*

(ii) $|A_i \cap W| < \ell$ *for each* $i \in N^*$*.*

Our first result is a corollary of Proposition 8. It states that, in the special case of approval-based multi-winner voting, CPSC, PJR and Generalised PSC are all equivalent.

Corollary 1. *For multi-winner voting with approval ballot, CPSC, PJR, and Generalised PSC are equivalent.*

Proof. Aziz and Lee (2020) proved that, for multi-winner voting with approval ballot, PJR and generalised PSC are equivalent.[4] We have shown that, for multi-winner voting, CPSC and generalised PSC are equivalent. ☐

Although the focus of the present paper has been on generalising the multi-winner PSC concept of Dummett (Dummett 1984) to the PB setting, Proposition 10 provides a surprising discovery in the reverse direction. In the special case of approval-based multi-winner voting, IPSC is a new PSC axiom that is closely related — albeit stronger — than PJR. In recent years, PJR and its related axioms have been intensely studied by the computational social choice community (see, e.g., Aziz et al. 2018, 2017a; Faliszewski et al. 2017; Aziz et al. 2017b; Elkind et al. 2017). Given Proposition 10 and the community's interest in PJR-like axioms, we formalise the IPSC axiom for the approval-based multi-winner setting. We also establish a number of results that illustrate the connection between IPSC and other axioms such as PJR, EJR and PAV.

Proposition 9 (Inclusion PSC (IPSC) for multi-winner voting with approval preferences)**.** *Suppose voters have dichotomous preferences. A committee W of size k satisfies* Inclusion PSC (IPSC) *if and only if there exists no set of voters $N' \subseteq N$ such that the following two conditions hold:*

(i) $|N'| \geq (|\cup_{i \in N'} A_i \cap W| + 1)n/k$*, and*

(ii) *there exists some* $c^* \in \cap_{i \in N'} A_i \backslash (\cup_{i \in N'} A_i \cap W)$*.*

Proof. Follows from Proposition 5 by setting $b(N') = |N'|$, $L = k$, $w(C') = |C'|$ for all $C' \subseteq C$, and simplifying. ☐

Proposition 10. *For multi-winner voting with approvals,*

(i) *IPSC implies PJR,*

(ii) *PJR does not imply IPSC,*

[4] Unlike the present paper, Aziz and Lee's (2020) model assumes that no voter is indifferent between all candidates; however, this assumption is not required to show the equivalence.

PB with Ordinal Prefs	PB with Approvals	**PB with Strict Pref**	Multi-winner with Ordinal Prefs	Multi-winner with Approvals	Multi-winner with Strict Prefs
CPSC	BPJR-L$^{(\#)}$	**CPSC**	generalised PSC$^{(*)}$	PJR$^{(*)}$	PSC$^{(*)}$
IPCS	**IPCS**	**IPCS**	**IPCS**	**IPCS**	**IPCS**

Table 2: Equivalent fairness concepts for social choice settings. The concepts and settings in bold are from this paper. $(*)$ implies that, for the given social choice setting, the fairness concept is equivalent to CPSC. $(\#)$ implies that, for the given social choice setting, the fairness concept combined with the maximal cost property is equivalent to CPSC.

(iii) *IPSC and EJR are incomparable.*

We next show that the well-studied voting rule *Proportional Approval Voting (PAV)* produces a committee that satisfies IPSC. Under PAV, a voter i that has j of their approved candidates elected, i.e., $j = |W \cap A_i|$, is assumed to attain utility $r(j) = \sum_{p=1}^{j} \frac{1}{p}$ if $j > 0$ and 0 otherwise.

Given an outcome W, the PAV-score of W is the sum of voter utilities, i.e., $\sum_{i \in N} r(|A_i \cap W|)$. The output of PAV is an outcome W^* that has maximal PAV-score, i.e., $W^* \in \arg\max\{\sum_{i \in N} r(|A_i \cap W|) \ : \ W \subseteq C$ and $|W| = k\}$.

Proposition 11. *PAV satisfies IPSC.*

Given that PAV implies IPSC, the above proposition shows that EJR and IPSC are compatible axioms. This follows immediately from combining the above result with the fact that PAV also implies EJR (Aziz et al. 2017a); however, IPSC and EJR do not characterize PAV. That is, there exists committees that satisfy both EJR and IPSC but are not PAV.

Proposition 12.

(i) *IPSC and EJR are compatible. That is, there always exists a committee outcome that satisfies both IPSC and EJR. In particular, the output of the PAV rule is such a committee.*

(ii) *A committee satisfying both EJR and IPSC need not be a PAV outcome.*

Part (ii) of Proposition 12 is a double-edged sword. On one hand, IPSC and EJR are insufficient in characterizing PAV. On the other hand, since PAV is computationally intractable, it suggests that computing committee outcomes that satisfy both axioms may be computationally tractable. Indeed, the following proposition proves that an outcome satisfying both axioms can be computed in polynomial-time. Interestingly, the algorithm that produces this outcome is a special case of the EAR algorithm (Aziz and Lee 2020) applied to dichotomous preferences. The algorithm in question is studied by Peters and Skowron (2020) who call it "Rule-X."

Proposition 13. *A committee satisfying both EJR and IPSC can be computed via a polynomial-time algorithm.*

Finally, we conclude by noting that there is no ranking that can be applied to EJR and IPSC in terms of PAV scores. That is, there are instances where an IPSC outcome provides higher PAV-score than an EJR outcome and vice-versa.

To summarize the results of this subsection, we provide a schematic illustration of the relationship between our axioms, PJR, EJR and PAV in Figure 2.

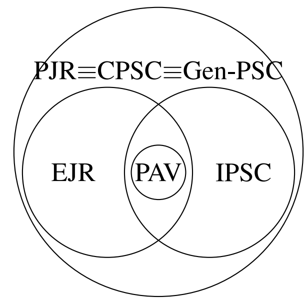

Figure 2: Schematic illustration of PJR, EJR, IPSC, CPSC and PAV for the approval-based multi-winner setting.

7 Conclusion

In this paper, we undertook a formal study of PB with ordinal preferences. Table 2 summarizes how some of the concepts are equivalent to each other in particular settings. We propose two axioms (CPSC and IPSC) that capture important aspects of the proportional representation. One of the concepts (IPSC) leads to a new concept even for the restricted setting of multi-winner voting. If voters have additive separable utilities over projects, the cardinal utility information can be used to derive the underlying ordinal preferences. Therefore, our axioms and rules also apply to settings where voters have additive separable utilities over projects. The formal study of PB from a (computational) social choice perspective is still in its infancy. We envisage further work on axioms and algorithms for fair participatory budgeting.

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