Cake Cutting Algorithms for Piecewise Constant Valuations

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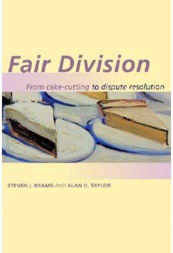
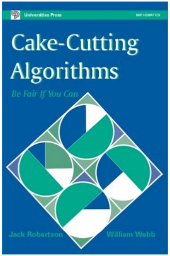
What is cake cutting?

- A metaphor for the allocation of a divisible one-dimensional heterogeneous good among multiple agents.
- **Goal:** finding an allocation that satisfies “fairness” and “efficiency” and also induces truthful reports.
- Application in scheduling of a valuable divisible resource such as server time.
- Originally studied by Polish mathematicians in the 1940s (Steinhaus, Knaster & Banach). Received a lot of attention recently from economists and computer scientists.





Interesting titles...

- A. D. Procaccia: **Cake cutting: not just child's play**. CACM 2013
- O. Arzi, Y. Aumann, and Y. Dombb. **Throw one's cake and eat it too**. SAGT 2011
- A. D. Procaccia **Thou shalt covet thy neighbor's cake** IJCAI 2009
- J. Edmonds and K. Pruhs. **Cake cutting really is not a piece of cake**. SODA 2006

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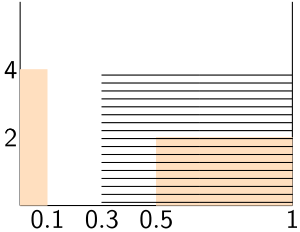
Cake representation and general valuation functions

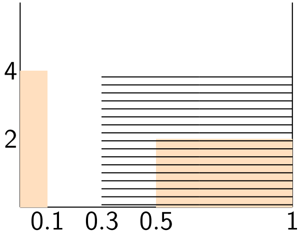
- A cake is represented by the interval $[0, 1]$.
- There is a set $N = \{1, \ldots, n\}$ of n agents.
- Each agent has a piecewise continuous value density function $v_i : [0, 1] \to [0, \infty)$.
- For any piece of cake $X \subseteq [0, 1]$, define valuation $V_i(X) = \int_X v_i(x)dx$.



Agent 1's valuation: in orange.
Agent 2's valuation: horizontal lines.
$V_1([0, 0.1]) = 0.1 \times 4$

Piecewise constant valuation functions

- A value density function v is piecewise constant if the cake can be partitioned into a finite number of intervals such that v is constant over each interval.
- A piecewise uniform valuation function v is a special case of a piecewise constant valuation function where v either takes on the value k or 0 for some $k > 0$.
- We consider a strategic version of the cake cutting problem where agents valuation function are private information. The algorithm receives as inputs each agent's reported valuation function.



Properties of allocations

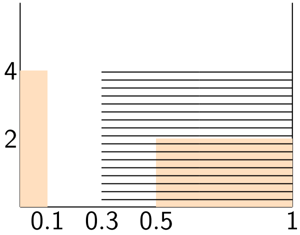
- An allocation is a partition of the cake into a set $A = \{X_1, \ldots, X_n, W\}$, where X_i is the piece of cake allocated to agent i, and W is a piece that is not allocated.

  An allocation X is:

- envy-free if $V_i(X_i) \geq V_i(X_j)$ for each pair of $i, j \in N$.[1]
- proportional if $V_i(X_i) \geq \frac{1}{n} V_i([0,1])$ for every $i \in N$.
- Pareto optimal if there does not exist X' such that $V_i(X_i') \geq V_i(X_i)$ for all $i \in N$ and $V_i(X_i') > V_i(X_i)$ for some $j \in N$.
- non-wastefulness if every part of the cake that is desired by at least one agent is allocated to some agent who desires it.

[1] Later on we will consider robust versions of envy-freeness and proportionality.

An example



$X_1 = [0, 0.1] \cup [0.7, 1]$, $X_2 = [0.3, 0.7]$ is an envy-free, proportional, and Pareto optimal allocation.

Properties of cake cutting algorithms

- A (deterministic) cake cutting algorithm is a mapping from the set of valuation profiles to the set of allocations.
- An algorithm satisfies property P (e.g., Pareto optimal) if it always returns an allocation that satisfies property P (e.g., Pareto optimal).
- A cake cutting algorithm is strategyproof if it is a dominant strategy for every agent to report his valuation function truthfully for all valuation profiles.

Research Question

Efficiency	Fairness	Incentives
Pareto optimal	Robust Envy-free	Group-SP
Non-wasteful	Robust Proportional	SP
Unanimous	Envy-free	SP in expectation
	Proportional	

When the agents have piecewise constant valuations,

1. for which (maximal) subsets of properties does there exist an algorithm that satisfies these properties?
2. which (minimal) subsets of properties are incompatible with each other?

Previous Work

1. Y. Chen and J. K. Lai and D. C. Parkes and A. D. Procaccia. Truth, justice, and cake cutting. GEB 2013.

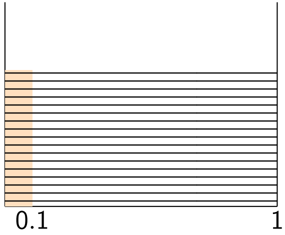
    A polynomial-time, strategyproof, Pareto efficient, and envy-free, and proportional algorithm for uniform valuations.

2. Y. J. Cohler and J. K. Lai and D. C. Parkes and A. D. Procaccia. Optimal envy-free cake cutting. AAAI 2011.

    A polynomial-time, envy-free, and proportional algorithm for constant valuations.

Challenge of strategyproofness

Challenge of strategyproofness: even the cut and choose protocol is not strategyproof!



Assume that the first agent only desires a specific small piece of cake, whereas the second agent uniformly values the cake. The first agent can obtain its entire desired piece, instead of just half of it, by carving that piece out.

Three new algorithms

- CCEA (Controlled Cake Eating Algorithm)
- MEA (Market Equilibrium Algorithm)
- MCSD (Mixed Constrained Serial Dictatorship)

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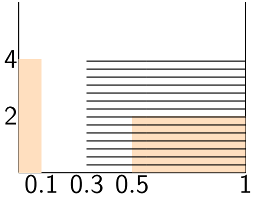
CCEA (Controlled Cake Eating Algorithm)

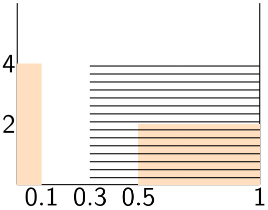
- Partition cake into intervals according to the discontinuity points. E.g.,

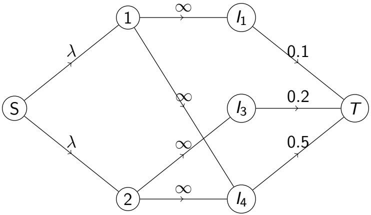


  The union of the discontinuity points are $\{0.1, 0.3, 0.5\}$. They form the intervals $[0, 0.1]$, $[0.1, 0.3]$, $[0.3, 0.5]$, and $[0.5, 1]$.
- Each agent 'eats' his most preferred interval at the same rate. Each agent gets as much fraction of an interval as he 'eats'
- Since agent may be indifferent between multiple intervals, we use parametric network flows to solve the problem. [2]

[2] This was first proposed by Katta and Sethuraman for the random assignment problem for an extension of the PS rule.

CCEA (Controlled Cake Eating Algorithm)





$$I_1 = [0, 0.1],\ I_2 = [0.1, 0.3],\ I_3 = [0.3, 0.5],\ I_4 = [0.5, 1].$$

$$X_1 = [0, 0.1] \cup [0.7, 1],\ X_2 = [0.3, 0.7].$$

CCEA (Controlled Cake Eating Algorithm)

Efficiency	Fairness	Incentives
Pareto optimal	Robust Envy-free/	Group-SP
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Unanimous	Envy-free	
	Proportional	

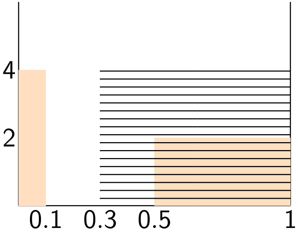
Table: Properties of CCEA

MEA (Market Equilibrium Algorithm)

- Write down a convex program to satisfy Pareto optimality and envy-freeness. This particular convex program was shown to be polynomial-time via a primal-dual algorithm (N. Devanur and C. H. Papadimitriou and A. Saberi and V. Vazirani. JACM 2008)
- Envy-freeness requires examining the dual program.
- x_{ij}: length of interval I_j that is allocated to agent i. ℓ_j: total length of interval I_j; v_{ij} is i's value for interval j; and u_i is the utility that agent i derives from his allocation.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} log(u_i) \\
\text{s.t.} \quad & u_i = \sum_{j=1}^{k} v_{ij} x_{ij} \quad \forall i = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij} \leq \ell_j \quad \forall j = 1, \ldots, k \\
& x_{ij} \geq 0 \quad \forall i, j.
\end{aligned}$$

MEA (Market Equilibrium Algorithm)



$X_1 = [0, 0.1] \cup [0.75, 0.1]$
$X_2 = [0.3, 0.75]$

(The allocation maximizes Nash welfare)

MEA

Efficiency	Fairness	Incentives
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Table: Properties of MEA

MEA and CCEA: piecewise uniform valuations

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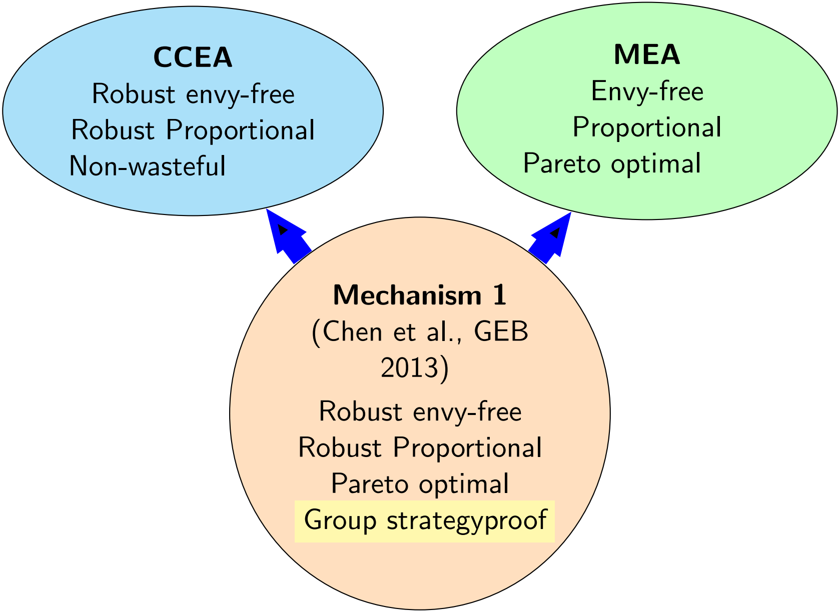
Table: Properties of MEA and CCEA for piecewise uniform

CCEA and MEA are equivalent for uniform valuations!



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Impossibilities I,II and III

Efficiency	Fairness	Incentives
Pareto optimal	Robust Envy-free/	Group-SP
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MCSD (Mixed Constrained Serial Dictatorship)

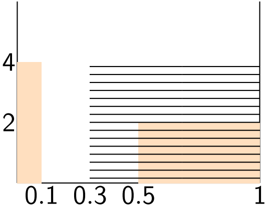
CRSD (Constrained Random Serial Dictatorship): a modification of random serial dictatorship where a serial ordering is drawn at random and each agent is allocated a piece of cake of length $1/n$ that maximizes his utility out of the remaining pieces with respect to the serial ordering.

MCSD (Mixed Constrained Serial Dictatorship)

- We de-randomize CRSD by combining the allocation of all of its n! possible realizations into a single allocation.
- However, the allocation is stated as outputting fractions of intervals.
- Another conversion step from the fractions of intervals into an actual allocation is required to ensure strategyproofness, and this is done via another randomization procedure.

SP not possible without final randomization; SP achieved with one random bit!

MCSD (Mixed Constrained Serial Dictatorship)



- Permutation $1, 2$: agent 1 asks for $[0, 0.1] \cup [0.5, 0.9]$; agent 2 asks for $[0.3, 0.5] \cup [0.9, 1]$.
- Permutation $2, 1$: agent 2 asks for $[0.3, 0.8]$; agent 1 asks for $[0, 0.1] \cup [0.8, 1]$.

- Permutation $1, 2$: agent 1 gets for $\frac{[0,0.1]}{2} \cup \frac{[0.5,0.9]}{2}$; agent 2 gets $\frac{[0.3,0.5]}{2} \cup \frac{[0.9,1]}{2}$.
- Permutation $2, 1$: agent 2 gets $\frac{[0.3,0.8]}{2}$; agent 1 gets $\frac{[0,0.1]}{2} \cup \frac{[0.8,1]}{2}$.

$X_1 = [0, 0.1] \cup [0.3, 0.4] \cup [0.5, 0.65] \cup [0.8, 0.9] \cup [0.9, 0.95]$
$X_2 = [0.4, 0.5] \cup [0.65, 0.8] \cup [0.95, 1]$

Randomized algorithms

CRSD

Strategyproof

Robust Proportional in Expectation

Unanimous

MCSD

Strategyproof in Expectation

Robust Proportional

Unanimous

MCSD (Mixed Constrained Serial Dictatorship)

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Summary

- CCEA (Controlled Cake Eating Algorithm)
- MEA (Market Equilibrium Algorithm)
- MCSD (Mixed Constrained Serial Dictatorship)

Algorithms	Restriction	DET	R-EF	EF	R-PROP	PROP	GSP	W-GSP	SP	PO	NW	UNAN	POLYT
CCEA	-	+	+	+	+	+	-	-	-	-	+	+	+
CCEA	pw uniform	+	+	+	+	+	+	+	+	+	+	+	+
MEA		+	-	+	-	+	-	-	-	+	+	+	+
MEA	pw uniform	+	+	+	+	+	+	+	+	+	+	+	+
MCSD	-	-	-	-	+	+	-	-	+	-	-	+	-
MCSD	pw uniform	-	-	-	+	+	-	-	+	+	-	+	-
MCSD	2 agents	-	+	+	+	+	-	-	+	-	-	+	+

Table: Properties satisfied by the cake cutting algorithms for pw (piecewise) constant valuations: DET (deterministic), R-EF (robust envy-freeness), EF (envy-freeness), R-PROP (robust proportionality), PROP (proportionality), GSP (group strategyproof), W-GSP (weak group strategyproof), SP (strategyproof), PO (Pareto optimal), NW (non-wasteful), UNAN (unanimity) and POLYT (polynomial-time).

Final Words

Open Problems

- Does there exist a deterministic strategyproof and proportional algorithm for piecewise constant valuations?
- Characterizations of the algorithms with respect to the properties they satisfy.

Nice Surveys

- A. D. Procaccia. Cake cutting: Not just child's play. Communications of the ACM. 2013.
- A. D. Procaccia. Cake Cutting Algorithms. Handbook of Computational Social Choice. Cambridge University Press (forthcoming).

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THANK YOU!