# A characterization of proportionally representative committees 

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#### Abstract

A well-known axiom for proportional representation is Proportionality for Solid Coalitions (PSC). We characterize committees satisfying PSC as the range of outcomes obtained by the class of Minimal Demand rules, which generalizes an approach pioneered by eminent philosopher Sir Michael Dummett.


Keywords: committee selection, multi-winner voting, proportional representation, single transferable vote.
JEL: C62, C63, and C78

## 1 Introduction

In multiwinner elections, a central concern is proportional representation of voters. Around the world, prominent electoral reform movements advocate for electoral systems that guarantee proportional representation. For example, the Electoral Reform Society in the United Kingdom has the goal of having "public authorities in the UK elected by proportional representation," ${ }^{1}$ and the FairVote organization in the United States is "committed to finding practical ways to advance [...] American forms of proportional representation. ${ }^{2}$ These concerns are not new-influential philosopher John Stuart Mill underscored the importance of proportional representation in Considerations on Representative Government:

[^0]> "In a really equal democracy, every or any section would be represented, not disproportionately, but proportionately. A majority of the electors would always have a majority of the representatives, but a minority of the electors would always have a minority of the representatives. Man for man they would be as fully represented as the majority." (Chapter VII of Mill, 1861)

When voters elicit ranked preferences over candidates, one particular axiom for proportional representation is Proportionality for Solid Coalitions (PSC). This axiom was advocated by eminent philosopher Sir Michael Dummett (1984, p. 283) and has been referred to as the most important requirement for proportional representation (Tideman, 1995; Tideman and Richardson, 2000; Woodall, 1994, 1997). ${ }^{3}$ Woodall (1994) who refers to a specific form of PSC as "DPC" underlines the importance of PSC for proportional representation as follows.
"DPC seems to me to be a sine qua non for a fair election rule. I suggest that any system that satisfies DPC deserves [...] to be regarded as a system of proportional representation [...]" (Section 4 of Woodall, 1994, emphasis in original)

PSC is the subject of many theoretical and empirical studies. Theoretical studies have focused on designing voting rules that satisfy PSC; these include single transferable vote (STV) (Tideman, 1995), Quota Borda System (QBS) (Dummett, 1997), SchulzSTV (Schulze, 2011a,b), and the Expanding Approvals Rule (EAR) (Aziz and Lee, 2020). ${ }^{4}$ STV is the most prominent of these rules; it is used for elections in Australia, Ireland, India, and Pakistan. STV has attracted significant attention in the literature from both a theoretical (Geller, 2005; Howard, 1990; Miller, 2007; Peleg and Peters, 2017; Ray, 1986; Van Deemen, 1993) and empirical perspective (Endersby and Towle, 2014; Farrell et al., 1996; Latner and McGann, 2005). However, few studies consider the structure imposed by the PSC axiom on election outcomes.

In this note, we present a characterization of PSC committees as the range of outcomes from a certain class of procedures, which we formalize and call Minimal Demand (MD) rules. The class of MD rules generalize an approach pioneered by Michael Dummett who proposed one particular rule (QBS) within our wider class. We also present an alternative way of viewing the range of outcomes that can be obtained by an MD rule in terms of a "Tree of MD rules," which represents a decision tree with decisions taken at each branching of a node. Our main result is the following: A committee satisfies PSC if and only if it is an outcome of a branch of the Tree of MD rules (equivalently, an outcome of an MD rule).

This characterization provides an intuitive and tractable method for researchers to analyze the demands of PSC on committee outcomes. In the public sphere, specific voting rules, such as STV, are often considered to be synonymous with proportional representation. Therefore, controversial, real-world election outcomes can attract criticisms of not just the voting rule but also the proportional representation principle. Our result allows

[^1]researchers to inform public debate by disentangling whether an election outcome is due to a specific voting rule or if it is a consequence of the proportional representation axiom (PSC). Our result also provides a unifying framework to analyze new and existing voting rules that satisfy PSC. For example, our characterization result shows that-not only does Dummett's QBS belong to our class of MD rules but-all of the previously mentioned voting rules (STV, Schulz-STV, and EAR) belong to this class. Our framework illustrates that the outcomes of different PSC rules correspond to different branches of the Tree of MD rules (equivalently, different ways that MD rules resolve "ties" when more than one candidate can potentially be elected at a given stage). More broadly, we hope that this characterization will be a useful stepping stone to understanding the interaction between PSC and other important axioms.

## 2 Preliminaries

We consider the standard social choice setting with a set of voters $N=\{1, \ldots, n\}$, a set of candidates $C=\left\{c_{1}, \ldots, c_{m}\right\}$, and a preference profile $\succ=\left(\succ_{1}, \ldots, \succ_{n}\right)$ such that each $\succ_{i}$ is a linear order over $C$. If $c_{s} \succ_{i} c_{t}$ then we say that voter $i$ prefers candidate $c_{s}$ to candidate $c_{t}$. We do not assume any relation between $n$ and $m$. Given $j \in\{1, \ldots, m\}$, voter $i^{\prime}$ s top- $j$ candidates will refer to the (unordered) set of $j$-most preferred candidates with respect to $\succ_{i}$. The goal is to select a committee $W \subset C$ of pre-determined size $k$.

The focus of this paper is on committees that satisfy the Proportionality for Solid Coalitions (PSC) axiom. Before formally defining PSC, we introduce the notion of a solid coalition (Dummett, 1984, p. 282), which is central to the PSC axiom. Intuitively, a set of voters $N^{\prime}$ forms a solid coalition for a set of candidates $C^{\prime}$ if every voter in $N^{\prime}$ prefers every candidate in $C^{\prime}$ to any candidate outside of $C^{\prime}$. Importantly, voters that form a solid coalition for a candidate-set $C^{\prime}$ are not required to have identical preference orderings over candidates within $C^{\prime}$ nor $C \backslash C^{\prime}$. There is no requirement on the relative sizes of $N^{\prime}$ and $C^{\prime}$.

Definition 1 (Solid coalition). A set of voters $N^{\prime}$ is a solid coalition for a set of candidates $C^{\prime}$ if for all $i \in N^{\prime}$ and for any $c^{\prime} \in C^{\prime}$

$$
\forall c \in C \backslash C^{\prime} \quad c^{\prime} \succ_{i} c
$$

The candidates in $C^{\prime}$ are said to be (solidly) supported by the voter set $N^{\prime}$; conversely, the voter set $N^{\prime}$ is said to be (solidly) supporting the candidate set $C^{\prime}$.

We now state the PSC definition (Dummett, 1984, p. 283). Informally, PSC requires that a committee $W$ "adequately" represents the preferences of solid coalitions that are "sufficiently large" by including some candidates that they support in relation to the size of the solid coalition. Here "adequately" and "sufficiently large" are defined according to a (possibly non-integer) parameter $q \in(n /(k+1), n / k]$ and leads to a hierarchy ${ }^{5}$ of PSC definitions denoted by $q$-PSC. ${ }^{6}$ If $q=n / k$, then the $q$-PSC property is often referred

[^2]to as Hare-PSC. If $q=n /(k+1)+\epsilon$, for sufficiently small $\epsilon>0, q$-PSC is often referred to as Droop-PSC. ${ }^{7,8}$ There are some reasons to prefer the Droop quota. First, for $k=1$ the Droop quota satisfies the majority principle (Woodall, 1997): a candidate that is most preferred by more than half of the voters is selected whenever such a candidate exists. Second, Droop-PSC is a stronger proportional representation requirement and implies Hare-PSC.

Definition 2 ( $q$-PSC). Let $q \in(n /(k+1), n / k]$. A committee $W$ satisfies $q$-PSC if for every positive integer $\ell$ and for every solid coalition $N^{\prime}$ supporting a candidate subset $C^{\prime}$ with size $\left|N^{\prime}\right| \geq \ell q$, the following holds

$$
\left|W \cap C^{\prime}\right| \geq \min \left\{\ell,\left|C^{\prime}\right|\right\}
$$

Given a subset $\tilde{W} \subseteq C$ and a solid coalition $N^{\prime}$ supporting $C^{\prime}$, we say that $N^{\prime}$ has an unmet PSC demand if $\left|\tilde{W} \cap C^{\prime}\right|<\min \left\{\left\lfloor\left|N^{\prime}\right| / q\right\rfloor,\left|C^{\prime}\right|\right\}$.

The $q$-PSC axiom captures intuitive features of proportional representation. In fact, Woodall (1997) refers to $q$-PSC as "the essential feature" of proportional representation. The axiom ensures representation of minority voters so long as they share similar preferences over candidates, i.e., they form a solid coalition, and the amount of representation given to a group of voters that form a solid coalition is (approximately) in proportion to their size. Importantly, the subset(s) of voters that constitute a group requiring representation are endogenously determined from the voters' preferences by the $q$-PSC axiom. In an interview with Fara and Salles (2006, p. 355), Michael Dummett stresses this "principle of identifying minorities by the way they voted" as follows:
"[minorities] identify themselves when they all vote for the same two or three candidates in perhaps different orders, but they put them all at the top. That identifies the minority, and if it's a sufficiently large minority [...], then it's bound to get represented."

In addition to capturing proportional representation concerns, PSC connects with stability notions from cooperative game theory (see, e.g., Scarf, 1967). Suppose that any subset of voters $N^{\prime}$ can choose not to participate in the election process of $k$ candidates, and not participating gives the voters the outside option of selecting their own committee $W^{\prime}$ —albeit of smaller size, $\left|W^{\prime}\right|=\left\lfloor\left|N^{\prime}\right| / q\right\rfloor$. When $q=n / k$, the size of committee $W^{\prime}$ reflects a "uniform (re)distribution" of resources. Informally speaking, PSC enforces a minimal requirement that the committee outcome, $W$, incentivizes participation from subsets of voters that are solid coalitions, where the outside option committee $W^{\prime}$ is naturally assumed to only contain the coalition's most preferred (or solidly supported) candidates.

For the remainder of the paper, we will explore the $q$-PSC axiom for fixed $q$; hence, abusing notation slightly, we refer to $q$-PSC as simply PSC. When necessary we will explicitly state the value of $q$.

[^3]We provide Example 1 to illustrate and motivate PSC. It explains the requirements that are imposed on the selected committee.

Example 1 (Illustration of PSC). Suppose there are $n=9$ voters and $m=4$ candidates: $a, b, c, d$. The target committee size is $k=3$. Hence, three candidates are to be selected. Suppose the preferences of the voters are as follows.

$$
\begin{aligned}
1-6: & a \succ b \succ c \succ d \\
7-8: & d \succ c \succ b \succ a \\
9: & c \succ a \succ b \succ d
\end{aligned}
$$

We consider the requirement imposed by Hare-PSC (i.e., $q$-PSC with $q=n / k=3$ ). Voters 1-6 form a solid coalition for candidates $a$ and $b$ and have size 6 , which exceeds $2 q=6$. Therefore, PSC requires that both $a$ and $b$ are selected among the three selected candidates. The idea behind this requirement is that if two-thirds of the voters most prefer $a$ and then $b$, and if they are assumed to have control over two-thirds of the committee, then they have the ability to enforce the selection of both $a$ and $b$.

Now suppose that the voters 1-6 do not have unanimous preferences and in fact two of them (voters 5-6) most prefer b and voter 4 prefers $d$ to $c$ as follows.

$$
\begin{aligned}
1-3: & a \succ b \succ c \succ d \\
4: & a \succ b \succ d \succ c \\
5-6: & b \succ a \succ c \succ d \\
7-8: & d \succ c \succ b \succ a \\
9: & c \succ a \succ b \succ d
\end{aligned}
$$

In this case, even though voters 1-6 have not coordinated their preferences perfectly, PSC still requires that $a$ and $b$ should be selected. This is because the voters 1-6 continue to form a solid coalition for candidates $a$ and $b$. In this way, PSC provides an implicit coordination device for any endogenous group of voters.

The above examples apply to various contexts. For example, if we scale up the voter population, the preference profile could pertain to town councils of varying sizes expressing preferences over the building of three public parks from among candidates $a, b, c, d$. The preferences could be based on proximity to the council or other factors.

$$
\begin{array}{ll}
\text { council } 1 \text { (population } 30 k \text { ) : } & a \succ b \succ c \succ d \\
\text { council } 2 \text { (population } 10 k \text { ) : } & a \succ b \succ d \succ c \\
\text { council } 3 \text { (population } 20 k \text { ) : } & b \succ a \succ c \succ d \\
\text { council } 4 \text { (population } 20 k \text { ) : } & d \succ c \succ b \succ a \\
\text { council } 5 \text { (population } 10 k \text { ) : } & c \succ a \succ b \succ d
\end{array}
$$

PSC will require that park $a$ and $b$ are funded among the three selected parks.

## 3 Minimal Demand Rules and Trees of Minimal Demand Rules

Dummett (1997, pp. 151-157) proposed the Quota Borda System (QBS) rule as follows. It examines the voters' top- $j$ candidates (for increasing $j$ ) and checks if there exists a corresponding solid coalition for a set of voters. If there is such a solid coalition of voters, then an appropriate number of candidates with the highest Borda count ${ }^{9}$ are selected so as to satisfy the corresponding PSC demand. ${ }^{10}$

We view Dummett's approach as a special case of a more general class of rules, which we call Minimal Demand (MD) rules. We formalize the class of MD rules below in 3 steps. Just like in QBS, MD rules examine voters' top- $j$ candidates (for increasing $j$ ) and checks if there exists a corresponding solid coalition for a set of voters. If there is such a solid set of voters, then a minimal subset of candidates is sequentially selected to satisfy the corresponding PSC demand.

For a given $q \in(n /(k+1), n / k]$, MD rules elect candidates as follows. Initialize $W=\emptyset$ and $j=1$.

Step 1. Partition the set of voters into equivalence classes where each class has the same top- $j$ candidates (i.e., the same (unordered) set of $j$-most preferred candidates). ${ }^{11}$

Step 2. If there exists an equivalence class of voters $N^{\prime} \subseteq N$ with top- $j$ candidates $C^{\prime}$ such that $\left|W \cap C^{\prime}\right|<\min \left\{\left\lfloor\left|N^{\prime}\right| / q\right\rfloor,\left|C^{\prime}\right|\right\}$, then for any such $c^{\prime} \in C^{\prime} \backslash W$ update $W$ to $W \cup\left\{c^{\prime}\right\}$. Repeat this step with the updated $W$ until no additional candidate can be added.

Step 3. If $j<m$, update $j$ to $j+1$ and repeat from Step 1. Otherwise, terminate and output $W$.

Later, in Example 2, we demonstrate how MD rules operate. However, before proceeding, we clarify some terminology. First, we note that Steps 1-3 describe a class of rules. A specific rule within the class of MD rules is defined by how candidates are selected in Step 2 when more than one candidate can potentially be elected (i.e., how "ties" are resolved). Abusing notation slightly, we will sometimes refer to the MD rule to mean an arbitrary MD rule. Second, for a given value $j^{\prime}$, the phrase stage $j=j^{\prime}$ of the MD rule will refer to the point in the MD rule where the index $j$ takes the value $j^{\prime}$.

MD rules consider voters' top- $j$ candidates (for increasing $j$ ) and make candidate selections whenever a PSC demand is unmet. From the description of the class of MD rules,

[^4]it is not obvious whether a specific MD rule is guaranteed to have outcome of size $k$. Lemma 1 verifies that any MD rule must terminate and output a committee of size $k .{ }^{12}$

Lemma 1. Every MD rule terminates and outputs a committee $W:|W|=k$.
Proof. The termination of any MD rule is trivial because the "if" condition in Step 2 can always be satisfied by adding all candidates in $C^{\prime}$.

We now show that every MD rule outputs $W$ : $|W|=k$. First, note that $|W| \geq k$ because, at the final stage $j=m$, all of the voters have the same top- $j$ candidates, which equals $C$. Hence, the "if" condition in Step 2 is satisfied for all $W:|W|<k$, and the rule cannot terminate with $|W|<k$. It remains to prove that an MD rule's output $W$ never exceeds size $k$. We begin by noting that, as an MD rule progresses through the $j$ stages, an injective correspondence from elected candidates to a "fractional subset" of voters can be constructed as follows: whenever a candidate, say $c_{W}$, is elected with equivalence class of voters $N^{\prime}$ (Step 2 of the MD rule), we match this candidate to any $q$ distinct-and thus far unmatched—voters from $N^{\prime} .{ }^{13}$ These $q$ distinct—and thus far unmatched—voters always exist since otherwise $N^{\prime}$ would have at least $\left\lfloor\left|N^{\prime}\right| / q\right\rfloor$ elected candidates contained in the voters' top- $j$ candidate set, which contradicts the "if" condition in Step 2 of the MD rule. Now, for the sake of a contradiction, suppose that $|W|=k^{\prime}>k$. The existence of the injective mapping described above implies that there must be at least $k^{\prime} q$ voters. But $q>n /(k+1)$ and, hence,

$$
k^{\prime} q>\frac{k^{\prime} n}{k+1} \geq n
$$

a contradiction.
Since the class of MD rules specify a candidate subset from which a candidate should be selected rather than specifying a single candidate (Step 2), there is a great deal of flexibility in how these "ties" are resolved, i.e., which candidate from the subset is selected. By considering different tie-breaking decisions, we can obtain different outcomes that arise from different MD rules. These outcomes can be represented in the form of a decision tree, which we will refer to as the Tree of MD rules.

The Tree of MD rules can be viewed as a tree corresponding to the range of possible outcomes that can be obtained by an MD rule, where each distinct outcome depends only on how the "ties" are resolved in Step 2. Each node along a path of the tree corresponds to a stage $j=1, \ldots, m$. The tree has depth $m$. If $k$ candidates have already been selected by stage $j$, we still go over all the stages. At each stage $j$ of the tree the rule only considers voters' top- $j$ candidates. Accordingly, only those PSC demands that pertain to the $j$-most preferred candidates of each voter are considered. Each path of the tree reflects an ordered

[^5]selection of candidates that coincides with an outcome of an MD rule. Each node in the path represents the selection of one or more candidates at that point keeping in view the candidates already selected at nodes higher up in the path; we include "null" nodes to represent stages where no candidate is selected.

Example 2 illustrates how the class of Minimal Demand (MD) rules work. Using the preference profile considered in Example 1, we describe every committee outcome that is obtainable via an MD rule. The example also provides an illustration of the Tree of MD rules (see Figure 1).

Example 2 (Illustration of MD rules and the Tree of MD rules). Suppose there are $n=9$ voters and $m=4$ candidates: $a, b, c, d$. The target committee size is $k=3$. Hence, three candidates are to be selected. Suppose the preferences of the voters are as follows.

$$
\begin{array}{rll}
1-3: & & a \succ b \succ c \succ d \\
4: & a \succ b \succ d \succ c \\
5-6: & b \succ a \succ c \succ d \\
7-8: & & d \succ c \succ b \succ a \\
9: & & c \succ a \succ b \succ d
\end{array}
$$

For illustration purposes, we consider Hare-PSC (i.e., $q=n / k=3$ ). We now proceed with demonstrating the process that MD rules follow. We first initialize $W=\emptyset$. In the first stage, $j=1$ and we consider solidly supporting coalitions of voters' top- $j$ candidates. There exists an equivalence class of voters $\{1, \ldots, 4\}$ who solidly support candidate $a$. This equivalence class has an unmet PSC demand: they are sufficiently large and can demand one candidate to be elected but are currently represented none $(W=\emptyset)$. At this point, to satisfy the unmet demand, candidate a is selected and added to $W$. Repeating Step 2 with the updated $W$, there are now no equivalence classes with unmet PSC demand. We proceed to stage $j=2$. When $j=2$, there exists an equivalence class of voters $\{1, \ldots, 6\}$ who solidly support $\{a, b\}$. This equivalence class has an unmet PSC demand: they are sufficiently large and can demand two candidates to be elected into $W$ but are currently only represented by one (i.e., candidate a). At this point, to satisfy the unmet demand, candidate $b$ is selected and added to $W$. Repeating Step 2 with the updated $W$, there are now no equivalence classes with unmet PSC demand. We proceed to stage $j=3$. When $j=3$, there is no equivalence class of voters with unmet PSC demand; we proceed to stage $j=4$. When $j=4$, the (trivial) equivalence class of all voters, $\{1, \ldots, 9\}$, solidly supports the entire candidate set $\{a, b, c, d\}$. This equivalence class has an unmet PSC demand: they are sufficiently large and can demand three candidates to be elected into $W$ but are currently represented by only two. In order to satisfy the unmet demand, candidate $c$ or $d$ must be selected and added to $W$. Therefore, there are two possible outcomes that arise from the class of $M D$ rules: $\{a, b, c\}$ and $\{a, b, d\}$. Figure 1 illustrates the corresponding Tree of MD rules, which provides a representation of the range of outcomes obtainable by MD rules. The tree has two distinct paths from the root node (i.e., the node at level $j=0$ ) to a leaf node (i.e., one of the two nodes at level $j=4$ ). The node at level $j=3$ is a "null" node, which illustrates that no candidate was elected at stage $j=3$.


Figure 1: Tree of MD rules.

## 4 PSC and the Tree of MD rules

We present some connections between PSC and the Tree of MD rules. Lemma 2 states that each possible path along the Tree of MD rules (equivalently, any outcome of an MD rule) leads to a PSC committee. To the best of our knowledge, it is the first formal argument that an MD rule always returns a PSC committee and the outcome being PSC does not depend on the specific "tie-breaking" choices made during Step 2 of the MD rule when more than one candidate can potentially be elected.

Lemma 2. Each path along the Tree of MD rules gives rise to a committee satisfying PSC.
Proof. Let $W$ be an outcome of an MD rule (by Lemma 1, $|W|=k$ ). For sake of a contradiction, suppose $W$ does not satisfy PSC: there exists a positive integer $\ell$ and set of voters $N^{\prime}$ with $\left|N^{\prime}\right| \geq \ell q$ solidly supporting a candidate subset $C^{\prime}$ such that $\left|W \cap C^{\prime}\right|<\min \left\{\ell,\left|C^{\prime}\right|\right\}$. But then, at stage $j=\left|C^{\prime}\right|$ of the MD rule, the voters in $N^{\prime}$ form an equivalence class (with top- $j$ candidates $C^{\prime}$ ) that satisfies the "if" condition in Step 2 . Thus, the MD rule cannot proceed to the next stage while guaranteeing the output $W$-a contradiction.

Lemma 2 implies the following. Any voting rule that proceeds by sequentially electing candidates is guaranteed to satisfy PSC so long as: (1) candidates are only added if they resolve a PSC violation, and (2) candidates that resolve violation of PSC with respect to voters' top- $j$ candidates are elected before candidates that resolve violations of PSC for larger $j$.

Lemma 3 provides a converse to Lemma 2: for every PSC committee, $W$, there exists a path of the Tree of MD rules that selects $W$ (equivalently, there exists an MD rule with outcome $W$ ).

[^6]Lemma 3. If $W$ is a committee satisfying PSC, then there exists a path of the Tree of MD rules that selects $W$.

Proof. Suppose $W$ satisfies PSC. We simulate an outcome of an MD rule that makes decisions along the tree and selects candidates from $W$. The proof is by induction on the stages of the MD rule. At each stage $j$, we restrict our selection of candidates to fulfill the PSC demands of voters (Step 2 of the MD rule) to those in $W$. First, note that since $W$ satisfies PSC, it does not satisfy the "if" condition in Step 2 for any $j$. Now, if we have selected $W_{j} \subseteq W$ by the $j$-th stage and the "if" condition in Step 2 is satisfied, then there exists $c \in W \backslash W_{j}$ that can be added at Step 2. Repeating this process leads to the outcome $W$ and, hence, there exists a path in the Tree of MD rules that selects $W$.

Lemma 3 implies the following. Any committee produced by a PSC voting rule (e.g., QBS, STV, Schulz-STV, and EAR) coincides with a path of the Tree of MD rules (equivalently, an outcome of an MD rule). Therefore, different PSC rules simply correspond to different ways of choosing branches of the tree (or equivalently, different ways of resolving "ties" within the class of MD rules) when more than one candidate can potentially be elected at a given stage.

Combining the two lemmas gives us the following equivalence theorem.
Theorem 1. The following are equivalent.
(i) A committee satisfies PSC.
(ii) A committee is an outcome of some path of the Tree of MD rules.

Proof. Lemma 2 shows that (ii) implies (i); Lemma 3 shows the converse.
As an application of Theorem 1, we show how the outcomes of three different voting rules (QBS, EAR, and STV) that satisfy PSC correspond to distinct paths of the Tree of MD rules. We borrow the following example from Aziz and Lee (2020).

Example 3 (Application of Theorem 1). Suppose there are $n=9$ voters and $m=8$ candidates: $c_{1}, c_{2}, c_{3}, d_{1}, e_{1}, e_{2}, e_{3}, e_{4}$. The target committee size is $k=3$. Suppose the preferences of the voters are as follows.

$$
\begin{aligned}
& c_{1}, c_{2}, c_{3}, e_{1}, e_{2}, e_{3}, e_{4}, d_{1} \\
& c_{2}, c_{3}, c_{1}, e_{1}, e_{2}, e_{3}, e_{4}, d_{1} \\
& c_{3}, c_{1}, d_{1}, c_{2}, e_{1}, e_{2}, e_{3}, e_{4} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1} \\
& e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}, d_{1}
\end{aligned}
$$



Figure 2: Tree of MD rules.

For illustration purposes, we consider Droop-PSC (i.e., $q=n /(k+1)+\epsilon=2.25+\epsilon$, where $\epsilon>0$ is sufficiently small). As shown in Examples 4, 5, and 7 in Aziz and Lee (2020), the QBS outcome is $W_{Q B S}=\left\{e_{1}, e_{2}, e_{3}\right\}$, the EAR outcome is $W_{E A R}=\left\{c_{1}, e_{1}, e_{2}\right\}$, and the STV outcome is $W_{S T V}=\left\{c_{2}, e_{1}, e_{2}\right\}$; for precise definitions of these voting rules, we refer the reader to Aziz and Lee (2020).

We now construct the Tree of MD rules. We start with $W=\emptyset$. In the first stage, $j=1$ and we consider solidly supporting coalitions of voters' top-j candidates. There exists an equivalence class of voters $\{4, \ldots, 9\}$ who solidly support candidate $e_{1}$. This equivalence class has an unmet PSC demand since $W=\emptyset$. To satisfy the unmet demand, candidate $e_{1}$ is selected and added to $W$. Repeating Step 2 of the MD rule process, there are now no equivalence classes with unmet PSC demand with respect to their top- $j$ candidates (where $j=1$ ). We proceed to stage $j=2$. When $j=2$, there exists an equivalence class of voters $\{4, \ldots, 9\}$ who solidly support candidates $\left\{e_{1}, e_{2}\right\}$. This equivalence class has an unmet PSC demand since they are of size 6 and, hence, can demand two candidates to be elected into $W$ but are currently only represented by one (i.e., candidate $e_{1}$ ). To satisfy the unmet demand, candidate $e_{2}$ is selected and added to $W$. Repeating Step 2 with the updated $W$, there are now no equivalence classes with unmet PSC demand. We proceed to stage $j=3$. When $j=3$, there are no equivalence classes with unmet PSC demand; the same holds true for $j=4,5,6$. When $j=7$, there exists an equivalence class of voters $\{1,2,4, \ldots, 9\}$ who solidly support candidates $\left\{e_{1}, e_{2}, e_{3}, e_{4}, c_{1}, c_{2}, c_{3}\right\}$. This equivalence class has an unmet PSC demand since they are of size 8 and, hence, can demand three candidates to be elected into $W$ but are currently only represented by two (i.e., candidates $e_{1}$ and $e_{2}$ ). To satisfy the unmet demand, one candidate from $\left\{e_{3}, e_{4}, c_{1}, c_{2}, c_{3}\right\}$ must be selected and added to $W$. This leads to 5 different ways to update $W$ : $W_{1}=\left\{e_{1}, e_{2}, e_{3}\right\} ; W_{2}=\left\{e_{1}, e_{2}, e_{4}\right\} ; W_{3}=\left\{e_{1}, e_{2}, c_{1}\right\} ; W_{4}=$ $\left\{e_{1}, e_{2}, c_{2}\right\} ; W_{5}=\left\{e_{1}, e_{2}, c_{3}\right\}$. In every case, after updating $W$, there are no equivalence classes with unmet PSC demand for all further stages and the process terminates. The outcome of this
process (i.e., the Tree of MD rules) is illustrated in Figure 2.
Combining Figure 2 and Theorem 1 tells us that there are exactly 5 PSC outcomes, all of which contain candidates $e_{1}$ and $e_{2}$ and do not contain candidate $d_{1}$. The QBS, EAR, and STV outcomes correspond to 3 distinct paths in the tree; the paths of these rules diverge at node $j=$ 7. Hence, QBS, EAR, and STV differ in terms of how they implicitly resolve the "tie" at stage $j=7$, where there are 5 valid candidates that can be selected to address the unmet PSC demand. For the QBS rule, the tie-breaking rule is explicit in the rule's description: the candidate with highest Borda count is selected. For EAR and STV, the tie-breaking rule is implicit and difficult to ascertain without implementing the voting rule itself-a perhaps interesting question is whether it is possible to characterize an explicit tie-breaking rule for these and other PSC voting rules.

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    ${ }^{1}$ "Electoral Reform Society 2021-2024 Strategy Summary" accessed via https://www.
    electoral-reform.org.uk/who-we-are/what-we-stand-for/ on 08/24/2021.
    2"Our Story" accessed via https://www.fairvote.org/our_story on 08/24/2021.

[^1]:    ${ }^{3}$ For broader discussion on axioms for multiwinner voting, see the chapter by Faliszewski et al. (2017).
    ${ }^{4}$ There is also a growing literature that explores issues of proportional representation in the context of participatory budgeting (see, e.g., Aziz and Lee, 2021; Aziz et al., 2018; Freeman et al., 2021).

[^2]:    ${ }^{5}$ If an outcome $W$ satisfies $q$-PSC, then $W$ satisfies $q^{\prime}$-PSC for all $q^{\prime}>q$ (Aziz and Lee, 2020, Lemma 2).
    ${ }^{6}$ The bounds on $q$ are necessary. If $q \leq n /(k+1)$, a $q$-PSC committee need not exist; if $q>n / k$, a $q$-PSC committee may fail a basic "unanimity" property (for details, see Footnote 10 of Aziz and Lee, 2020).

[^3]:    ${ }^{7}$ In particular, $\epsilon$ is required to be positive and small enough so that for any positive integer $\ell \leq k,\left\lceil\ell \cdot\left(\frac{n}{k+1}+\right.\right.$ $\epsilon)\rceil \leq \ell \frac{n}{k+1}+1$; such an $\epsilon$ always exists. For any $\epsilon$ and $\epsilon^{\prime}$ satisfying this condition, the PSC axiom with respect to $q=n /(k+1)+\epsilon$ and $q^{\prime}=n /(k+1)+\epsilon^{\prime}$ are equivalent (for a proof sketch, see Footnote 12 of Aziz and Lee, 2020).
    ${ }^{8}$ Droop PSC is sometimes referred to as Droop's proportionality criterion (DPC).

[^4]:    ${ }^{9}$ The Borda count is calculated as follows. For each voter's preference list, the lowest-ranked candidate receives 0 points, the second-lowest candidate receives 1 point, the third-lowest candidate receives 2 points, and so on. A candidate's Borda count is the total number of points that they receive.
    ${ }^{10}$ In an earlier work, Dummett (1984, p. 284) proposed a slightly more general voting rule called the Quota Preference Score (QPS) rule. The QPS rule proceeds similarly to QBS, but when a solid coalition of voters exists, QPS selects candidates according to some (arbitrary) preference score rather than the Borda count.
    ${ }^{11}$ Note that each voter-partition is a solid coalition supporting their top- $j$ candidates.

[^5]:    $\overline{{ }^{12} \text { One can explicitly add a stopping condition to an MD rule to terminate when }|W|=k \text {. However, this }}$ only defers the issue: one must then prove that this stopping condition does not affect the rule's ability to produce a PSC outcome. Lemma 1 provides an explicit argument that any MD rule terminates with exactly $k$ candidates. Because the QBS rule of Dummett $(1984,1997)$ belongs to the class of MD rules, Lemma 1 provides, to the best of our knowledge, the first formal proof that the QBS rule finds an outcome of size $k$; Lemma 2 (to be proven later) then implies that the QBS rule satisfies PSC.
    ${ }^{13}$ Since $q$ is possibly non-integer valued, we are treating the voters as if they are divisible. This is only for the purposes of this proof; nothing in this paper requires voters to be divisible.

[^6]:    ${ }^{14}$ In this example, each node corresponds to the election of at most one candidate, and each path corresponds to a distinct committee. However, in general, multiple candidates can be elected at a given node if the PSC demand requires it (Step 2 of the MD rule), and multiple paths may correspond to the same committee outcome. The latter arises because the Tree of MD rules illustrates not only which candidate(s) are elected, but also the order that they are elected.

