Ex-post Stability under Two-Sided Matching: Complexity and Characterization

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ABSTRACT

A probabilistic approach to the stable matching problem has been identified as an important research area with several important open problems. When considering random matchings, ex-post stability is a fundamental stability concept. A prominent open problem is characterizing ex-post stability and establishing its computational complexity. We investigate the computational complexity of testing ex-post stability. Our central result is that when either side has ties in the preferences/priorities, testing ex-post stability is NP-complete. The result even holds if both sides have dichotomous preferences. We also consider stronger versions of ex-post stability (in particular robust ex-post stability and ex-post strong stability) and prove that they can be tested in polynomial time.

KEYWORDS

Matching theory, Stability Concepts, Fairness, Random Assignment

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1 INTRODUCTION

"*One important question is about the characterization of ex post stability when matchings are allowed to be random*"—Kesten and Unver [19].

We consider the two-sided matching problems in which agents have preferences over items and items have priorities over agents. For deterministic matchings, the most widely applied stability concept (often referred to simply as stability, pairwise stability or weak stability) requires that there should be no agent i and item o such that both prefer each other over their existing match. A stronger version of the concept (referred to as strong stability) requires that there should be no agent i and item o such that both weakly prefer each other over their existing match with one of the relations being strict. Both concepts are well-understood in the context of deterministic matchings but less so in the context of probabilistic matchings.

Whereas much of the matching literature focuses on deterministic matching, there is increased focus on probabilistic matchings. There are several reasons randomization is useful including allowing a richer outcome space (may be essential to achieve fairness properties such as anonymity and (ex-ante) equal-treatment-of-equals, or to satisfy certain distributional constraints in expectation) and also to capture time sharing arrangements [3, 14, 22, 24]. A random matching specifies the probability with which each agent gets a particular item. In graph-theoretical terms, it is a fractional matching. Kesten and Unver [19] mention that "*The research on school-choice lotteries is a relatively new area in market design theory, and there are many remaining open questions.*" Research on random matching under preferences has also been highlighted as an important research direction in the Dagstuhl Workshop on Matching Under Preferences: Theory and Practice (2021) [4]. When there is a need to both consider ex-ante probabilistic constraints as well as the requirement to achieve stability ex-post, a fundamental algorithmic problem that arises is the following one:

> *Can a given random matching be represented as a probability distribution over stable integral / deterministic matchings?*

The problem above also captures the problem of checking whether a given random matching is ex post stable or not.[1] This problem will be the central focus of our paper. The problem is intrinsically linked to the open problem highlighted by Kesten and Unver [19] concerning the characterization of ex post stability when matchings are allowed to be random.

Results. We show that when either agents have ties in their preferences or items have ties in their priorities, testing ex-post stability is NP-complete. In particular, we prove two complexity results: (1) deciding whether a random matching p is ex-post stable is NP-complete, even if one side has strict and the other has dichotomous preferences and (2) deciding whether a random matching p is ex-post stable is NP-complete, even if both sides have dichotomous preferences.

The results also give an explanation on why a simple and computationally tractable characterization of ex-post stability has eluded researchers. Woeginger [25] writes that "*the combinatorics of NP-complete problems usually is complicated and rather messy. If one proves theorems about properties and the behavior of NP-complete problems, then this usually involves lots of tedious case analysis.*" Our

[1] The complexity of the problem was highlighted as an open problem by Jay Sethuraman at the Dagstuhl Workshop on Matching Under Preferences: Theory and Practice [4].

Restriction	ex-post stability	strong ex-post stability	robust ex-post stability
strict prefs, strict priorities	in P	in P	in P
dichotomous prefs, dichotomous priorities	NP-complete (Th 5.4)	in P	in P
strict prefs, dichotomous priorities	NP-complete (Th 5.1)	in P	in P
dichotomous prefs, strict priorities	NP-complete	in P	in P
-	NP-complete	in P (Th 6.8)	in P (Th 6.2)

Table 1: Complexity of testing stability

results sharply contrast with the fact that the set of ex-post matching matchings can be represented by a linear number of linear constraints when both the preferences and priorities are strict.

We then turn our attention to stronger versions of ex post stability. We show that there is a polynomial time algorithm for testing ex-post strong stability and also find a decomposition into a convex combination of integral strongly stable matchings (if it exists). We also prove a similar result for *robust ex post stability* that is another stability concept. Our complexity results are summarized in Table 1.

2 RELATED WORK

The theory of stable matchings has a long history with several books written on the topic [17, 21, 23].

In the theory of matching under preferences, Roth et al. [22] presented several results regarding the stable matching polytope (when there are no ties) that also provide insights into random stable matching. Teo and Sethuraman [24] presented another paper that provides connections between linear programming formulations and stable matchings.

Bogomolnaia and Moulin [9] presented a seminal paper on random matchings when the items do not have any priorities. In this paper, we focus on the setting when items also have priorities. Our focus is also on stability concepts. Bogomolnaia and Moulin [10] then considered two-sided matching under dichotomous preferences.

Kesten and Unver [19] initiated a mechanism design approach to the stable random matching problem where they explore the compatibility of stability and efficiency and propose algorithms that satisfy ex-ante stability (a property that is stronger than ex-post stability). Afacan [1] considered a more general model in which objects have probabilities for prioritizing one agent over another. They present a weak stability concept called claimwise stability and propose an algorithm to achieve it. Aziz and Klaus [6] explore a hierarchy of stability concepts when considering random matchings and explored their relations and mathematical properties. Caragiannis et al. [12] considered stability under cardinal preferences. Aziz and Brandl [5] presented a general random allocation algorithm that can handle general feasibility constraints including those that are as a result of imposing stability concepts.

Chen et al. [13] considered the classical as well as a concept based on cardinal utilities [12] and presented additional complexity results when stability is combined with other objectives such as maximum size or maximum welfare. Aziz et al. [7] examined the complexity of testing ex post Pareto optimality and proved that the problem is coNP-complete.

Regarding the practical applications, the problem of socially optimal decomposition of probabilistic allocations came up in at least two contexts, where the probabilities are coming from lotteries. Ashlagi and Shi [2] proposed to improve community cohesion in a school choice mechanism by finding a convex combination of such deterministic assignments, where the students from the same neighbourhood are matched to the same schools with high chances. Such solutions can decrease the busing costs as well, which has been a crucial objective for the city of Boston, where the redesign was proposed. Bronfman et al. [11] used a similar approach to implement a new matching algorithm in the resident allocation programme of Israel, where the focus of the optimal decomposition was to keep the married couples together.

3 PRELIMINARIES

We consider the classic matching setting in which there are n agents and n items. The agents have preferences over items and items have priorities over agents. The preference relation of an agent $i \in N$ over items is denoted by $\succsim_i$ where $\succ_i$ denotes the strict part of the preference and $\sim_i$ denotes the indifference part. The priority relation of an item $o \in N$ over agents is denoted by $\succsim_o$ where $\succ_o$ denotes the strict part of the preference and $\sim_o$ denotes the indifference part.

A ***random matching*** p is a bistochastic $n \times n$ matrix $[p(i,o)]_{i \in N, o \in O}$, i.e.,

$$\text{for each pair } (i,o) \in N \times O,\ p(i,o) \geq 0, \tag{1}$$

$$\text{for each } i \in N,\ \sum_{o \in O} p(i,o) = 1, \text{ and} \tag{2}$$

$$\text{for each } o \in O,\ \sum_{i \in N} p(i,o) = 1. \tag{3}$$

Random matchings are often also referred to as *fractional matchings* [24]. For each pair $(i,o) \in N \times O$, the value $p(i,o)$ represents the probability of item o being matched to agent i and agent i's ***match*** is the probability vector $p(i) = (p(i,o))_{o \in O}$. A random matching p is ***deterministic*** if for each pair $(i,o) \in N \times O$, $p(i,o) \in \{0,1\}$. Alternatively, a deterministic matching is an integer solution to linear inequalities (1), (2), and (3).

By [8], each random matching can be represented as a convex combination of deterministic matchings: a ***decomposition*** of a random matching p into deterministic matchings P_j $(j \in \{1,\ldots,k\})$

equals a sum $p = \sum_{j=1}^{k} \lambda_j P_j$ such that for each $j \in \{1, \ldots, k\}$, $\lambda_j \in (0, 1]$ and $\sum_{j=1}^{k} \lambda_j = 1$.

*Definition 3.1 (**Stability for deterministic matchings**).* A deterministic matching p has ***no justified envy*** or is ***stable*** if there exists no agent i who is matched to item o' but prefers item o while item o is matched to some agent j with lower priority than i, i.e., there exist no $i, j \in N$ and no $o, o' \in O$ such that $p(i, o') = 1$, $p(j, o) = 1$, $o \succ_i o'$, and $i \succ_o j$.

A deterministic matching p is stable if it satisfies the following inequalities: for each pair $(i, o) \in N \times O$,

$$p(i,o) + \sum_{o':o' \succsim_i o; o' \neq o} p(i,o') + \sum_{j:j \succsim_o i; j \neq i} p(j,o) \geq 1. \qquad (4)$$

If one breaks all preference and priority ties, then the well-known deferred-acceptance algorithm [15] computes a deterministic matching that is stable.

*Definition 3.2 (**Ex-post stability**).* A random matching p is ***ex-post stable*** if it can be decomposed into deterministic stable matchings.

4 EX POST STABILITY: ALGORITHM AND CHARACTERIZATION UNDER ABSENCE OF TIES

We first warm up with an observation that when both preferences and priorities are strict, then ex post stability admits both a simple characterization, concise geometric description and also a polynomial-time algorithm to test ex post stability.

*Definition 4.1 (**Fractional stability and violations of fractional stability**).* A random matching p is ***fractionally stable*** if for each pair $(i, o) \in N \times O$,

$$p(i,o) + \sum_{o':o' \succsim_i o; o' \neq o} p(i,o') + \sum_{j:j \succsim_o i; j \neq i} p(j,o) \geq 1, \qquad (4)$$

or more compactly,

$$\sum_{o':o' \succsim_i o; o' \neq o} p(i,o') \geq \sum_{j:j \prec_o i} p(j,o). \qquad (5)$$

A ***violation of fractional stability occurs if there exists a*** *pair $(i, o) \in N \times O$ such that*

$$\sum_{j:j \prec_o i} p(j,o) > \sum_{o':o' \succsim_i o; o' \neq o} p(i,o'). \qquad (6)$$

Next, we highlight that fractional stability does not imply ex-post stability.[2]

Example 4.2 (Fractional stability does not imply ex-post stability). Let $N = \{1, 2, 3\}$ and $O = \{x, y, z\}$. Consider the following preferences and priorities (the brackets indicate indifference):

$\succ_1$:	$[x\ y\ z]$			$\succ_x$:	$[2\ 3]$	1
$\succ_2$:	y	x	z	$\succ_y$:	$[1\ 2\ 3]$	
$\succ_3$:	$[x\ y\ z]$			$\succ_z$:	$[1\ 2\ 3]$	

Consider random matching q, which is fractionally stable because agents 1 and 3 only get best items and from agent 2's perspective no agent with a lower priority consumes his best item y, which he receives with probability $\frac{1}{2}$, and agent 2, who does have a lower priority for item x does not consume more that $\frac{1}{2}$ of x,

$$q = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Note that random matching q has a unique decomposition into the deterministic matchings

$$p^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad p^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

such that $q = \frac{1}{2}p^1 + \frac{1}{2}p^2$. However, deterministic matching p^1 is unstable because agent 2 justifiably envies agent 1. Hence, random matching q is not ex-post stable.

We note that ex post stability can be tested in linear-time if preferences / priorities on both sides are strict

Proposition 4.3. *Ex post stability can be tested in linear-time if preferences on both sides are strict. Furthermore, if a given a random matching is ex post stable, there exists a polynomial-time algorithm to represent the random matching as a lottery over deterministic stable matchings.*

Proof. Under strict preferences and priorities, ex post stability and fractional stability are equivalent [6]. So we just need to check for the linear constraints capturing fractional stability. In strict preference setting, any fractional stable matching (equivalently ex post stable matching) can be efficiently decomposed to a convex combination of deterministic stable matchings [24]. □

5 EX POST STABILITY: COMPLEXITY UNDER THE PRESENCE OF TIES

Next, we move to the general setting in which there may be ties in the preferences or priorities. Our first observation is as follows. Suppose that C denote the set of all stable matching of a given instance with weak preferences. The convex hull of all points in C is a subset of the polytope defined by the inequalities (1) to (4). Next, we prove that checking whether a random matching p is ex-post stable is NP-complete

Theorem 5.1. *Deciding whether a random matching p is ex-post stable is NP-complete, even if one side's preferences/priorities are strict and the other's are dichotomous.*

Proof. To show NP membership, it is enough to show that there always exist a polynomial size witness for yes instances. But it is true, since if a random matching p is in the convex hull of the characteristic vectors of stable matchings, than by Caratheodory's theorem it can be expressed as a convex combination of at most $n^2 + 1$ stable matchings also.

To show NP-hardness, we reduce from X3C. In this problem, we are given a family of 3-sets $C_1, .., C_{3n}$ over elements $a_1, .., a_{3n}$ and the question is whether there is an exact 3-cover among the 3-sets. This problem is NP-complete, even if each element a_i is contained in exactly three 3-sets [16].

[2] The example was first presented by Aziz and Klaus [6].

X3C

Input: A finite set $X = \{a_1, .., a_{3n}\}$ containing exactly 3n elements; a collection $C = \{C_1, .., C_{3n}\}$ of subsets of X each of which contains exactly 3 elements.

Task: Does C contain an exact cover for X, i.e. a subcollection of 3-element sets $D = (D_1, ..., D_n)$ such that each element of X occurs in exactly one subset in D?

Let I be such an instance of X3C. We construct an instance I' for our problem as follows.

For each element a_i we add an item a_i. For each set C_j, we add 6 items x_j^1, x_j^2, x_j^3 and y_j^1, y_j^2, y_j^3 and also 6 agents c_j^1, c_j^2, c_j^3 and d_j^1, d_j^2, d_j^3. Then, we add $3n$ collector agents $z_1, , .., z_{3n}$. Finally, we add two more items o_1 and o_2 and two more agents s_1 and s_2. Let $C_j = \{a_{j_1}, a_{j_2}, a_{j_3}\}$, $j_1 < j_2 < j_3$ be the j-th set in I. We refer to a_{j_1} as the first element in C_j, a_{j_2} as the second and a_{j_3} as the third.

Let the preferences be the following. For the agents:

$$\begin{aligned} c_j^l &: \quad a_{j_l}, y_j^l, x_j^{l-1}, x_j^l, others \\ d_j^l &: \quad x_j^l, y_j^l, others, \\ s_1 &: \quad o_2, (Y), o_1, others \\ s_2 &: \quad o_1, o_2, others \\ z_j &: \quad (X), others \end{aligned}$$

where $j = 1, .., 3n$, $l = 1, 2, 3$ taken (mod 3) and $Y = \{y_j^l | \ j = 1, .., 3n, \ l = 1, 2, 3\}$, $X = \{x_j^l | \ j = 1, .., 3n, \ l = 1, 2, 3\}$ and (S) for a set S indicates that the elements of S are ranked in an arbitrary strict order.

For the item we have:

$$\begin{aligned} a_i &: \quad [c^1(a_i), c^2(a_i), c^3(a_i)], [others] \\ x_j^l &: \quad [c_j^l, c_j^{l+1}], [others] \\ y_j^l &: \quad [d_j^l, s_1], [others] \\ o_1 &: \quad [every\ agent] \\ o_2 &: \quad [every\ agent] \end{aligned}$$

where $Z = \{z_1, .., z_{3n}\}$, $i = 1, .., 3n$, $j = 1, .., 3n$, $l = 1, 2, 3$ and $c^k(a_i)$ is c_j^l, iff the k-th appearance of a_i is in the l-th position of the set C_j and the brackets indicate indifferences. Let the fractional matching p be:

(1) $p(c^k(a_i), a_i) = \frac{1}{3}$ for $i = 1, .., 3n$, $k = 1, 2, 3$
(2) $p(c_j^l, x_j^l) = p(c_j^l, y_j^l) = \frac{1}{3}$, $j = 1, .., 3n$, $l = 1, 2, 3$
(3) $p(d_j^l, x_j^l) = \frac{1}{3}$, $p(d_j^l, y_j^l) = \frac{2}{3}$, $j = 1, .., 3n$, $l = 1, 2, 3$
(4) $p(s_1, o_1) = p(s_2, o_2) = \frac{1}{3}$, $p(s_2, o_1) = p(s_1, o_2) = \frac{2}{3}$
(5) $p(z_k, x_j^l) = \frac{1}{9n}$ for each $j, k = 1, .., 3n$, $l = 1, 2, 3$.

This completes the construction of I'. The random matching restricted to a gadget of a set C_j is illustrated in Figure1.

Proposition 5.2. *If p is ex-post stable, then there exists an exact 3-cover.*

Proof. If p can be written as a convex combination of stable matchings, then, because $p(s_1, o_1) > 0$, there has to be one matching in which the edge (s_1, o_1) is included. Let this matching be M.

M is stable, therefore s_1 does not block with any items from Y. This can only happen, if each item from Y is matched to someone with at least as high priority, so $(y_j^l, d_j^l) \in M$ for each $j = 1, .., 3n$, $l = 1, 2, 3$. We also know that each c_j^l agent must be matched in M, so she is matched to either x_j^l or her element item.

We claim that for each j, either all of c_j^1, c_j^2, c_j^3 are matched to items from $A = \{a_1, .., a_{3n}\}$, or none of them are.

Suppose that it is not the case. Then, there is a j and an l, such that c_j^l is matched to x_j^l, but c_j^{l-1} is not matched to x_j^{l-1}. But then, x_j^{l-1} must be matched to an agent from Z in M, and therefore (c_j^l, x_j^{l-1}) blocks M, contradiction.

Also, observe that each a_i must be matched with a c_j^l agent in M, since otherwise they would block with $c^1(a_i)$.

Therefore, if we take those C_j sets, for which c_j^1, c_j^2, c_j^3 are matched to a_i items, they must form an exact 3-cover.

□

Now, we move on to the other direction.

Proposition 5.3. *If there exists an exact 3-cover in I, then p is ex-post stable.*

Proof. We prove that $p = \frac{1}{9n}(\sum_{k=1}^{3n} M_1^k + \sum_{k=1}^{3n} M_2^k + \sum_{k=1}^{3n} M_3^k)$, where each M_i^k is stable. For the sake of simplicity, suppose the exact cover of I is $C_1, .., C_n$. (by the symmetry of the construction and the fact that each a_i is in exactly 3 sets, we can suppose this by reindexing the sets). Then, for each a_i, $c^1(a_i) \in \{C_1, .., C_n\}$ and $c^2(a_i), c^3(a_i) \notin \{C_1, .., C_n\}$.

Now we define the $9n$ matchings that will be the support of p.

For each k, let edges of M_1^k be $(c^1(a_i), a_i), (d_j^l, y_j^l)$ for $j \leq n$, $i = 1, .., 3n$, $l = 1, 2, 3$ and $(c_j^l, x_j^l), (d_j^l, y_j^l)$ for $j > n$. Furthermore (s_1, o_1) and (s_2, o_2) are also matched in M_1^k. Then, let x_t be the t-th x_j^l agent who has not obtained a partner yet, $t = 1, .., 3n$. Then, we match x_t to z_{t+k} in M_1^k, where $t + k$ is taken modulo $3n$. The matchings M_1^k are illustrated in Figure 2 and 5.

Now, we observe that removing $C_1, .., C_n$, the remaining sets will satisfy that each a_i is included in exactly 2 of them, since $C_1, .., C_n$ is an exact 3-cover.

For each k, let the edges of M_2^k be $(c_j^l, x_j^l), (d_j^l, y_j^l)$ for $j \leq n$ and $(c^2(a_i), a_i)$, $i = 1, .., 3n$. The c_j^l agents that are not matched yet are matched to the corresponding y_j^l. The d_j^l agents are matched to y_j^l, if that item is not matched to c_j^l agents and to x_j^l otherwise. Then, we match $(s_1, o_2), (s_2, o_1)$ in M_2^k. Finally, let x_t be the t-th x_j^l agent who has not obtained a partner yet, $t = 1, .., 3n$. Then, we match x_t to z_{t+k} in M_2^k, where $t + k$ is taken modulo $3n$. The matchings M_2^k are illustrated in Figure 3 and 6

For each k, let the edges of M_3^k be $(c_j^l, y_j^l), (d_j^l, x_j^l)$ for $j \leq n$ and $(c^3(a_i), a_i)$ $i = 1, .., 3n$. The c_j^l agents that are not matched yet are matched to the corresponding y_j^l. The d_j^l agents are matched to y_j^l, if that item is not matched to c_j^l agents and to x_j^l otherwise. Then, we match $(s_1, o_2), (s_2, o_1)$ in M_3^l. Finally, let x_t be the t-th x_j^l agent who has not obtained a partner yet, $t = 1, .., 3n$. Then, we match x_t

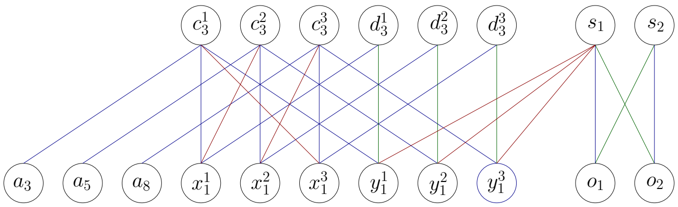

Figure 1: The gadget for a set $C_3 = \{a_3, a_5, a_8\}$ with the important edges. Red edges have weight 0, blue edges have weight $\frac{1}{3}$, green edges have weight $\frac{2}{3}$.

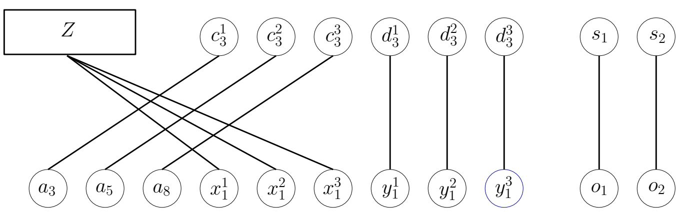

Figure 2: The induced edges of M_1 on a set gadget of a set that is part of the exact cover.

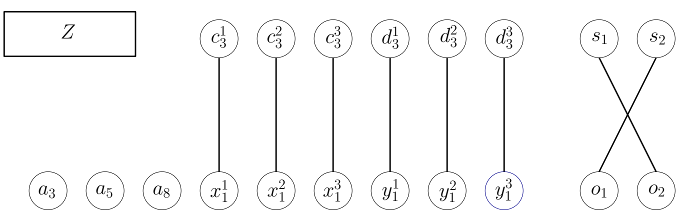

Figure 3: The induced edges of M_2 on a set gadget of a set that is part of the exact cover.

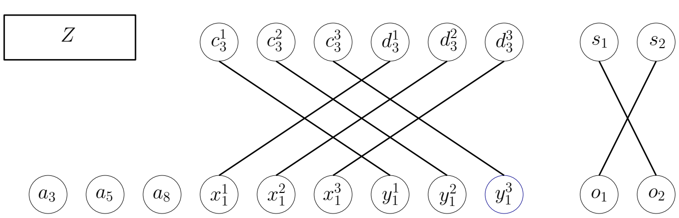

Figure 4: The induced edges of M_3 on a set gadget of a set that is part of the exact cover.

to z_{t+k} in M_3^k, where $t + k$ is taken modulo $3n$. These matchings are illustrated in Figure 4 and Figure 7.

It is easy to see that the edges with weight $\frac{1}{3}$ are included in exactly $3n$ matchings, the ones with weight $\frac{2}{3}$ are included in exactly $6n$ matchings, the edges with weight $\frac{1}{9n}$ are included in exactly one matching and all other edges are not included in any matching from $\{M_1^k, M_2^k, M_3^k \mid k = 1, .., 3n\}$. Therefore, $p = \frac{1}{9n}(\sum_{k=1}^{3n} M_1^k + \sum_{k=1}^{3n} M_2^k + \sum_{k=1}^{3n} M_3^k)$.

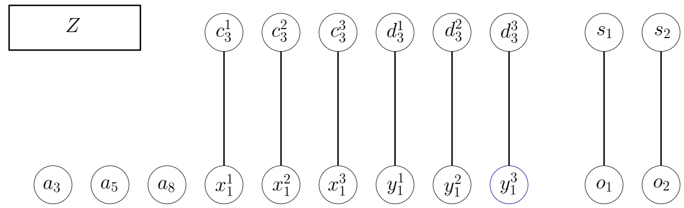

Figure 5: The induced edges of M_1 on a set gadget of a set that is not part of the exact cover.

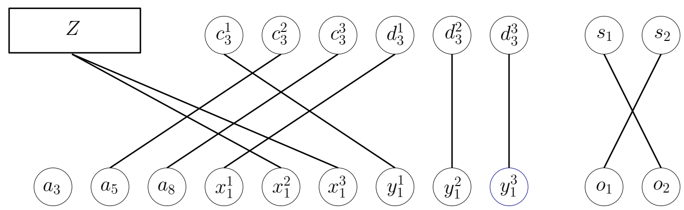

Figure 6: The induced edges of M_2 on a set gadget of a set that is not part of the exact cover.

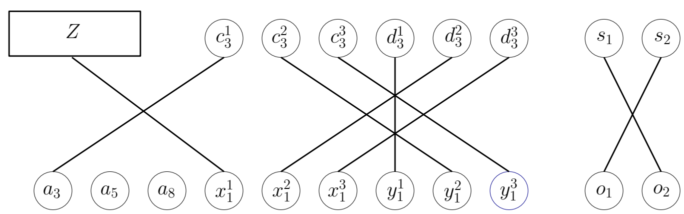

Figure 7: The induced edges of M_3 on a set gadget of a set that is not part of the exact cover.

Let k be an arbitrary index from $\{1, .., 3n\}$. It only remains to show that M_1^k, M_2^k and M_3^k are stable matchings. Let us start with M_1^k.

Each a_i and y_j^l item and also o_1 and o_2 are matched to one of their best agents in M_1^k, so they cannot participate in any blocking. For an item x_j^l, either it is matched to one of its top agents, or it is matched to someone from Z. But, even if it is with a collector agent from Z, all higher priority agents for it (c_j^l and c_j^{l+1}) are matched to better items (a_i-s), so there is no blocking with x_j^l items either. Therefore, M_1^k is stable.

The cases of M_2^k and M_3^k are similar, we only show stability of M_2^k. Again, each a_i item as well as o_1 and o_2 are matched to one of their highest priority options, so they cannot be part of a blocking pair. Each y_j^l agent is matched to either c_j^l or d_j^l. There could only be a potential block, if y_j^l is matched to c_j^l. But, since s_1 is matched to o_2, it cannot block with s_1, and since each d_j^l is matched to an at least as good item, it cannot block with d_j^l either. The x_j^l items also cannot block with anyone, since if they are not with one of their first choices (which are the only strictly higher priority ones than the agents of $Z \cup \{d_j^l\}$), then each of their top agents (c_j^l and c_j^{l+1}) are matched with someone strictly better (an a_i or y_j^l).

This shows that p is indeed ex-post stable. □

This completes the proof of the theorem.

□

Now we show that the problem remains hard even if both sides have dichotomous preferences.

Theorem 5.4. *Deciding whether a random matching p is ex-post stable is NP-complete, even if the preferences / priorites of both sides are dichotomous.*

Proof. We reduce from x3c. The construction, the random matching p and the matchings M_i^k, $i = 1, 2, 3$, $k = 1, .., 3n$ are identical as in the proof of theorem 5.1, only the preferences are modified. Let the new preferences be the following. For the agents:

$$\begin{aligned} c_j^l &: \quad [a_{j_l}, y_j^l, x_j^{l-1}], [others] \\ d_j^l &: \quad [every\ item], \\ s_1 &: \quad [o_2, Y], [others] \\ s_2 &: \quad [every\ item] \\ z_j &: \quad [every\ item] \end{aligned}$$

where $j = 1, .., 3n$, $l = 1, 2, 3$ taken $(mod\ 3)$ and $Y = \{y_j^l | \ j = 1, .., 3n,\ l = 1, 2, 3\}$.

For the items we have:

$$\begin{aligned} a_i &: \quad [c^1(a_i), c^2(a_i), c^3(a_i)], [others] \\ x_j^l &: \quad [c_j^l, c_j^{l+1}], [others] \\ y_j^l &: \quad [d_j^l, s_1], [others] \\ o_1 &: \quad [every\ agent] \\ o_2 &: \quad [every\ agent] \end{aligned}$$

where $Z = \{z_1, .., z_{3n}\}$, $i = 1, .., 3n$, $j = 1, .., 3n$, $l = 1, 2, 3$ and $c^k(a_i)$ is c_j^l, iff the k-th appearance of a_i is in the l-th position of the set C_j and the brackets indicate indifferences.

Proposition 5.5. *If p is ex-post stable, then there exists an exact 3-cover.*

Proof. The proof is exactly the same as it was in theorem 5.1.

□

Now, we move on to the other direction.

Proposition 5.6. *If there exists an exact 3-cover in I, then p is ex-post stable.*

Proof. Again, we prove that $p = \frac{1}{9n}(\sum_{k=1}^{3n} M_1^k + \sum_{k=1}^{3n} M_2^k + \sum_{k=1}^{3n} M_3^k)$, where each M_i^k is stable and is defined the same.

Let k be an arbitrary index from $\{1, .., 3n\}$. It only remains to show that M_1^k, M_2^k and M_3^k are stable matchings. Notice, that with the new preferences, in any matching, only agents from $\{c_j^l \mid j = 1, .., 3n, l = 1, 2, 3\} \cup \{s_1\}$ could block, since others are totally indifferent.

Let us start with M_1^k. From the construction, it follows that s_1 does not block with any agent from Y, since all of them are matched to a same priority agent. A c_j^l agent could only block, if it is assigned to x_j^l. But then, x_j^l, y_j^l and a_{j_l} are all assigned to at least as good agents, so no pair can block.

In the case of M_2^k, s_1 is with o_2, so it is not part of any blocking pair. A c_j^l agent could again only block, if it is with x_j^l. But that can only happen with those set agent that correspond to the exact cover. But then, all of their better items are with agents that have at least as high priorities.

In M_3^k, each agent is with a top choice, so it is obviously stable.

This shows that p is indeed ex-post stable. □

This completes the proof of the theorem.

□

We also obtain the following statement that may be of independent interest.

Theorem 5.7. *Deciding whether there is a stable matching that is consistent with a random matching p is NP-complete.*

Proof. We use the same construction from theorem 5.1, with the only difference, that instead of $p(s_1, o_1) = p(s_2, o_2) = \frac{1}{3}$, $p(s_2, o_1) = p(s_1, o_2) = \frac{2}{3}$ we have $p(s_1, o_1) = p(s_2, o_2) = 1$, $p(s_2, o_1) = p(s_1, o_2) = 0$. Then, any consistent matching with p must contain (s_1, o_1), so by the same argument, there exists an exact 3-cover.

And if there is an exact 3-cover, then the matching M_1^1 will be a stable matching consistent with p. □

6 STRONGER VERSIONS OF EX-POST STABILITY

In this section, we consider stronger versions of ex-post stability.

6.1 Robust Ex-post Stability

Robust ex-post stability, is a natural strengthening of ex-post stability [6].

*Definition 6.1 (**Robust ex-post stability**).* A random matching p is ***robust ex-post stable*** if all of its decompositions are into deterministic and stable matchings.

It follows easily that if we restrict attention to deterministic matchings, then all the stability concepts for random matchings coincide with stability and no envy (Definition 3.1).

Theorem 6.2. *For robust* ex post *stability, checking whether a current allocation p is stable is polynomial-time solvable.*

Proof. For each $i \in N$ and $o \in O$, we check whether there exists an integral matching q consistent with p such that i is not matched to o under q and (i, o) form a blocking pair for q. This can be checked by testing whether there exists an allocation in which i is matched to some item o' such that $o \succ_i o$ and $p(i, o')$; o is matched to some agent $j \in N$ such that $i \succ_o j$ and $p(j, o) > 0$; and each other agent k is matched to some item o'' such that $p(k, o'') > 0$. This test can be solved in polynomial time by checking whether the underlying bipartite graph with the admissible edges admits a perfect matching. □

6.2 Ex-post Strong Stability

In this section, we consider a stability concept called ex-post strong stability which is based on a concept called *strong stability*.

*Definition 6.3 (***Strong stability***).* A deterministic matching p is ***strongly stable*** if it satisfies the following two conditions.

(1) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{o' \sim_i o} p(i, o') \geq 1$
(2) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{i' \sim_o i} p(i', o) \geq 1$.

Clearly strong stability implies stability. Note that under strict preferences, strong stability and (weak) stability are equivalent. A strongly matching may not exist but there is a linear-time algorithm to check if it exists or not and to find one if it exists [18].

The notion of strong stability for integral matchings lends itself to two natural stability concept for the case of random / fractional matchings.

*Definition 6.4 (***Ex post strong stability***).* A matching p is ***ex post strongly stable*** if it can represented as a convex combination of integral strongly stable matchings.

*Definition 6.5 (***Fractional strong stability***).* A fractional matching p is ***fractional strong stable*** if it satisfies the following two conditions.

(1) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{o' \sim_i o} p(i, o') \geq 1$
(2) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{i' \sim_o i} p(i', o) \geq 1$

Clearly ex post strong stability implies ex post stability.

Proposition 6.6. *Strong fractional stability implies fractional stability.*

Proof. Suppose, the first condition of strong fractional stability is satisfied: for all (i, o), $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{o' \sim_i o} p(i, o') \geq 1$. Then, for all (i, o), $\sum_{o' \succsim_i o; o' \neq o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + p(i, o) \geq 1$ which means that fractional stability is satisfied. □

Next, we establish an equivalence between ex-post strong stability and fractional strong stability.

Lemma 6.7. *The following are equivalent. A fractional matching*

(1) satisfies fractional strong stability
(2) is in the convex convex hull of deterministic strongly stable matchings
(3) satisfies ex-post strong stability.

Proof. (1) $\iff$ (2). Theorem 13 of Kunysz [20] shows that the polytope capturing fractional strong stable matchings is equivalent to the convex hull of deterministic strongly stable matchings.

(2) $\implies$ (3). If a matching is in the convex hull of deterministic strongly stable matchings, then by Caratheodory's theorem, it can represented by a convex combination of the end points of the convex hull (consisting of deterministic strongly stable matchings). Hence it satisfies ex-post strong stability.

(3) $\implies$ (2). If a fractional matching is ex-post strongly stable, then by definition, it can be represented as a convex combination of some deterministic strongly stable matchings. Hence, it can be represented as a convex combination of the set of all deterministic stable matchings. Hence, it is in the convex hull of the set of deterministic stable matchings. □

Theorem 6.8. *For weak preferences and priorities, there exists a polynomial-time algorithm to test strong ex post stability and in case the answer is yes, there is a polynomial-time algorithm to find its representation as a convex combination of strongly stable deterministic matchings.*

Proof. Strong ex post stability can be checked in polynomial time as follows. Strong ex post stability is equivalent to strong fractional stability (Lemma 6.7). Strong fractional stability can be checked by considering $2|N| \times |O|$ inequalities used in the definition of strong fractional stability. For a matching that satisfies strong fractional stability, it lies in the convex hull of the set of deterministic strongly stable matchings. Such a matching can be represented by a convex combination of strongly stable deterministic matchings by an algorithm of Kunysz [20] that uses a similar argument as that of Teo and Sethuraman [24]. □

In the proof of Theorem 6.8, we invoke an algorithmic result of Kunysz [20]. For the sake of completeness and exposition, we give a description of the algorithm of Kunysz [20] . The proposed algorithms by Teo and Sethuraman [24] and its extension by Kunysz [20] are based on self-duality of polytope defined by the fractional strong stability. By using the self-duality and complementary slack property it was shown that if p is an optimal solution and $p(i, o) > 0$, then

(1) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{o' \sim_i o} p(i, o') = 1$
(2) $\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + \sum_{i' \sim_o i} p(i', o) = 1$
(3) $\sum_{i'} p(i', o) = 1$
(4) $\sum_{o'} p(i, o') = 1$

For each i and o, consider interval $I_i = (0, 1]$ and $I_o = (0, 1]$ that results into $2n$ intervals. Corresponding to each $p(i, o)$, consider an interval of length $p(i, o)$ and by abusing the notation denote the interval by $p(i, o)$. The intervals are also arranged in desecrating preference of i. This means that if $o \succ o'$, then interval $p(i, o)$ appears before $p(i, o')$. Notice that indifferent preferences are arranged arbitrary next to each other. Since we have that $\sum_{o'} p(i, o') = 1$, then $\cup_{o'} p(i, o') = (0, 1]$. Similarly, define sub-intervals $p(i, o)$ for each $I_o = (0, 1]$ and arrange them in increasing order.

First, consider the case where preferences are strict. Then, let $u \in (0, 1]$ be an arbitrary number. Then, we get stable integral matching M_u as follows: i gets matched to o if u belongs to interval $p(i, o) \subseteq I_i$. Moreover, o gets matched to i if u belongs to interval $p(i, o) \subseteq I_o$. Notice that by the fact that sub-intervals in I_i and I_o are arranged in opposite way and

$$\sum_{o' \succ_i o} p(i, o') + \sum_{i' \succ_o i} p(i', o) + p(i, o) = 1.$$

On can observe that M_u is an integral matching. By sub-intervals construction, each I_i and I_o is partitioned to at most n intervals which are determined by $n + 1$ district numbers. Since there are $2n$ intervals, there are at most $2n(n + 1)$ such numbers. Sort them as $0 = x_0 < x_1 < \ldots < x_s = 1$, where $s < 2n(n + 1)$. Teo and Sethuraman showed that

$$p = \sum_{t=1}^{s} (x_t - x_{t-1}) \cdot M_{x_t}$$

Kunysz slightly modified the construction to handle the case where there is a weak preferences. In this case one may not be able to construct an integral matching M_u, $u \in (0, 1]$. Instead he defined an auxiliary bipartite graph H_u and then showed that there exists matching M_u in H_u. Then, finding the convex composition follows as Teo and Sethuraman's algorithm.

7 CONCLUSION

We undertake a study of testing stability of random matchings. Subtle differences between various stability concepts and restrictions on preferences / priorities lead to remarkably different complexity results. Our central result is that testing ex-post stability is NP-complete. The computational hardness result also explains why a combinatorially simple and tractable characterization has eluded mathematicians and economists. We also consider stronger versions of ex post stability and present polynomial-time algorithms for testing them. A natural research direction is to understand sufficient conditions on the preferences and priorities under which testing ex-post stability is polynomial-time solvable. Yet another research problem is understanding the conditions under which stability concepts coincide. Parametrized algorithms for the computationally hard problems is yet another research direction.

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