Probabilistic Rationing with Categorized Priorities: Processing Reserves Fairly and Efficiently

Haris Aziz
UNSW Sydney
haris.aziz@unsw.edu.au

Abstract

In recent years, a market design approach for rationing problems with multi-category priorities has been considered for various applications including healthcare, immigration, and school choice. We consider a probabilistic or fractional approach to rationing that is geared towards achieving symmetry axioms such as anonymity and neutrality in conjunction to primary axioms such as eligibility compatibility, respect of priorities, and non-wastefulness. We present new algorithms for the problem that have advantages over the simultaneous reservation rule of Delacrétaz (ACM EC 2021) with respect to fairness, efficiency, and simplicity.

Introduction

We consider a rationing problem in which agents are interested in obtaining a unit of a resource. The resources could be immigration slots, school seats, or heathcare treatments. The resources are divided into categories with each category having a specified number of units and own priority list over the agents. The goal is to allocate the resources among the agents in a principled manner. The model that we consider captures important resource allocation and market design problems with applications to school admissions [Dur *et al.*, 2018], immigration [Pathak *et al.*, 2020a], and healthcare rationing [Pathak *et al.*, 2020b; Aziz and Brandl, 2021].

When making decisions about who gets which category's unit, a fundamental and highly relevant question that arises is about the criteria used to make allocation decisions. Three basic requirements include (1) compliance with eligibility requirements (a unit from a category should be given to an agent who is eligible for the category); (2) respect of priorities (if an agent does not get a full unit, then none of her eligible categories is giving any part to a lower priority agent); and (3) non-wastefulness (there is no agent who can use an unused fraction of a unit from an eligible category). Another requirement that is desirable is that the outcome should maximize the number of allocated units.

One of the main approaches to solve the problem is to process the agents or the categories in some given order which leads to a violation of ex-ante equity concerns such as an anonymity or neutrality. A natural idea to address these concerns is to design rules or algorithms that take symmetry concerns into account and perform some kind of simultaneous reservation. Such simultaneous reservation does not hinge critically in the processing order of the categories or agents. The main problem we want to address in the paper is the following one: *What is a fair and efficient method for simultaneously processing resources for rationing with heterogenous categorized priorities?*

In this paper, we present new algorithms for rationing scarce resources. In particular, we explore probabilistic rationing in systems with categorized priorities. A probabilistic approach is central for achieving the goal of treating agents and categories in a symmetric way. It is also useful for capturing fractional-sharing arrangements in which agents use portions of resources from multiple categories.

Contributions Our main contribution is to propose two new allocation rules and establish their relative merits in terms of fainess and efficiency axioms over existing rules. We first propose a rule called *Rationing Eating (RE)* that not only satisfies the main axioms for rationing problems but also satisfies several fairness and symmetry axioms with respect to agents and categories. In particular, it satisfies neutrality, category sd-efficiency, and category sd-envy-freeness. The latter property also constitutes a relative merit over the SR rule of Delacrétaz (2020, 2021). Category sd-envy-freeness can be especially important in the context of immigration problems where the categories are various profession categories that have their own rankings over eligible immigrants and we want to achieve fairness across categories. Another desirable feature of the RE algorithm is that it is simple with a running time that is linear in the number of agents and categories. Simplicity, transparency, and verifiability have been discussed as important requirements of decision-making systems. RE is also provably strategyproof: no agent can have an incentive to lower their priority in some category or to hide their eligibility.

We then propose a second rule called *Maximum Rationing Eating (MRE)*. In contrast to RE and the SR rule of Delacrétaz [2021], MRE finds a matching of *maximum size*. It works by first calling the Hopcroft-Karp algorithm to compute a maximum size matching. It then changes the instance suitably and calls the *Vigilant Eating Rule (VER)* of Aziz and Brandl [2020] with the specific constraint of maximum size matching. MRE does not satisfy category sd-envy-freeness that RE satisfies. We show that the maximum size property and category sd-envy-freeness are incompatible by proving a general impossibility result.

Related Work

As mentioned before, a standard approach for the problem is to treat reserves from categories in a sequential manner [Kominers and Sönmez, 2016; Dur *et al.*, 2020; Aygün and Bó, 2020; Aygun and Turhan, 2020]. These approaches violate axioms pertaining to neutrality or fairness towards categories. The myopic picks can also lead to outcomes that do not satisfy the maximum size property.

Pathak *et al.* [2020b] framed the rationing problem with category priorities as a two-sided matching problem in which agents are simply interested in a unit of resource and the resources are reserved for different categories. They presented two characterizations of integral outcomes that satisfy eligibility compliance, non-wastefuless, and respect of priorities. Their central method (Smart Reserves) assumes homogenous priorities whereas we focus on more general heterogeneous priorities. They also studied a Deferred Acceptance class of rules for the problem that can capture sequential processing of categories if the category processing ordering is imposed as the preferences of the agents over categories.

In many of the rules for rationing, a baseline ordering is imposed on the agents which is used to make selection decisions [Aziz and Brandl, 2021; Pathak *et al.*, 2020b]. Such approaches are sensitive to what baseline order over the agents is used. The asymmetry in the way agents are treated can be countered by generating the baseline order uniformly at random. However, such an approach has its own issues especially when we want to capture fractional sharing and *compute* the shares. Such approaches also do not result in outcomes that are ex-ante Pareto optimal from the perspective of the categories.

Next, we discuss the work that is closest to our approach. Delacrétaz [2020] discussed that if an approach is dependent on the processing order of the categories, then different processing orders result in different outcomes. Delacrétaz (2020, 2021) proposed a solution that is not dependent on the processing order. In particular he focussed on a particular form of neutrality called category neutrality. In order to avoid confusion from the standard category neutrality axioms and to better capture the essence of the concept, we will refer to the property of Delacrétaz [2021] as category uniformity. The idea behind the approach of Delacrétaz [2021] is as follows. In each round categories allocate a unit to their highest priority agent who does not have a full unit. If an agent gets more than one unit in aggregate over eligible categories, then each category's contribution is reduced until the agent has one unit overall. The algorithm stops when no category has additional capacity. Delacrétaz [2021] shows that his base algorithm does not terminate. Following an approach of Kesten and Unver [2015], this issue is addressed by repeated calls to linear programs to test for cyclic situations. Delacrétaz [2021] shows that the outcome of the algorithm satisfies three basic axioms extended to the case for fractional matchings. He also shows that the outcome is nearly an integral matching: the number of agents who get an amount strictly between zero and one is at most the number of categories. A particular guiding principle of his algorithm is that an agent is given the same contribution from categories if possible. The axiom (that we will refer to as category uniformity) requires that if an agent i is not allocated the same share from two categories, then the category allocating less to the agent allocates all of its units to agent i and higher priority agents. Delacrétaz [2020] writes that the axiom "*is needed to ensure that all categories are treated the same so that their relative importance only depends on their quotas*." We show a symmetric approach towards categories can be captured via another route that is simpler and computationally faster. In contrast to category uniformity that is incompatible with the maximal size property, we prove that one of our rules satisfies various fairness properties designed for categories but also additionally satisfies the maximum size property.

There are other models concerning matching under diversity constraints. One approach is to apply minimum and maximum quotas in a soft or hard manner [Abdulkadiroğlu, 2005; Ehlers *et al.*, 2014; Fragiadakis *et al.*, 2016; Aziz *et al.*, 2020]. The paper is related to an active area of research on matching with distributional constraints [see, e.g., Kojima, 2019]. Other related works on probabilistic stable matchings under two-sided preferences include [Teo and Sethuraman, 1998; Kesten and Unver, 2015; Aziz and Klaus, 2019; Chen *et al.*, 2020].

Setup

We adopt the essential features of the healthcare rationing model [Pathak *et al.*, 2020b; Delacrétaz, 2021; Aziz and Brandl, 2021]. There are q identical and indivisible units of some resource, which are to be allocated to the agents in a set N with $|N| = n$. Each category c has a quota $q_c \in N$ with $\sum_{c\in C} q_c = q$ and a priority ranking $\succ_c$, which is a linear order on $N \cup \{\emptyset\}$. We will assume strict priorities as is the standard assumption in most of the literature. Let N_c be the agents eligible for category c. An agent i is *eligible* for category c if $i \succ_c \emptyset$. We say that $I = (N, C, (\succ_c), (q_c))$ is an instance (of the rationing problem). For convenience, we will write $(\succ_c)$ and (q_c) for the profile of priorities and quotas in the sequel.

A matching μ specifies a fraction $\mu(i, c)$ for each $i \in N$ and $c \in C$ with the following feasibility constraints: (1) $0 \leq \sum_{i\in N} \mu(i, c) \leq q_c$ and (2) $0 \leq \sum_{c\in C} \mu(i, c) \leq 1$. An *allocation rule* takes as input a problem instance and returns a matching.

Example 1. *Suppose there are two agents and two categories with one reserved unit each.*

$$N = \{1, 2\}, \quad C = \{c_1, c_2\}, \quad q_{c_1} = 1, q_{c_2} = 1.$$

The priority ranking of c_1 *is* $1 \succ_{c_1} 2 \succ_{c_1} \emptyset$ *and the priority ranking of* c_2 *is* $1 \succ_{c_2} \emptyset \succ_{c_2} 2$. *Figure 1 illustrates this instance of the rationing problem.*

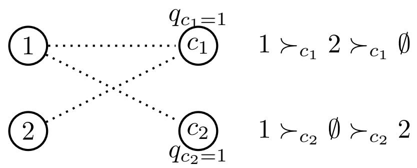

Figure 1: The problem instance described in Example 1. A dotted line between an agent and a category indicates that the agent is eligible for the category.

Next, we consider standard axioms that were considered by Pathak *et al.* [2020b] in the context of integral outcomes and by Delacrétaz [2021] in the context of fractional matchings. We use the more general framework of fractional matchings. The first axiom we consider requires that matchings comply with eligibility requirements. It specifies that a patient should take a fraction of a category for which the agent is eligible.

Definition 1 (Compliance with eligibility requirements)**.** *A matching* μ complies with eligibility requirements *if for any* $i \in N$ *and* $c \in C$, $\mu(i,c) > 0 \implies i \succ_c \emptyset$.

The second axiom concerns the respect of priorities of categories. It rules out that an agent is matched with some category c while some other agent with a higher priority for c is unmatched. The axiom can be viewed as a concept that captures fairness towards the agents.

Definition 2 (Respect of priorities)**.** *A matching* μ respects priorities *if for any* $i, j \in N$ *and* $c \in C$ *such that* $i \succ_c j$, $\sum_{c' \in C} \mu(i,c') < 1 \implies \mu(j,c) = 0$.

Respect of priorities can also be seen as applying ex-ante fairness [Aziz and Klaus, 2019; Kesten and Unver, 2015] in our setting. Next, non-wastefulness requires that if an agent is unmatched despite being eligible for a category, then all units reserved for that category are matched to other agents.

Definition 3 (Non-wastefulness)**.** *A matching* μ *is* non-wasteful *if for any* $i \in N$ *and* $c \in C$, $i \succ_c \emptyset$ *and* $\sum_{c' \in C} \mu(i,c') < 1 \implies \sum_{j \in N} \mu(j,c) = q_c$.

We will refer to the three axioms above as the basic axioms. Not all non-wasteful matchings allocate the same number of units. In particular, some may not allocate as many units as possible. A stronger efficiency notion prescribes that the number of allocated units is maximal subject to compliance with the eligibility requirements.

The *size* of a matching μ is $\sum_{i \in N} \sum_{c \in C} \mu(i,c)$.

Definition 4 (Maximum size matching)**.** *A matching* μ *is a* maximum size matching *if it has maximal size among all matchings complying with eligibility requirements.*

A fractional matching rule is *anonymous* if its outcome depends only on the profile of quotas, eligibility information, and priorities and not on the identity of the agents. A random assignment rule is *neutral* if its outcome depends only on the profile of quotas, eligibility information, and priorities and does not depend on the identity of the categories.

For a matching μ, we will denote by $\mu(c)$ the allocation of category c that specifies what fraction of each agent is given to c. For a matching μ, we will denote by $\mu(i)$ the allocation of agent i that specifies what fraction of each category is given to i. We will denote by $|\mu(i)|$ the term $\sum_{c \in C} \mu(i,c)$. We will refer to $|\mu(i)|$ as the size of i's allocation under μ.

Let $(\succ_c)$ and $(\succ'_c)$ be priority profiles and $i \in N$. We say agent i's priority decreases from $(\succ_c)$ to $(\succ'_c)$ if for all $j, k \neq i$ and $c \in C$,

$$j \succ_c k \longleftrightarrow j \succ'_c k$$
$$j \succ_c i \longrightarrow j \succ'_c i \text{ and } j \succ_c i \longrightarrow j \succ'_c i$$

That is, the priority rankings over agents other than i are the same in both profiles and i can only move down in the priority rankings from $(\succ_c)$ to $(\succ'_c)$. We also say that i's priority decreases from $I = (N, C, (\succ_c), (q_c))$ to $I' = (N, C, (\succ'_c), (q_c))$. Strategyproofness requires that if i is unmatched for I, then i is also unmatched for I'.

Definition 5 (Strategyproofness)**.** *An allocation rule* f *is strategyproof if the aggregate allocation of* i *under* I *is at least the aggregate allocation of* i *under* I' *whenever* i*'s priority decreases from* I *to* I'.

Note that although agents do not have preferences over which category they use, they have the power to lower their priority in some ranking (for example by hiding their eligibility for a category). We are interested in mechanisms that do not incentive agents to hide or underreport their priority in the priority ranking of some category. The definition of strategyproofness is a probabilistic generalization of strategyproofness used in previous work (see, e.g., Aziz and Brandl [2021]).[1]

Rationing Eating (RE) Rule

When considering simultaneous processing of reserves, a natural idea is to consider some form of the eating approach that underlies the probabilistic serial (PS) rule of Bogomolnaia and Moulin [2001]. In the PS rule, agents simultaneously and at the same rate eat their most preferred items until they are fully consumed. PS naturally extends to the case where agents have capacities or find some items unacceptable. In these cases, agents only eat items acceptable to them and stop eating if their capacity has reached.

The original PS rule was generalized to arbitrary constraints Aziz and Brandl [2020]. The idea can be applied to the rationing problem as follows. Agents simultaneously consume fractions of units of eligible categories while ensuring the constraints capturing axioms such eligibility compliance and respect of priorities. However, one immediate challenge that arises is how to efficiently capture respect of priorities as a non-convex feasibility constraint that can be handled in polynomial time.

Instead of pursuing this route, we use the idea of the probabilistic serial rule but from an *inverted* perspective. We treat categories as pseudo-agents and the agents as pseudo-items. The pseudo-agents aka categories now have preferences over the pseudo-items that are derived from the priorities of the

[1] Note that if agents are also allowed to manipulate and become eligible for categories or improve their priorities, then any reasonable rule would be manipulable.

corresponding categories. Each category also has an upper-limit on how many agents it wants. By using this idea, we run the probabilistic serial rule over the pseudomarket (without any additional feasibility constraint that captures respect of priorities). Although, we do not incorporate any priority respecting constraint in the algorithm, we will show that the outcome satisfies respect of priorities. Interestingly, we will show that our proposal also satisfies strategyproofness in the rationing context (in the random assignment context [Bogomolnaia and Moulin, 2001], PS is not strategyproof).

Our approach is formalized as Algorithm 1. It can easily be explained as follows. Categories simultaneously 'eat' their most preferred/highest priority eligible agent at a uniform rate. A category c moves to the next priority agent that is still not finished if an agent has been consumed. A category c stops if all agents are finished or it has eaten q_c agents. In the outcome matching μ, the amount $\mu(i, c)$ is the time category c was eating agent i.

Algorithm 1 The Rationing Eating Rule

Input: $I = (N, C, (\succ_c), (q_c))$
Output: A fractional matching

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Construct an item allocation instance I' = (C, N, (≻_c), (q_c)) where C is viewed as the set of agents, N is the set of items, (≻_c) represent the preferences of agents in C over items in N. Each agent c ∈ C has a upper capacity of q_c.
μ ⟵ PS(I').
return Return μ.
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Our first observation is that RE constitutes a new rule that may give a different outcome from the SR rule of Delacrétaz [2021] (see example in the appendix).

Next, we establish the important axiomatic properties of RE. A matching μ satisfies *category sd-envy-freeness* if for any $c, d \in C$, the following holds. For any allocation $\mu'(d)$ that is a suballocation of $\mu(d)$ with $|\mu'(d)| \leq q_c$, it is the case that $\mu(c) \succsim_c^{sd} \mu'(d)$ where $\succsim_c^{sd}$ is the *first order stochastic dominance lottery (sd)* extension defined as follows: $\mu(c) \succsim_c^{sd} \mu'(d)$ if and only if for all $i \in N$: $\sum_{j \succsim_c i} \mu(i, c) \geq \sum_{j \succsim_c i} \mu'(i, d)$. Also $\mu(c) \succ_c^{sd} \mu'(d)$ if $\mu(c) \succsim_c^{sd} \mu'(d)$ and $\mu'(d) \not\succsim_c^{sd} \mu(c)$.

A matching μ satisfies *category sd-efficiency* if there exists no other matching μ' such that $\mu'(c) \succsim_c^{sd} \mu(c)$ for all $c \in C$ and $\mu'(c) \succ_c^{sd} \mu(c)$ for some $c \in C$.

Theorem 1. *RE satisfies (1) eligibility requirements, (2) respect of priorities, (3) non-wastefulness, (4) anonymity, (5) neutrality, (6) category sd-envy-freeness, (7) category efficiency, (8) and matches at most $|C|$ agents with weight in $(0, 1)$.*

Proof Sketch. We deal with each case separately.

(1) Eligibility requirements: At any point in the algorithm, a category only tries to increase the corresponding share with agents who are eligible.

(2) Respect of priorities: Suppose for contradiction that there exist $i, j \in N$ and $c \in C$ such that $i \succ_c j$, $\sum_{c' \in C} \mu(i, c') < 1$ and $\mu(j, c) > 0$. But this is not possible as category c would have tried to get more of i before considering j.

(3) Non-wastefulness: Suppose an outcome violates non-wastefulness. This means that there is an $i \in N$ and $c \in C$, $i \succ_c \emptyset$, $\sum_{c' \in C} \mu(i, c') < 1$, but $\sum_{j \in N} \mu(j, c) < q_c$. But this is not possible as the algorithm would not have terminated with this outcome as categories continue to increase share with an eligible agent who is not fully matched until the quota is met or all agents are matched.

(4) Anonymity: the algorithm does not use the specifications of the agent names.

(5) Neutrality: the algorithm does not use the specifications of the category names.

(6) Category sd-envy-freeness: at each step, any category who has not reached its quota is eating a most preferred agent. Hence, the outcome satisfies category sd-envy-freeness.

(7) Category sd-efficiency: follows from the fact that the outcome of multi-unit eating PS is sd-efficient Kojima [2009].

(8) Matches at most $|C|$ agents with weight in $(0, 1)$. If an agent is matched with aggregate weight in $(0, 1)$, then it must an agent who was still being eaten by some category when the algorithm terminated. There can be at most $|C|$ such agents.

□

Next, we prove strategyproofness of RE which is considerably more challenging to prove. In order to do so, we first explore connections with a rule called *round robin (RR) sequential allocation*. The round robin (RR) sequential allocation rule allocates indivisible items. Agents take turns in a round robin manner and in their turn, they pick the most preferred available and acceptable item if the agent capacity is not reached. The rule is well-known within the class of 'picking sequences' rules (see, e.g., [Bouveret and Lang, 2011]).

Next we point out that the PS rule can be viewed as first dividing the divisible resources into small enough indivisible items and then running RR. For n agents and m items, consider running PS on all possible $(2^m(m!))^n$ preference profiles for n agents and m items where 2^m reflects the possibilities of acceptable sets of items for an agent. In each profile i, let $t_i^1, \ldots, t_i^{k_i}$ be the k_i different time points in the PS algorithm run for the i-th profile when at least one item is finished. We claim that each of these time points is rational. We prove the claim by induction.

Proof. Suppose that the first k time points are rational. Then, consider the item o that is next to be consumed at the $k+1$-st time point. Since all previous time points are rational, a rational amount of o has been consumed. The remaining amount is allocated uniformly among the agents who eat it till it is consumed. Hence, the $k+1$-st time point is rational as well. □

Let $g = \text{GCD}(\{t_i^{j+1} - t_i^j : j \in \{1, \ldots, k_i - 1\}, i \in \{1, \ldots, m!^n\})$ where GCD denotes the greatest common divisor. Since in each profile i, $t_i^{j+1} - t_i^j > 0$ for all $j \in \{0, \ldots, k_i - 1\}$, we have that g is finite and greater than zero. The time interval length g is small enough such that each run of the PS rule can be considered to have m/g stages of duration g. Each stage can be viewed as having n sub-stages so that in each stage, agent i eats g/n units of a item in sub-stage i of a stage. In each sub-stage only one agent eats g/n units of the most favoured item that is available. Hence we now view PS as consisting of a total of mn/g sub-stages and the agents keep coming in order $1, 2, \ldots, n$ to eat g/n units of the most preferred item that is still available.

Next, we present a reduction f from an instance $I = (N, q, O, \succ,)$ where O is a set of divisible items to $I' = (N, q, O', \succ')$ where O' is a set of indivisible items. The agent set remains unchanged. Each $o \in O$ has corresponding items $\{o^1, \ldots, o^{n/g}\}$ items in O'. So $O' = \bigcup_{o \in O}\{o^1, \ldots, o^{n/g}\}$. The preferences of the agents are as follows. The preference $o_j \succ_i o_k$ implies that $o_j^a \succ_i' o_k^b$ for all $a, b \in \{1, \ldots, n/g\}$. For indivisible items pertaining to an item $o \in O$, agents prefer the item with a lower index more than the one with higher index: $o^j \succ_i' o^k$ for $o \in O$ and $j < k$. We prove that the allocations $PS(I)$ and $RR(f(I))$ give the same outcome if we view the indivisible items $o^1, \ldots, o^{n/g}$ as portions of the original items o.

We prove a series of lemmas that are helpful in establishing that RE is strategyproof.

Lemma 1. *The allocations $PS(I)$ and $RR(f(I))$ give the same allocation.*

Lemma 2. *Consider an instance $I = (N, q, O, \succ)$ and its corresponding instance $f(I) = I' = (N, q, O', \succ')$. Then if o^j is not allocated under $RR(I')$, then neither are $o^j, \ldots, o^{g/n}$.*

Proof. An item o^j and o^k for $k > j$ are identical for all agents except that $o_j \succ_i o^k$ for all $i \in N$. Hence, if some agent i picks o^k, then it should already have picked o^j. ☐

Lemma 3. *Let O_1 be the set of allocated items under RR applied to instance $I' = (N, q, O', \succ')$. Suppose some agent i moves an item o later in the preference list right before item o_2 which results in preference profile $\succ''$. Suppose all other agents $j \in N \setminus \{i\}$ find o_2 to be a clone of o such that $o \succ_j o_2$. Suppose o_2 is unallocated under $\succ'$. Then, for the set of allocated items O_2 for the instance $I'' = (N, q, O', \succ'')$, o_2 is unallocated and one of the following holds: 1. $O_2 = O_1$ 2. $O_2 = O_1 \setminus \{o\}$ 3. $O_2 = (O_1 \setminus \{o\}) \cup \{a\}$ for some $a \in O_1$*

The proof is based on a long and detailed case analysis and is deferred to the appendix. The lemmas above help establish Lemma 4.

Lemma 4. *Suppose that an agent i lowers an item o in its preference list. Then consider the original preference profile $\succ$ and the modified profile $\succ'$. The amount of item o consumed in $PS(\succ')$ is at most the amount of item o consumed in $PS(\succ)$.*

Lemma 5. *RE is strategyproof.*

Proof. We have already proved in Lemma 4 that if an item is placed lower in the preference list, then under PS, at most as much of the item is consumed. Since in RE, the categories are the 'agents' and the agents are the 'items', an agent lowering itself in a priority list of a category results in at most as much of the agent being eaten by the categories. ☐

Maximum Rationing Eating (MRE) Rule

In this section, we present a new rule that simultaneously processes reserves but does so without compromising on the maximum size property. Our first observation is that the key axiomatic property (category uniformity) of the SR rule of Delacrétaz [2021] is incompatible with maximum size property. Take Example 1: category uniformity requires that agent 1 gets half a unit from each category. But then the outcome cannot be maximum size. The following proposition can be seen as highlighting this limiting aspect of the category uniformity property proposed by Delacrétaz [2021].

Proposition 1 (Impossibility result)**.** *Category uniformity is incompatible with the maximum size property.*

We design a new rule called Maximum Rationing Eating (MRE) that can be viewed as a careful modification of the RE rule. We will show that although the modification leads to category sd-envy-freeness not holding, it allows us to obtain the maximum-size matching property. Note that for bipartite graphs with integral quotas, a maximum size fractional matching has the same size as the maximum size integral matching. We first compute the maximal size of a matching. We then use the same approach as RE to build an instance of indivisible item allocation problem. For the instance, instead of applying PS, we apply the *Vigilant Eating Rule (VER)* of Aziz and Brandl [2020] with the specific constraint that the outcome should have maximum size. VER is a more complex eating algorithm that is parametrized with respect to feasibility constraints and only allows eating if eating still allows for no feasibility constraint being violated in the returned allocation. VER can handle arbitrary constraints but for linear convex constraints, it is guaranteed to take polynomial time. It also computes an outcome that is sd-efficient among the outcomes satisfying the constraints. Oue MRE algorithm is specified as Algorithm 2.

Algorithm 2 The Max Size Rationing Eating Rule

Input: $I = (N, C, (\succ_c), (q_c))$
Output: A fractional matching

For I, use the Hopcroft-Karp algorithm to compute the maximum size $ms(I)$ of a matching that satisfies eligibility requirements.
Construct an item allocation instance $I' = (C, N, (\succ_c), (q_c))$ where C is viewed as the set of agents, N is the set of items, $(\succ_c)$ representing the preferences of agents in C over items in N. Each 'agent' $c \in C$ has an upper capacity of q_c.
$\mu \longleftarrow VER(I')$ with the constraint that $|\mu| = ms(I)$. {VER is the rule of Aziz and Brandl [2020].}
return μ.

The next theorem establishes the properties of MRE.

	MRE	RE	Simultaneous Reserves (Delacrétaz, 2021)	REV (Aziz & Brandl, 2021)	Smart Reserves (Pathak et al., 2020)	DA / Sequential Categories
compliance with eligibility requirements	✓	✓	✓	✓	✓	✓
respect of priorities	✓	✓	✓	✓	✓	✓
maximum size	✓	–	–	✓	✓	–
anonymity	✓	✓	✓	–	–	✓
neutrality	✓	✓	✓	✓	✓	–
category sd-envy-freeness	–	✓	–	–	–	–
category sd-efficiency	✓	✓	?	–	–	✓
category uniformity	–	–	✓	–	–	–
handles heterogeneous priorities	✓	✓	✓	✓	–	✓

Table 1: Properties satisfied by prioritized rationing algorithms.

Theorem 2. *MRE satisfies (1) eligibility requirements, (2) respect of priorities, (3) non-wastefulness, (4) anonymity, (5) neutrality, (6) the maximum-size property, (7) category sd-efficiency, (8) and matches at most $|C|$ agents with weight in $(0, 1)$.*

Proof. We deal with each case separately.

(1) Eligibility requirements: At any point in the algorithm, a category only tries to increase the corresponding share with agents who are eligible.

(2) Respect of priorities. Suppose for contradiction that there exist $i, j \in N$ and $c \in C$ such that $i \succ_c j$, $\sum_{c' \in C} \mu(i, c') < 1$ and $\mu(j, c) > 0$. But this is not possible as category c would have tried to get more of i before considering j. In particular, category c can get at least ϵ more of i where $0 < \epsilon \leq \min((1 - \sum_{c' \in C} \mu(i, c')), \mu(j, c))$ and ϵ less of j without violating the constraint of maximum size.

(3) Non-wastefulness: Suppose an outcome violates non-wastefulness. This means that there is an $i \in N$ and $c \in C$, $i \succ_c \emptyset$, $\sum_{c' \in C} \mu(i, c') < 1$, but $\sum_{j \in N} \mu(j, c) < q_c$. But this is not possible as the algorithm would not have terminated with this outcome as categories continue to increase share with an eligible agent who is not fully matched until the quota is met or all agents are matched.

(4) Anonymity: the algorithm does not use the specifications of the agent names.

(5) Neutrality: the algorithm does not use the specifications of the category names.

(6) Max-size property: since we impose max-size as a constraint of VER, it follows that this constraint is satisfied.

(7) Category sd-efficiency: By sd-efficiency of VER, the outcome is category sd-efficient among all *maximum size* matchings. It implies that the outcome of MRE is category sd-efficient among all *maximum size* matchings. Next, we claim that the outcome is sd-efficient among *all matchings*. Suppose a matching λ sd-dominates the μ the outcome matching. Then λ must have a smaller size than μ. But then some category gets less agents than before so λ does not sd-dominate μ, a contradiction. So we have established that μ is sd-efficient among all matchings.

(8) Matches at most $|C|$ agents with weight in $(0, 1)$. If an agent is matched with aggregate weight in $(0, 1)$, then it must an agent who was still being eaten by some category when the algorithm terminated. There can be at most $|C|$ such agents.

□

Since, MRE does not satisfy category sd-envy-freeness, it leads to the question of whether there is some rule that simultaneously satisfies compliance with eligibility requirements, category sd-envy-freeness and the maximum size property. This is impossible in view of the following.

Proposition 2 (Impossibility result)**.** *Compliance with eligibility requirements, category sd-envy-freeness and maximum size property are incompatible.*

Proof. Consider Example 1. There is a unique matching that is maximum size and it does not satisfy category sd-envy-freeness. □

Discussion

The topic of allocation of reserved units under category capacities and priorities has tremendous applications. We added two new rules to the toolkit of rationing under categories and established their relative merits. The relative merits of the rules in comparison with previously presented rules are shown in Table 1. Since the SR rule is considerably more complex (requires linear programing to address convergence issues) than other rules, it is not clear whether it satisfies sd-efficiency or strategyproofness.

The probabilities that we obtain of giving an agent a unit from a particular category needs to be used to obtain an actual integral matching. This can easily be done by invoking the Birkhoff's decomposition algorithm [Birkhoff, 1946; Lovász and Plummer, 2009]. In the next statements, we point out that some of the properties of the fractional matching also hold for the matchings in the decomposition.

Proposition 3. *Let $\mu = \lambda_1 \mu_1 + \cdots + \lambda_k \mu_k$ be the MRE outcome represented as a convex combination of integral matchings. Then each μ_i satisfies (1) compliance with eligibility requirements, (2) maximal size property, (3) respect of priorities, and (4) non-wastefulness.*

We note however the integral matchings in the convex combination may not satisfy the hard capacity constraints (a category may give one more unit than its capacity).

Proposition 4. *Let $\mu = \lambda_1 \mu_1 + \cdots + \lambda_k \mu_k$ be the RE outcome represesented as a convex combination of integral matchings. Then each μ_i satisfies (1) compliance with eligibility requirements, (2) respect of priorities, and (3) non-wastefulness.*

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Difference between RE and SR

Example 2.

$$c_1: \quad 1 \succ_{c_1} 2 \succ_{c_1} 3 \succ_{c_1} 4$$
$$c_2: \quad 3 \succ_{c_2} 2 \succ_{c_2} 1 \succ_{c_2} 4$$
$$c_3: \quad 1 \succ_{c_3} 3 \succ_{c_3} 2 \succ_{c_3} 4$$
$$q_{c_1} = 1, q_{c_2} = 1, q_{c_3} = 1.$$

For the problem instance I, the outcome of our rule RE as well as the outcome of the SR rule of Delacrétaz [2021] as follows.

$$SR(I) = \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$RE(I) = \begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1/2 & 0 & 6/12 \\ 1/3 & 1/4 & 5/12 \\ 0 & 3/4 & 3/12 \\ 1/6 & 0 & 10/12 \end{pmatrix} \end{array}$$

Proof of Lemma 3

Proof. Suppose agent i does not get o under $\succ'$. Then $O_1 = O_2$. Hence, we consider the case in which i gets o under $\succ'$ Since the items are picked in the same manner until i's turn to pick o under $\succ'$, let us consider it as the first turn. We prove that for each turn k, the statement of the lemma holds up till that turn. For the base case, if i still picks o under $\succ''$ in the first turn. Then (1) holds up till the first turn. If i picks some item $b \neq o$ under $\succ''$ in her first turn, then (3) holds up till the first turn. Finally, if i does not pick any item under $\succ''$ in her first turn, then (2) holds up till the first turn.

For the induction step, suppose that (1), (2), or (3) holds after k turns. Let us consider the $k+1$st turn. Suppose agent j has the turn.

1. If (1) holds for k, then the set of available items at this point is the same under $\succ'$ and $\succ''$. Therefore, the next agent j will pick the same item under both profiles. Hence, (1) hold after $k+1$ turns.
2. If (2) holds after k turns, then at the $k+1$st turn agent j has one more item (item o) available under $\succ''$. The following are all the scenarios can happen.
   (a) Agent j picks item c under $\succ'$ and picks c under $\succ''$: (2) holds after $k+1$ turns
   (b) Agent j picks no item under $\succ'$ and no item under $\succ''$: (2) holds after $k+1$ turns.
   (c) Agent j picks item c under $\succ'$ and picks o under $\succ''$. This means that $o \succ''_j c$ which implies that $o \succ'_j c$. Also $o \succ''_j o_2 \succ''_j c$. Hence, under $\succ'$, agent j picks o_2 in the $k+1$st turn, which is a contradiction as o_2 is unallocated under $\succ'$. Thus, this case does not arise.
   (d) Agent j picks no item under $\succ'$ and picks o under $\succ''$: (1) holds after $k+1$ turns.
3. Suppose (3) holds after k steps. The set of items allocated after k steps are $O^* \cup \{o\}$ under $\succ'$ and $O^* \cup \{a\}$ under $\succ''$ where $O^* \subseteq O' \setminus \{a, o\}$. The following are all the scenarios can happen.
   (a) Agent j picks item c under $\succ'$ and picks c under $\succ''$: (3) holds after $k+1$ steps
   (b) Agent j picks no item under $\succ'$ and picks no item under $\succ''$: (3) holds after $k+1$ steps
   (c) Agent j picks a under $\succ'$ and picks o under $\succ''$: (1) holds after $k+1$ steps
   (d) Agent j picks a under $\succ'$ and picks no item under $\succ''$: (2) holds after $k+1$ steps
   (e) Agent j picks no item under $\succ'$ and picks o under $\succ''$. We show that this case is impossible. Since j picks o under $\succ''$, it also finds o_2 acceptable because agents find o_2 to be a clone of o. Since o_2 is unallocated under $\succ'$, j would picked it up at the $k+1$st turn under $\succ'$, which is a contradiction to the case.
   (f) Agent j picks c under $\succ'$ and picks o under $\succ''$. This means that $o \succ''_j c$.
      - Suppose $j \neq i$. But this is a contradiction as under $\succ'$, agent j would have picked o_2 as $o \succ'_j o_2 \succ'_j c$ if agent $j \neq i$.
      - Now suppose $j = i$. Suppose i picks c under $\succ'$ and picks o under $\succ''$. This means that $o \succ''_i o_2 \succ''_i c$ which implies that $o \succ'_i o_2 \succ'_i c$. The latter implies that i picks o_2 before c under $\succ'$, a contradiction as o_2 is unallocated under $\succ'$.
   (g) Agent j picks a under $\succ'$ and picks $c \neq o$ under $\succ''$. Since o is available under $\succ''$, it follows that $c \succ_j o \succ_j o_2$. In particular, $c \neq o_2$. Hence, (3) holds after $k+1$ steps.

□

Proof of Lemma 4

Proof. Consider an instance I and its corresponding instance $f(I)$. We are focussing on agent i lowering item o in its preference list. We capture the effect on the outcome under PS indirectly by focussing on the outcomes of RR for instance $f(I)$. We analyse the effect of agent i lowering item o in its preference list in instance I by moving the set of items $\{o^1, \ldots, o^{n/g}\}$ all together lower down in the preference list under instance I'. Suppose o is moved down in the preference list to a position just after item a. Then we move the corresponding indivisible items in $\{o^1, \ldots, o^{n/g}\}$ to a position just after the indivisible items $a^1, \ldots, a^{n/g}$. Instead of understanding the effect of moving all these items together, we carefully move items $o^{n/g}, \ldots, o^1$ one by one. In each such operation except case (4), we know from Lemma 3 that at most as many items pertaining to o are picked as before.

Hence, the item b or a do not pertain to o so the count of the items pertaining to o does not increase. We also know from Lemma 2 that if o^j is not allocated under $RR(I')$, then neither are $o^j, \ldots, o^{g/n}$.

By moving all items pertaining to o lower down the preference list, we have simulated the effect of moving item o in the preference list $\succ$ for instance I. Hence, the statement of the lemma follows. $\square$