Matching Market Design with Constraints

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Abstract

Two-sided matching is an important research area that has had a major impact on the design of real-world matching markets. One consistent feature in many of the real-world applications is that they impose new feasibility constraints that lead to research challenges. We survey developments in the field of two-sided matching with various constraints, including those based on regions, diversity, multi-dimensional capacities, and matroids.

1 Introduction

Matching market design has provided a rich and fertile theory for designing centralized mechanisms for the ubiquitous issue of allocation. The research area has provided a ground for cross-fertilization of ideas, tools, and methods from economics (social choice, mechanism design etc.), computer science (theoretical computer science and artificial intelligence), mathematics (game theory and discrete applied mathematics), and operations research (in particular combinatorial optimization).

In matching markets, the central problem is to match agents to institutions (such as schools or hospitals). In these markets, not only do agents have preferences over institutions, but these institutions also have priorities over the agents [Roth and Sotomayor, 1992]. The goal is to match the agents to the institutions while keeping into consideration the preference and priorities.

In this survey, we consider matching market design under various constraints. In the seminal paper on this topic [Gale and Shapley, 1962], there was already a practical constraint that was considered: the institutions had upper capacities on how many agents they can accommodate. As the theory of matching market design has developed, a consistent stream of results has focused on general two-sided matching models with various types of constraints (see, e.g., Kojima [2019]). Some of their results are aimed at abstraction, unification and generalization of existing methods for two-sided matching. Many results, however, are directly motivated by concrete problems that lead to new feasibility or distributional constraints to a standard two-sided matching problem. For example, there may be an upper bound on a set of urban hospitals to ensure there are enough doctors matched to rural hospitals. Another example is putting a lower bound on the number of underprivileged students at a university as an affirmative action policy. We survey some of the recent results and trends on two-sided matching with constraints. We cover both some of the active work in the economic theory community as well as a growing body of work on the topic within computer science.

2 Basic Model of Gale and Shapley

We first establish the base model of matching under preferences. The two-sided matching model typically has a set of agents on one side and a set of institutions on the other. Depending on the application context, the agents are students, doctors, or applicants. Similarly, depending on the application context, the institutions are schools, hospitals, or employers. The basic model has been referred to as the College Admissions model [Gale and Shapley, 1962] or the Hospitals/Residents problem [Manlove, 2013].

Each institution has an upper capacity on how many agents can be matched to it. Each agent has a preference relation over the institutions. Each institution has a priority relation over the agents. These preference and priority relations are typically ordinal rankings.

An outcome of a two-sided matching problem is an allocation or matching that matches each agent to either some institution or keeps her unmatched. When matching agents to institutions, it is natural to take the preferences and priorities into account. How well this information is taken into account is formalized through axioms that capture the desirability of the matching or the algorithm that finds the matching. Next, we discuss some of the most important axiomatic properties.

A minimal efficiency concept is non-wastefulness that requires that no agent should prefer to be matched to an institution that still has some vacant space. A widely-used fairness concept is fairness [Kamada and Kojima, 2020] (also called justified envy-freeness [Abdulkadiroğlu and Sönmez, 2003]) that requires that there should be no agent who would prefer to be matched to some institution and there is an agent matched to who has lower priority than in ’s priority ranking. If non-wastefulness or fairness are violated, then the agent and institution with respect to which the violation occurs are viewed as forming a blocking pair.

The combination of non-wastefulness and fairness (i.e., the lack of blocking) is also referred to as stability in the literature. Stability, however, is a loaded term and there are several variants and generalizations of stability for more complex models and many of them are simply referred to as stability or weak stability.

In two-sided matching, one of the most impactful results is that the Agent Proposing Deferred Acceptance Algorithm satisfies compelling properties. The algorithm returns a matching that
is stable and agent-optimal [Gale and Shapley, 1962]. It is also polynomial-time and strategyproof (an agent cannot get a better outcome by misreporting the preferences). The algorithm works in the following way: agents propose to their most preferred institutions. Based on the proposals, the institution choose the highest priority individuals up to the capacity. The rejected agents then propose to their next most preferred institution until they run out of options. For a history of the Deferred Acceptance (DA) algorithm, the reader is recommended to read the paper by Roth [2008].

3 Generalized Deferred Acceptance

The general model that we present here has three main extensions. First, we allow different contract terms to be used for an agent-institution pair, so for a contract \( x = (i, h, c) \), \( c \) denotes the contractual relation, \( i \in N \) is an agent, and \( h \in H \) is an institution. For instance, in the college admission setting the contractual relation can mean the tuition and other terms, as in Hungary [Biró et al., 2010] and [Biró et al., 2020], or the number of years served, as in the US cadet-branch matching [Sönmez and Switzer, 2013]. Let \( X \) denote the set of all contracts, and for a set of contract \( X \subset X \) let \( X_i \) and \( X_h \) denote the subsets of contracts restricted to \( i \) and \( h \), respectively. Secondly, the institutions have choice functions \( Ch_i \) that for every set of contracts \( X \), \( h \) selects a subset \( Y \subset X_h \). Note that in some matching with contract models, e.g., in [Hatfield and Milgrom, 2005], as the primitive of the model it is assumed that every institution \( h \) has strict preference over the sets of contracts involving \( h \) that determines its choice function. However, this is a simplification, that does not cover even some basic settings that is present in some applications.1 Thirdly, we shall define the collective choice function of the institutions \( Ch_H \), that select \( Ch_H(X) \subset X \) for every \( X \subset X \). Most of the models in the literature assume that \( Ch_H \) is simply the union of the individual choice functions \( Ch_h \) for all \( h \in H \), however, this can be more complex in models with distributional constraints.2 We summarize the main properties and results for this generalized setting.

The following properties are crucial for choice functions.

- **Substitutability (SUB):** \( X \subset X \subset X \) implies \( X \setminus Ch_H(X) \subset X \setminus Ch_H(X) \), so if a contract is rejected from a set, then it is also rejected from an extended set.
- **Irrelevance of Rejected Contracts (IRC):** \( Ch_H(X) \subset X \subset X \) implies \( Ch_H(X) = Ch_H(Y) \), so if a contract is selected from a set, then it will also be selected when some rejected contracts are removed from the choice set.
- **Law of Aggregate Demand (LAD):** \( Y \subset X \) implies \( |Ch_H(Y)| \leq |Ch_H(X)| \), so the number of selected contracts can only increase from an extended set of contracts.3

1As an example we mention the choice function of the Hungarian universities, where ties are present due to equal scores. When two students are applying for the last seat at a program the decision is to reject them both, however, if only one of these students are applying then she is admitted. In Chile, both of them are admitted even if the quota is violated. See more about these choice functions in [Biró and Kiselgof, 2015] and [Fleiner and Jankó, 2014].

2An example is again from the Hungarian university admissions, where common quotas are applied for sets of programs on the top of the individual capacities of the programs, where the programs considered share the same ranking over the students [Biró et al., 2010].

3Note that Fleiner [2003] used the following terminologies: contracts

Fleiner [2003] showed that for the general many-to-many matching model with substitutable choice functions on both sides, a so-called three-stable3 matching always exists by Tarski’s fixpoint theorem. Three-stability means the existence of three sets \( X, A, B \), subsets of \( X \), such that \( X = A \cap B \), \( X = A \cup B \), and \( Ch_H(A) = X = Ch_N(B) \).

Furthermore, Fleiner [2003] showed that the three-stable solutions form a lattice if the IRC property holds, where the two variants of the Generalized Deferred Acceptance algorithm produce the agent-optimal and institute-optimal solutions.

<table>
<thead>
<tr>
<th>Algorithm 1 Generalized Deferred Acceptance (GDA)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> Instance ( I, Ch_N, Ch_H ), a set of contracts ( X )</td>
</tr>
<tr>
<td><strong>Output:</strong> An outcome ( Z \subset X )</td>
</tr>
<tr>
<td>( Re \leftarrow \emptyset ), ( Y \leftarrow X ), ( Z \leftarrow \emptyset )</td>
</tr>
<tr>
<td>while ( Y \neq Z ) do</td>
</tr>
<tr>
<td>( Y \leftarrow Ch_N(Y \setminus Re) ) {Agents select contracts.}</td>
</tr>
<tr>
<td>( Z \leftarrow Ch_H(Y) ) {Institutions select contracts.}</td>
</tr>
<tr>
<td>( Re \leftarrow Re \cup (Y \setminus Z) ) {Update rejected contracts.}</td>
</tr>
<tr>
<td>end while</td>
</tr>
<tr>
<td>return ( Z )</td>
</tr>
</tbody>
</table>

Note that when the IRC property holds three-stability is equivalent to stability [Fleiner and Jankó, 2014], where stability is defined as follows. An outcome \( X \subset X \) is stable if (i) \( X = Ch(N)(X) = Ch_H(X) \) and (ii) there exist no contract \( x \in X \) such that \( x \in Ch_H(X \cup \{x\}) \) and \( x \in Ch_N(X \cup \{x\}) \) hold. However, without the IRC condition a stable matching may not exist even for substitutable choice functions in the many-to-one case, as Aygün and Sönmez [2013] demonstrated.

Hatfield and Milgrom [2005] considered a restricted many-to-one setting, where they assumed strict preferences for the institutions (in which case the IRC property holds automatically). Furthermore they assumed that the choice functions are unitary, meaning that each hospital selects at most one contract by every agent.3 Using the same proof technique as Fleiner [2003] they showed that if the SUB and LAD properties hold, then GDA produces an agent-optimal stable matching, and as a new result they also showed that this mechanism is strategyproof for the agents.

The above described general framework with contracts is useful to derive similar results in the context of distributional constraints using the variants of the DA algorithm. However, in most papers in this literature the primitives are simpler, based on the Gale-Shapley model with no contracts. Every agent \( i \) has a preference order \( \succ_i \) over the institutions and her choice function \( Ch_i \) simply selects the best institute from a set offered. Similarly, every institution \( h \) has a

\( = \) edges, \( SUB = \) co-monoton, \( IRC = \) path-independence, \( LAD = \) increasing.

3The name of three-stability was given in [Fleiner and Jankó, 2014] for this stability concept, where the connections between different stability notions were clarified.

3In a recent paper, Hatfield and Kominers [2016] showed that indeed this unitary assumption in [Hatfield and Milgrom, 2005] was unnecessary and misleading because there are many natural choice functions in practice that satisfy the SUB condition, but fail to satisfy it when the unitary requirement is imposed. For these cases the stability and agent-optimality still holds by [Fleiner, 2003], and Hatfield and Kominers showed that strategyproofness also holds for the Generalized DA.
strict order over the acceptable agents \(\succ_h\), together with a quota \(q_h\) and from a set of agents \(C_h\), selects the quota-many best agents.

When additional feasibility constraints are imposed in two-sided matching, several challenges come up [Kojima, 2019]. A stable matching or even a feasible matching may not exist even and checking the existence of such a matching can be computationally intractable. In the subsequent sections, we discuss various types of constraints that have been considered in the context of two-sided matching under preferences.

4 Lower quotas

The challenge of feasibility is immediately seen if we impose lower capacities at institutions besides the upper capacities. There are two main cases for this special feature.

In college admission or course allocation setting, the lower quota is considered as a minimum number of students needed to open the program. Stability is defined in a standard way for open programs, and for each closed program, the natural non-wastefulness condition is that the number of students preferring a stable matching or even a feasible matching may not exist. This model was first proposed in [Kamada and Kojima, 2010].

Regional constraints have been studied in several papers [Kamada and Kojima, 2010; Aziz et al., 2019]. Kamada and Kojima [2017a] focused on the case of disjoint regions and showed that even for this restriction, a natural notion of stability which they refer to as strong stability is not guaranteed to be satisfied by at least some matching. Aziz et al. [2020] showed that even for disjoint regions, checking whether a strongly stable matching exists or not is NP-hard. Kamada and Kojima [2017a] proposed a weak stability concept for the setting. We call a matching \(X\) weakly stable if, for any blocking pair \((i, h)\) for matching \(X\), the following two conditions are satisfied. (i) \(j \succ_h i\) for all applicants \(s.t. \ (j, h) \in X_h \) (ii) \(X \cup \{(i, h)\}\) is not feasible.

We can construct \(Ch_H\) such that it satisfies the SUB condition as well as required constraints, as discussed in Kojima et al. [2018]. For the simplest case, assume each region \(r\) has a priority order among all contracts related to the institutions in \(r\). \(Ch_H(X)\) chooses at most \(q_r\) contracts from \(\bigcup_{h \in H} X_h\) as a whole, and at most \(q_h\) contracts from \(X_h\), according to the priority. It is easy to verify such a choice function satisfies the SUB, IRC, and LAD conditions. By using this choice function, Algorithm 1 satisfies strategyproofness, and the obtained matching satisfies the stability defined in Section 3 as well as weak stability.

Kamada and Kojima [2017a] also presented an algorithm to compute a weakly stable matching for the problem for any class of region constraints. The algorithm is not strategyproof and in fact, it is an open problem whether weak stability and strategyproofness are compatible.

For a general model of matching with regional constraints in which regions also have priorities over the distributions of agents within the sub-regions within the region, Kamada and Kojima [2017a] considered an intermediate notion of stability that they refer to as stability. They show that a matching guaranteeing the notion is guaranteed to exist if and only if the set of region constraints forms a nested structure.

Biró et al. [2010] and Goto et al. [2016] also presented positive existence and computational results for nested constraints. Goto et al. [2016] allow both lower and upper quotas on the regions.

5 Regional Constraints

Regional constraints came up in the context of residency allocation, i.e., matching medical residents (doctors) to hospitals. The background was that in order to ensure that sufficient doctors work in rural regions, an upper capacity was placed on urban regions. In general, regional constraints place subsets of institutions such as hospitals into regions and then place upper capacities on the regions. Regional constraints have also been referred to as common quotas [Biró et al., 2010].

There are various levels of regional constraints: (1) disjoint (2) nested (in which for any two regions, either they are disjoint or one is contained in the other), and (3) arbitrary. For the disjoint case, we assume there exists a set of regions \(R\), which is a partition of institutes \(H\). Each region \(r \in R\) has its upper quota \(q_r\). In addition to the constraints for the basic model, we require for any \(X' \subseteq X\), if \(X = Ch_H(X')\), then \(\sum_{h \in R} |X_h| \leq q_r\) holds. Note that with regional constraints, we are no longer able to represent choice function \(Ch_H\) as a union of individual choice functions \(Ch_h\) for \(h \in H\).

The hospital-resident matching problem with regional constraints has been studied in several papers [Kamada and Kojima, 2015; Biró et al., 2010; Goto et al., 2016; Aziz et al., 2019].

6 Diversity Constraints

Diversity concerns are prevalent in many decision making problems including that of two-sided matching. Each agent in a two-sided matching problem may have certain types that indicate they have special attributes (such as being highly talented) or satisfy some affirmative action categories (such as being a historically disadvantaged group). Each institution may have its own diversity constraints and may make selection decisions based on a combination of its priority list over agents as well as information about the types satisfied by the agents.

A standard technique to address diversity constraints are to impose lower and upper quotas on the number of members of given types at the institution. These quotas may be treated as hard constraints or soft constraints (goals). For example, assume agents are divided into disjoint types \(T = \{t_1, \ldots, t_k\}\), and institution \(h\) imposes a hard-type-specific upper quota \(q_{h,t}\) for each type \(t \in T\) (as well as its overall upper quota \(q_h\)). Then, we require for any \(X' \subseteq X\), if \(X = Ch_H(X')\), then \(|X_{h,t}| \leq q_{h,t}\) holds, where \(X_{h,t} = \{(i, h) | (i, h) \in X, i \text{ belongs to type } t\}\).

Ehlers et al. [2014a] wrote one of the most influential papers on the topic where they considered the impact of soft and hard diversity constraints. Echenique and Yenmez [2015] also assumed
that each student has one type and examined the structure of choice functions that satisfy the SUB condition.

Kurata et al. [2017] were the first to consider the setting in which each student has multiple types and each school imposes soft quotas on each type. There are several subsequent works with similar approaches. For example, Aygün and Turhan [2020] and Correa et al. [2019] focus on a model in which student are allowed to have multiple types and colleges divide the seats into groups. Aygün and Bö [2020] study the college admission with multidimensional privileges where each student may have multiple types and each seat within the same school may have different priorities over students. One of the main routes to achieving desirable desiderata such as stability and strategyproofness in this domain is to design suitable choice functions of institutions that satisfy the SUB and LAD conditions. By doing so, the general results of Hatfield and Milgrom [2005] can be applied in these contexts.

Gonczarowski et al. [2019] proposed an algorithm for the Israeli “Mechinot” gap-year problem in which each student has multiple types and schools impose soft lower quotas and hard upper quotas. There has also been some recent work on matchings with constraints on the ratios of types [Nguyen and Vohra, 2019].

In a recent book chapter, Heo [2019] surveys market design as well as experimental work on affirmative action in two-sided matching. The problem has also been examined from a complexity perspective [Biró et al., 2010; Chen et al., 2020; Huang, 2010; Aziz et al., 2019]. Aziz et al. [2019] and Sun [2020] showed that under hard constraints, certain problems for diversity constraints can be reduced to problems under general regional constraints.

7 Multi-dimensional

Capacity Constraints and Agent Sizes

Matching with Sizes

When the agents have sizes, then a stable matching may not exist, and the corresponding decision problem is NP-hard, as McDermid and Manlove [2010] showed for the case of sizes one and two, motivated by the resident allocation problem with couples. In this model, the couples are accepting joint offers only at the hospitals occupying pairs of positions. However, we note that the usual model setting and practice allow also the couples to apply for positions in different hospitals, for a survey see [Biró and Klijn, 2013]. The stable matching problem with sizes was also studied in the context of unsplittable flow problem, motivated by job scheduling to machines [Dean et al., 2006; Cseh and Dean, 2016].

Refugee Constraints

Next, we overview multi-dimensional upper capacity constraints inspired by the problem of matching refugee families. Delacrétaz et al. [2017] formalized refugee allocation as a centralized matching market design problem. A locality can feasibly host a set of families if it can meet the multi-dimensional requirements of the families that could involve services such as hospital beds, children’s day care, special medical services, etc. These feasibility requirements can be captured by multi-dimensional feasibility constraints. For example a family of four members may require four beds and 2 school seats.

Under multi-dimensional sizes and capacity constraints, the synergies between the agents lead to the choice functions of institutions not satisfying the SUB condition. Delacrétaz et al. [2017] pointed out that for the refugee allocation problem, the standard stability concept may lead to non-existence of a stable matching. Hence they focus on a weaker stability notion called quasi-stability for which they propose algorithms. Quasi-stability captures the idea that any family and locality pair cannot block an outcome if the family would have the lowest priority in the new locality, even though the new locality can provide sufficient services to accommodate it.

Aziz et al. [2018] considered a different stability concept that is also a natural weakening of stability for two-sided matching and proposed an algorithm to achieve the property. The concept requires that only those agents can have justified envy for another if they require less resources in each category. Aziz et al. also studied the computational complexity of computing matchings satisfying various notions of stability.

Andersson and Ehlers [2016] focused on a restricted version of the refugee allocation problem with unidimensional service demands and capacity vectors but with a feature that captures language compatibility of families and hosts. For this setting, they presented an algorithm that finds a stable maximum matching. Bansak et al. [2018] and Gölz and Procaccia [2018] studied refugee allocation as an optimization problem without considering stability issues.

In a later section, we discuss how refugee constraints are a special case of more general constraints called heredity constraints. New algorithms that apply for heredity constraints can also be applied for the refugee matching problem.

Budgetary Capacity Constraints

Next, we consider related models in which budgetary constraints are involved that are essentially single-dimensional constraints. Ismaili et al. [2019] considered the case where hospitals hire doctors. To hire a doctor, a hospital need to pay a certain wage, which can vary according to the expertise of the doctor. A hospital wish to hire best doctors within a given budget limit. Assume each contract is represented as \((i,h,w)\), where \(w\) is the wage for agent \(i\) paid by institution \(h\). We require for any \(X_i \subseteq X\), if \(X = \sum_{(i,h,w) \in X_i} w\) is at most the budget limit of \(h\). It is easy to see that the SUB condition can be violated; if a hospital prefers hiring two low-wage doctors over hiring one high-wage doctor, two low-wage doctors can be complementary.

Ismaili et al. [2019] presented several impossibility results for the general case, as well as positive results for some special cases. There also exist works that deal with budget constraints where hospitals have additive utilities [Kawase and Iwasaki, 2017; Kawase and Iwasaki, 2018].

Kawase and Iwasaki [2020] considered a model in which institutions have cardinal preferences rather than ordinal. They focus on three types of cardinal preferences: cardinality-based (utility is equal to the size of the match), additive, and submodular. The choice functions of the institutions maximize total utility subject to various kinds of packing constraints such as capacity (upper quota), an intersection of multiple matroids, or a multi-dimensional knapsack. They considered the problems of checking existence of a feasible matching as well as a stable matching. The cardinal preference allows them to also explore approximately stable outcomes.\(^6\)

\(^6\)Note that the constraints involved in refugee matching markets are also based on multi-dimensional knapsacks.
8 Matroidal Constraints

In the classical stable matching problem, each institution has a capacity constraint. These constraints can be generalized to matroidal constraints. In addition to capacity constraints, matroids also capture more complex constraints including hierarchical or nested capacity constraints that are discussed in the section on regional constraints.

For a finite set of contracts $\mathcal{X}$, let $\mathcal{F}$ denote a family of subsets of $\mathcal{X}$, each of which is feasible according to the constraints. We say a pair $(\mathcal{X}, \mathcal{F})$ is a matroid if it satisfies the following conditions. (i) $\emptyset \in \mathcal{F}$ (ii) If $X' \in \mathcal{F}$ and $X'' \subseteq X'$, then $X'' \in \mathcal{F}$. (iii) If $X', X'' \in \mathcal{F}$ and $|X'| > |X''|$, then there exists some $x \in X' \setminus X''$ such that $X'' \cup \{x\} \in \mathcal{F}$. It is easy to verify that for the standard model, where only constraints are the upper quota of each institute, the family of feasible contracts related to each institute constitutes a matroid. Also, for the case of disjoint region, the family of feasible contracts related to each region constitutes a matroid. Matroidal constraints have been examined in a series of papers (see, e.g., Fleiner [2001], Fleiner and Kamiyama [2016], Kamiyama [2019a], Kamiyama [2019b], [Kamiyama, 2020a]). Kojima [2019] discussed that if the feasibility constraints are matroidal and if there is a total order on the individuals, then the greedy algorithm of selecting agents while not violating feasibility constraints gives rise to a choice function that satisfies the SUB condition.

Note the matroidal feasibility constraints require that the empty set must be feasible. Hence, feasibility constraints that involve lower bounds on the quotas cannot be directly captured by matroidal constraints. However, both Fleiner and Kamiyama [2016] and Kojoi [2017] consider a generalized approach based on matroids that captures stability and fairness, respectively, for lower quotas with matroid feasibility constraints.

Kojima et al. [2018] showed that (i) Algorithm 1 is strategyproof for agents, (ii) the resulting matching is stable (according to the definition in Section 3) and optimal for each agent among all stable matchings, and (iii) the time complexity of the algorithm proportional to the square of the number of possible contracts, assuming $Ch_M$ is defined based on matroidal constraints (as well as institutions’ aggregated preferences have some simple structures, e.g., maximizing the sum of values associated with individual contracts).

Kojima et al. [2018] showed that a wide variety of constraints, including nested regions [Biró et al., 2010; Goto et al., 2016], diversity requirements in school choice with soft constraints [Ehlers et al., 2014b], the student-project allocation problem [Abraham et al., 2007], and the cadet-branch matching problem [Sonmez and Switzer, 2013], can be represented as matroidal constraints and existing mechanisms/algorithm that had been developed independently/separately can be represented as a unified mechanism/algorithm described as Algorithm 1.

9 Heredity Constraints

In this section, we consider feasibility constraints that satisfy the heredity property. The heredity property requires that if a matching is feasible then it remains feasible if the numbers of agents matched to each institution weakly decreases. Heredity constraints are more general than imposing upper bounds on regions/subsets of institutions. They are also more general than matroidal constraints. However, they do not capture certain types of diversity constraints. Heredity constraints have been considered in some recent papers [Kamada and Kojima, 2017b; Goto et al., 2017; Aziz et al., 2020; Kamada and Kojima, 2020].

Adaptive Deferred Acceptance

Goto et al. [2017] proposed a general algorithm called Adaptive Deferred Acceptance (ADA) that can be applied to any heredity constraints. This algorithm satisfies strategyproofness, non-wastefulness, and a fairness property (that is weaker than the weak stability concept of Kamada and Kojima [2017b]). Next, we provide some high level ideas behind ADA. ADA utilizes a serial order among agents called mater-list. ADA works by repeatedly calling the standard Deferred Acceptance algorithm by adding agents one by one according to the master-list. In ADA, an institution becomes forbidden if it cannot accommodate any additional agent without some feasibility constraint being violated. In each stage, there are multiple rounds, each adding students according to the master-list as long as no institution becomes forbidden. When an institution becomes forbidden, the algorithm finalizes the current matching and moved to the next stage where the upper quotas and the distributional constraints are updated by taking into account of all the contracts that have been finalized so far. The formal description of ADA is given as Algorithm 2.

Algorithm 2 Adaptive Deferred Acceptance (ADA)

Input: master-list $L = (i_1, i_2, \ldots)$ and upper quotas $q_H$

Output: matching $Y$

Initialization: $q_h^1 \leftarrow q_h$ for each $h \in H$, $Y \leftarrow \emptyset$. Proceed to Stage 1.

Stage $k$: Proceed to Round 1.

Round $t$ Select $t$ agents from the top of $L$. Let $Y'$ denote the matching obtained by the standard deferred acceptance for the selected agents under $(q_h^k)_{h \in H}$.

(i) If all agents in $L$ are already selected, then $Y \leftarrow Y \cup Y'$, output $Y$ and terminate the mechanism.

(ii) If no institute is forbidden, then proceed to Round $t + 1$.

(iii) Otherwise, $Y \leftarrow Y \cup Y'$. Remove $t$ agents from the top of $L$. For each institute $h$ that is forbidden, set $q_h^{k+1} \leftarrow q_h^k - |Y'_h|$. Proceed to Stage $k+1$.

Cutoff Stable Algorithm

Aziz et al. [2021] showed that as long as the feasibility constraints satisfy the heredity property, cutoff stability (a natural notion of stability that is stronger than weak stability of Kamada and Kojima [2017b]) is satisfied. The idea behind cutoff stability is as follows. Cutoff stability requires that fairness as well as what is called cutoff non-wastefulness are satisfied. Cutoff non-wastefulness requires that either an agent cannot leave her match and join a more
preferred institution $h$ without violating some feasibility constraint or if it can, then there exists another agent that is at higher priority at $h$ that cannot leave her match and join $h$ without violating some feasibility constraint. Aziz et al. [2021] presented a simple but widely applicable algorithm that works for any matching problem with feasibility constraints. If the constraints satisfy the heredity property, the algorithm returns a cutoff stable outcome.

Let $d: H \rightarrow [0, 1, \ldots, |N| + 1]$ be the cutoff score function, where $d(h)$ is the cutoff at institution $h$. Without loss of generality we assume that each institution $h$ assigns a score to each agent $i$ in accordance with her preference list, that is $i$ has score $|N| - k + 1$ if she is ranked $k$-th by institution $h$. Given cutoff scores $d$, we say that agent $i$ is admissible to institution $h$ if her score achieves the cutoff. Cutoff scores $d$ induce matching $M$, if every agent is matched to the best institution of her preference where she is admissible.

Let $d^{-h}$ denote the cutoff scores after decreasing the cutoff of $h$ by one, and keeping the other cutoffs the same, i.e., $d^{-h}(h) = d(h) - 1$, and $d^{-h}(h') = d(h')$ for every $h' \neq h$. We say that cutoffs $d$ are minimal if we cannot decrease the cutoff score of any institution without making the induced matching infeasible. More formally for every institution $h$, either $d(h) = 0$ or the matching induced by $d^{-h}$, which we call $M^{-h}$, is infeasible. The formal definition of the algorithm presented by Aziz et al. [2021] is given as Algorithm 3.

**Algorithm 3** Cutoff algorithm for heredity constraints.

**Input:** lists $\succ_h$ for all $h \in H$ and $\succ_i$ for all $i \in N$; feasibility function $f$; institution order $P^* = (h_1, \ldots, h_k)$

**Output:** Matching $M$ and corresponding cutoffs $d_M$

1. Initialize $M$ to empty and $d_M(h) = |N| + 1$ for every institution $h$.
2. while Cutoff $d_M$ are not minimal do
3. Locate the first $h_j$ in the list $P^*$ such that $M^{-h_j}$ is feasible.
4. Let $M = M^{-h_j}$ and $d_M = d_M^{-h_j}$.
5. end while

**Intra-Institution Heredity Constraints**

Kamada and Kojima [2020] considered a more restricted version of heredity constraints that are *intra-institution* and showed that an outcome that satisfies the following properties exists: fairness, feasibility, agent-optimal among all matchings that satisfy fairness and feasibility. The intra-institution heredity constraints are still more general than refugee matching constraints.

10 Conclusions and Discussion

We surveyed recent work on two-sided matching with constraints. We focused on the work that takes stability and fairness as central concerns. In recent year, work on other objectives such as popularity has also been extended to more complex feasibility constraints (see, e.g., Kamiyama [2020b]).

We restricted our presentation to many-to-one matching problems in which each agent is matched to at most one institutions. There are many results in the literature that pertain to many-to-many matchings, a model motivated by several relevant applications such as the resident allocation problem with couples [Biró and Klijn, 2013], the assignment of papers to reviewers [Garg et al., 2010], and course allocation [Budish et al., 2017].

In this survey, we have focused on two-sided matching problems with fairness concerns. Another stream of work is on exchange problems, a basic version of which is the housing markets. Abdulkadiroğlu and Sönmez [1999] and Guillen and Kesten [2012] considered a housing market problem with existing tenants, where some agents initially own their houses, while the others do not, and some houses are not initially owned by any agent. In such a case, we require individual rationality, i.e., each agent must be assigned to a weakly better house than her initial endowment. Recent works considered a more general setting with multiple identical objects (i.e., multiple identical seats of an institution) and various distributional constraints. Hamada et al. [2017] developed two strategyproof mechanisms, one is based on Top Trading Cycles mechanism [Shapley and Scarf, 1974], and the other one is based on Deferred Acceptance, when regional lower/upper quotas are imposed. Suzaki et al. [2018] dealt with more general matroidal constraints.

When agents have cardinal preferences and institutions do not have priorities, market-based mechanisms à la the mechanism of Hylland and Zeckhauser [1979] are desirable. In recent years, such approaches have been extended to a rich class of constraints [Echenique et al., 2019].

Another perspective that we did not cover in detail is that of treating stability itself as a feasibility constraint. There is a growing line of work on this optimization approach, for instance, on college admissions with special features [Ágoston et al., 2016], and stable project allocation under distributional constraints [Ágoston et al., 2018]. For example, Aziz and Brandl [2020] generalized probabilistic serial (a desirable rule for matching problems without priorities [Bogomolnaia and Moulin, 2001]) to any class of closed constraints. In particular, they showed that their generalization leads to compelling rules for two-sided probabilistic matching in which priorities are taken into account, and stability requirements are imposed as feasibility constraints. There is also a stream of papers that consider constraints for which a random assignment can be realized by a probability distribution of deterministic assignments satisfying the constraints [Budish et al., 2013; Akbarpour and Nikzad, 2020].

References


