Portioning Using Ordinal Preferences: Fairness and Efficiency

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A divisible public resource is to be divided among projects. We study rules that decide on a distribution of the budget when voters have ordinal preference rankings over projects. Examples of such portioning problems are participatory budgeting, time shares, and parliament elections. We introduce a family of rules for portioning, inspired by positional scoring rules. Rules in this family are given by a scoring vector (such as plurality or Borda) associating a positive value with each rank in a vote, and an aggregation function such as leximin or the Nash product. Our family contains well-studied rules, but most are new. We discuss computational and normative properties of our rules. We focus on fairness, and introduce the SD-core, a group fairness notion. Our Nash rules are in the SD-core, and the leximin rules satisfy individual fairness properties. Both are Pareto-efficient.

1. Introduction

The members of an organization need to divide its budget among several projects. They have different opinions about the relative value of different projects, and would like to *vote* over the budget. What kind of voting rule could they use?

Some cities let citizens vote over the use of the city budget, giving rise to *participatory budgeting* (see Aziz and Shah [2021], De Vries et al. [2021]). In deployed applications (such as in Paris, Madrid, and Warsaw), the projects are *indivisible*, and can be either fully funded or not at all, such as refurbishing a school or adding a bike lane. We focus on *divisible* projects on which an arbitrary fraction of the budget could be spent, such as 'education' or 'transport' or 'parks'. The result of the vote can be visualized as a pie chart showing which percentage of the budget is spent on each expense. The 'budget' need not be monetary, and we refer to this general scheme as *portioning*. There are many applications:

- A conference board deciding how much time to assign to talks, poster sessions, invited talks, and coffee breaks.
- Coauthors deciding how much space to devote to various topics in a textbook or article with fixed total length.
- A parliamentary election deciding what percentage of parliament seats should go to each party (see Section 7).
- A company deciding which charities to direct its annual donations to, letting employees vote over which charities should receive a donation.

We ask voters to report their preferences over projects as *rankings*, the most common format considered in social choice. If a project is ranked more highly, the voter thinks it is more worthwhile and should receive a larger fraction of the budget. The rules we study in this paper make an important assumption: voters' preferences over budget allocations are implicitly interpreted to be *separable*. Thus, a voter ranking project a above project b is assumed to prefer giving a larger share to a than to b in all possible contexts. In particular, voters cannot express complementarity or supplementarity between projects, nor that they want the share of a to be 50%, or that projects a and b should ideally receive the same share. Of the four example applications we mentioned, separability is plausible for the latter two, but less plausible for the first two (since people typically want a varied conference program, and textbooks about more than one topic).[1]

The space of sensible aggregation rules is large, so let us illustrate some important design considerations by an example.

An Example A company wants to decide about its annual donation to charities. Its five employees have different preferences about which ones should receive money. The charities under consideration are a, b, c, d, e. Frances, George and Helena all think $a \succ b \succ c \succ d \succ e$; Ingrid thinks $e \succ b \succ c \succ d \succ a$; and John thinks $c \succ a \succ e \succ d \succ b$.

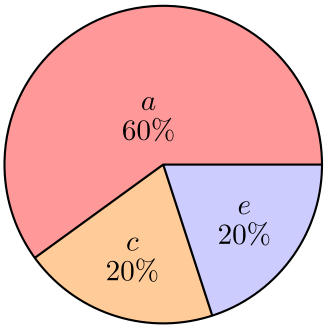

Figure 1: Random Dictator

One simple way to split the money is to allocate each person the same share of the total amount of the donation (20%) and let them decide which charity it should be given to (see Figure 1). To the social choice theorist, this rule sounds familiar: it is formally identical to *Random Dictatorship*, whose output is usually seen not as a division of a budget, but as a probability distribution. Indeed, any probabilistic social choice function can be repurposed to divide budgets; but these are often not attractive for portioning since many of them were designed as tie-breaking devices.

The output of random dictatorship can be a good choice, especially if our five employees strongly prefer their top choice to any other charity. But it is also plausible that Frances, George, Helena and Ingrid agree that b is good common ground. Random Dictatorship, using plurality scores, ignores this (plurality scores assign one point to the best alternative and zero to the other ones). Instead, we could impute *Borda scores* on our company. For m alternatives, Borda scores assign $m-1$ points to the best alternative, $m-2$ to the second best, and so on. For example, Frances, George, and Helena give 4 'utility' points to a, 3 to b, 2 to c, 1 to d, and 0 to e. *Proportional Borda* then allocates money in proportion to the total Borda score of the charities (see Figure 2). This leads to a significant money share for b. On the other hand, the company now also donates to d, which is dominated: everyone agrees that c is better than d!

To be more efficient, it makes sense to maximize a notion of social welfare. Suppose the utility enjoyed by a company employee is the weighted average of the Borda scores of the charity the company donates to, where the weights come from the fraction of money spent on each charity. It

[1] If voters prefer mixtures, this could be accommodated by including projects that are themselves mixtures, so that the outcome is a "mixture of mixtures", as discussed by Brandl and Brandt [2020, p. 802].

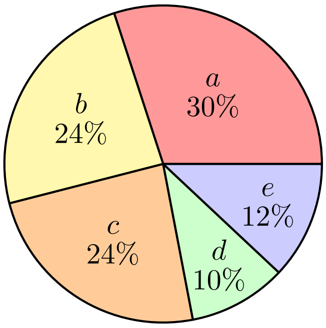

Figure 2: Proportional Borda

remains to decide how to aggregate these utilities. *Utilitarianism* picks a distribution where the *sum* of utilities is greatest. In our example, *Borda-utilitarianism* — resulting from the choice of the Borda scoring vector and utilitarianism — gives 100% of the donation to a (see Figure 3).

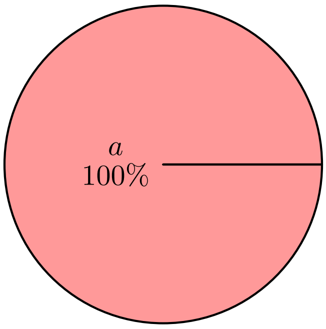

Figure 3: Borda-utilitarianism

This is Pareto-efficient: no other distribution is better for all agents. But it is unfair to Ingrid, who sees all the funding to her least-preferred choice. Many rules suffer from this phenomenon of overriding some voters' preferences: For example, the 'maximal lotteries' rule (see the related work section) also only funds a since it is the *Condorcet winner*, that is, for any other candidate x, a majority of voters prefer a to x.

To avoid frustration, we may take a more egalitarian approach, and aim to give every company employee a significant share. *Egalitarianism* consists in picking the distribution maximizing the utility of the worst-off employee. In our example, *Borda-egalitarianism* — resulting from the choice of the Borda scoring vector and egalitarianism — gives every employee an average Borda-utility of 2.4. The outcome is represented in Figure 4.

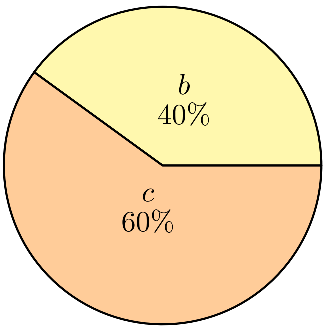

Figure 4: Borda-egalitarianism

We can also maximize *Nash social welfare*, the product of utilities. This is often seen as a compromise between maximizing utilitarian and egalitarian welfare notions. While egalitarian rules perform well when we wish to be fair to each *individual*, Nash rules tend to be fair to *groups*. In our example, Frances, George and Helena form a large group with the same preferences, liking a most, and accordingly Borda-Nash gives almost half the budget to a (see Figure 5).

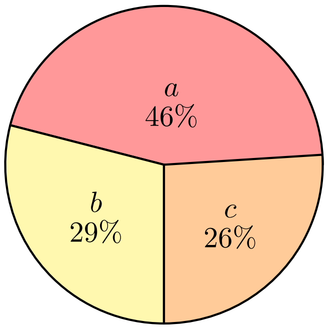

Figure 5: Borda-Nash

If there were more than those 3 employees with the same preferences in the company, Borda-Nash would increase the budget share of a. In contrast, Borda-egalitarianism avoids funding a to benefit Ingrid (who ranks a last), and the output of egalitarian rules does not change with the number of employees with identical preferences. Depending on the context, either of these behaviors might be more appropriate.

Our Contributions We introduce a class of aggregation rules called *positional social decision schemes*. Rules in this class first convert each input ranking into scores for the alternatives, using a scheme such as plurality or Borda scores. Then, these scores are lifted and used to score distributions (using a weighted average). Finally, the rules select a distribution of the budget maximizing social welfare given those scores, where different notions of welfare can be used; classically, we consider utilitarian, egalitarian (leximin), and Nash welfare. Our class contains known rules such as random dictatorship, but most have not been studied.

We begin by noting basic properties of the rules in our class, giving closed forms and equivalent definitions in some cases. We also show that the rules in this class can be calculated or approximated in polynomial time. For rules based on Nash welfare, we show that their output can involve irrational percentages; we prove that those rules are guaranteed to be rational if the scoring vector used is plurality or veto, but that no other scoring vector guarantees rational output.

We then formalize intuitive notions of fairness in the budgeting context. The axioms we propose require that procedures do not ignore individuals: every voter should have at least some of the budget allocated to favored causes. We also give some group fairness notions. Our strongest axiom is the *SD-core* which, roughly, requires that a group of $\alpha\%$ of the voters can control what happens with $\alpha\%$ of the budget. We show that the rules in our class based on Nash welfare satisfy the SD-core, while the egalitarian rules satisfy the individual fairness notions.

We study the performance of our rules on standard social choice properties, such as Pareto-efficiency, strategyproofness, monotonicity, and participation. In the conclusion, a table summarizes which properties are satisfied by the various rules.

We close by giving a detailed example of applying our rules to the problem of allocating seats in a parliament under a party-list system.

Related Work Bogomolnaia et al. [2005] introduced the portioning problem, motivated by time-sharing. They assume *dichotomous preferences*, and agents report a subset of the alternatives (an approval vote), rather than rankings. They study the compatibility of Pareto-efficiency and strategyproofness, with positive results (for example, spending the entire budget on the approval winner satisfies both requirements). However, after adding a fairness axiom, they get an impossibility result. Related impossibilities are proved by Duddy [2015] and Brandl et al. [2021]. Aziz et al. [2020] introduce some new rules based on welfare maximization, and introduce new fairness axioms (including a core notion), and a weakened strategyproofness axiom. Guerdjikova and Nehring [2014] give an axiomatic characterization of the Nash product rule. Michorzewski et al. [2020] study utilitarian welfare guarantees of various rules in this setting. Brandl et al. [2022] study cases when the budget is owned by the voters. Freeman et al. [2021] consider portioning when each agent has a preferred ideal distribution and wants the chosen distribution to be close to the ideal.

Fain et al. [2016] study portioning in a *cardinal* model which allows agents to give a full utility function over alternatives (which may also feature decreasing returns). They study the core and connect it to the Lindahl equilibrium from the study of public goods, and prove that a core solution always exists. For a broad class of utility functions, they show that a core solution can be found in polynomial time by solving a suitable convex program. They also use differential privacy to design a mechanism for this setting which satisfies approximate versions of efficiency, truthfulness, and the core.

With rankings as input, this setting has been studied in the formally isomorphic guise of *probabilistic social choice* [see Brandt, 2019 for a recent survey]. In this literature, the outcome distribution is interpreted as a random device, which is used to eventually implement a single outcome. This makes notions of fairness and proportionality less relevant, and it is seen as desirable for a rule to randomize as little as possible. For example, the *maximal lotteries* rule [Kreweras, 1965, Brandl et al., 2016], while attractive according to consistency axioms, spends the entire budget on the Condorcet winner if it exists. This is often undesirable in a budgeting context. Some papers on probabilistic social choice also discuss fairness concerns and the portioning application [see, e.g., Aziz et al., 2018c, Aziz and Stursberg, 2014].

We can view positional decision schemes as choosing a utility function for each voter and then performing welfare maximization for different social welfare functions. Following the literature on implicit utilitarian voting [Procaccia and Rosenschein, 2006, Boutilier et al., 2015], Ebadian et al. [2022] take a different view. They assume that each voter has a true utility function that is unknown to the voting rule but that is compatible with the reported ranking. The goal is to find a portioning that approximately maximizes social welfare, no matter what the true utilities turn out to be. For utilitarian welfare, they show that an $O(\sqrt{m})$-approximation is possible, and for Nash welfare, they show that an $O(\log m)$-approximation is possible, where m is the number of alternatives. They also consider a notion of the core that is related to our SD-core; we could call their notion the *strong* SD-core. A distribution fails to be in the (weak) SD-core if a coalition of voters has a deviation that they all prefer for all possible utility functions compatible with the reported rankings. The latter is a difficult condition to satisfy, making the SD-core a somewhat weak property. For Ebadian et al. [2022], a distribution fails to be in the strong SD-core if a coalition of voters has a deviation that they prefer with respect to *some* compatible utility functions. This is so strong that no distribution can be in the SD-core, except for trivial profiles. Thus, Ebadian et al. [2022] study an approximation they call the α-core, and show that there always exists a distribution satisfying it for $\alpha = O(\log m)$.

In a related setting, mirroring how many cities do participatory budgeting, projects are indivisible and come with a fixed cost; they can either be fully funded or not at all [Goel et al., 2019, Benade et al., 2021, Aziz and Shah, 2021]. Several recent papers have studied fairness in this setting [Aziz et al., 2018b, Fain et al., 2018, Conitzer et al., 2017, Peters et al., 2021, Fairstein et al., 2022, Los et al., 2022]. It is related to *multi-winner elections* (which can be seen as the special case where all projects have the same cost), for which fairness and proportionality are well-studied [Aziz et al., 2017, Faliszewski et al., 2017, Lackner and Skowron, 2022]. The setting is also known as *combinatorial public projects* [Papadimitriou et al., 2008].

The literature on cake-cutting and item allocation is mostly *unrelated* to our work: in those settings, goods are allocated to agents for their exclusive use. In our setting, resources are spent on projects which can be enjoyed by all agents. On a technical level, the idea of scoring followed by aggregation has been explored in fair division [Brams and King, 2005, Darmann and Schauer, 2015, Baumeister et al., 2016], and work on the fair allocation of goods to several groups of agents raises similar issues [see, e.g., Manurangsi and Suksompong, 2019; Bade and Segal-Halevi, 2018; Segal-Halevi and Suksompong, 2019].

2. Positional social decision schemes

Let $X = \{x_1, \dots, x_m\}$ be a set of alternatives and $N = \{1, \dots, n\}$ be a set of voters. Let $\mathcal{L}(X)$ be the set of linear orders over X. For $\succ \in \mathcal{L}(X)$, the *rank* of alternative x_j is $r(\succ, x_j) = 1 + |\{x_i \in X : x_i \succ x_j\}|$. For example, if x_j is the top-ranked alternative in $\succ$ then $r(\succ, x_j) = 1$. A *profile* $P = (\succ_1, \dots, \succ_n) \in \mathcal{L}(X)^n$ is a collection of linear orders, one for each voter. We write abc as shorthand for $a \succ b \succ c$. Let $\Delta(X) = \{p : X \to [0,1] : \sum_{x \in X} p_x = 1\}$ be the set of (probability) distributions over X. We use notation like $\frac{1}{2}x_1 + \frac{1}{2}x_2$ to specify a distribution, and write x_j for the distribution with $p_{x_j} = 1$. Given a distribution p, for brevity we sometimes write p_j for p_{x_j}. We say that $z : X \to [0,1]$ is a *partial distribution* if $\sum_{x \in X} z_x \leqslant 1$. A *social decision scheme* (SDS) [Gibbard, 1977] is a function F assigning to each $P \in \mathcal{L}(X)^n$ a nonempty subset of $\Delta(X)$ of selected distributions. Usually, this set is a singleton, but there can be several tied distributions.

A *scoring vector* for m alternatives is a vector $\mathbf{s} = (s_1, \dots, s_m)$ of numbers with $s_1 \geqslant s_2 \geqslant \dots \geqslant s_m$ and $s_1 > s_m$. We usually assume $s_m = 0$. A scoring vector $\mathbf{s}$ is *strictly decreasing* if $s_j > s_{j+1}$ for all $j < m$. The Borda vector is $\mathbf{s}_{\text{Borda}} = (m-1, m-2, \dots, 0)$; the plurality vector is $\mathbf{s}_{\text{plurality}} = (1, 0, \dots, 0)$; the veto (or antiplurality) vector is $\mathbf{s}_{\text{veto}} = (1, \dots, 1, 0)$.

For a fixed profile P, we write $\mathbf{s}[i,j] = s_{r(\succ_i, x_j)}$ for the $\mathbf{s}$-score that voter $i \in N$ assigns to alternative $x_j \in X$. These scores can be lifted to (partial) distributions by taking a weighted average (i.e., taking the expected score): We say that the $\mathbf{s}$-score of a distribution p for i is $\mathbf{s}[i,p] = \sum_{j=1}^{m} p_j \mathbf{s}[i,j]$. Finally, define the *utility vector* $\mathbf{s}[p] = (\mathbf{s}[1,p], \dots, \mathbf{s}[n,p])$.

A *welfare ordering* is a weak order $\geqslant_W$ of utility vectors $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{\geqslant 0}$. We denote by $>_W$ the strict part of $\geqslant_W$. The most common choices for $\geqslant_W$ are:

- *utilitarianism*, which orders vectors by their sum;
- *egalitarianism*, which orders vectors by their minimum;
- *leximin*, which sorts the components of the utility vector increasingly and then orders sorted vectors lexicographically;
- the *Nash product* [Nash, 1950], which orders vectors by their product.

By combining a scoring vector and a welfare ordering, we can define a positional social decision scheme.

Definition 1 (Positional Social Decision Schemes)**.** *For a scoring vector* $\mathbf{s}$ *and a welfare ordering* $\geqslant_W$*, define the social decision scheme* $F_{\mathbf{s},\geqslant_W}$ *so that for all profiles* P*,*

$$F_{\mathbf{s},\geqslant_W}(P) = \{p \in \Delta(X) : \mathbf{s}[p] \geqslant_W \mathbf{s}[q] \textit{ for all } q \in \Delta(X)\}.$$

For the specific $\geqslant_W$ mentioned, we usually call these rules $\mathbf{s}$-utilitarianism, $\mathbf{s}$-egalitarianism, $\mathbf{s}$-leximin, and $\mathbf{s}$-Nash.

Example 1 (The case of two alternatives). Consider the profile $P = (ab, ab, ba)$ over two alternatives, with $\mathbf{s} = (1, 0)$. For each $r \in [0,1]$, consider the distribution $p^{(r)} = ra + (1-r)b$, whose associated utility vector is $(r, r, 1-r)$.

- $\mathbf{s}$-utilitarianism selects the distribution $p^{(r)}$ maximizing $\sum_{i=1}^{n} \mathbf{s}[i, p^{(r)}] = 1 + r$, which is the deterministic distribution a.

- $\mathbf{s}$-egalitarianism selects the distribution $p^{(r)}$ maximizing $\min_{i=1\dots n} \mathbf{s}[i, p^{(r)}] = \min(r, 1-r)$, which is $\frac{1}{2}a + \frac{1}{2}b$. The same distribution is selected by $\mathbf{s}$-leximin.

- $\mathbf{s}$-Nash selects the distribution $p^{(r)}$ maximizing $\prod_{i=1}^{n} \mathbf{s}[i, p^{(r)}] = r^2(1-r)$, which is $\frac{2}{3}a + \frac{1}{3}b$.

More generally, if a profile consists of q votes for ab and $n-q$ votes for ba, then: $\mathbf{s}$-utilitarianism selects a if $q > n-q$ and b if $q < n-q$, and selects all possible distributions if $q = n-q$; $\mathbf{s}$-egalitarianism and $\mathbf{s}$-leximin select $\frac{1}{2}a+\frac{1}{2}b$ whenever $1 \leqslant q < n$; and $\mathbf{s}$-Nash selects $\frac{q}{n}a+(1-\frac{q}{n})b$. $\square$

For normative analysis, it is useful to extend voters' rankings of the alternatives to (partial) preferences over distributions. We assume *linear preferences*: there is an unknown utility function $u_i : X \to \mathbb{R}$ consistent with $\succ_i$ such that i prefers those distributions p with higher expected utility $\sum_{x \in X} u_i(x) p_x$. A classical way of ranking distributions despite not knowing u_i uses *stochastic dominance* (SD): If p and q are (possibly partial) distributions, we write

$$p \succcurlyeq_i^{\mathrm{SD}} q \iff \textstyle\sum_{x_k \succ_i x_j} p_{x_k} \geqslant \sum_{x_k \succ_i x_j} q_{x_k} \text{ for all } x_j \in X.$$

As usual, we then write $p \succ_i^{\mathrm{SD}} q$ if $p \succcurlyeq_i^{\mathrm{SD}} q$ but $q \not\succcurlyeq_i^{\mathrm{SD}} p$, and $p \sim_i^{\mathrm{SD}} q$ if $p \succcurlyeq_i^{\mathrm{SD}} q$ and $q \succcurlyeq_i^{\mathrm{SD}} p$.

If we label alternatives such that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m$, we can equivalently write this definition as

$$p \succcurlyeq_i^{\mathrm{SD}} q \iff \begin{cases} p_1 \geqslant q_1, \text{ and} \\ p_1 + p_2 \geqslant q_1 + q_2, \text{ and} \\ \vdots \\ p_1 + \cdots + p_{m-1} \geqslant q_1 + \cdots + q_{m-1}. \end{cases}$$

Note carefully that our definition does not include the inequality $p_1 + \cdots + p_m \geqslant q_1 + \cdots + q_m$, which usually is part of the definition of SD. This inequality is trivially satisfied if both p and q are distributions summing to 1. But since we omit this inequality, we can SD-compare partial distributions. According to our definition of SD, each agent interprets a partial distribution z with $\sum_{x \in X} z_x = \alpha < 1$ as if it was the full distribution $z + (1-\alpha)x_m$, which places the remaining probability mass $1-\alpha$ on the agent's least-preferred alternative. This extension to partial distributions will be crucial for the definition of the SD-core in Section 4. Here are some examples for an agent with $x_1 \succ_i x_2 \succ_i x_3 \succ_i x_4$:

$$(0.8, 0.1, 0.1, 0) \succ_i^{\mathrm{SD}} (0.7, 0.2, 0.1, 0) \succ_i^{\mathrm{SD}} (0.7, 0.2, 0, 0.1) \succ_i^{\mathrm{SD}} (0.7, 0, 0, 0) \sim_i^{\mathrm{SD}} (0.7, 0, 0, 0.3).$$

The definition of SD-preference is interesting due to the following standard equivalence [Bogomolnaia and Moulin, 2001]: distribution p is weakly SD-preferred to q by voter i if and only if p gives i a weakly higher $\mathbf{s}$-score than does q, for all scoring vectors $\mathbf{s}$ with $s_m = 0$. (The condition that $s_m = 0$ is necessary to implement our way of doing SD-comparisons of partial distributions: $s_m = 0$ implies that voters are indifferent between not spending part of the budget or spending it on their worst alternative.) The equivalence is formalized in the following proposition.

Proposition 1. *Let p and q be (possibly partial) distributions.*

(a) *$p \succcurlyeq_i^{\mathrm{SD}} q$ if and only if $\mathbf{s}[i,p] \geqslant \mathbf{s}[i,q]$ for all scoring vectors $\mathbf{s}$ with $s_m = 0$.*

(b) *If $p \succ_i^{\mathrm{SD}} q$ then for each strictly decreasing scoring vector $\mathbf{s}$ with $s_m = 0$, we have $\mathbf{s}[i,p] > \mathbf{s}[i,q]$.*

Proof. Label alternatives such that $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m$.

Suppose $p \succcurlyeq_i^{\mathrm{SD}} q$, and let $\mathbf{s}$ be a (weakly decreasing) scoring vector with $s_m = 0$. Since $p \succcurlyeq_i^{\mathrm{SD}} q$, we have $p_1 \geqslant q_1$, $p_1 + p_2 \geqslant q_1 + q_2$, $\dots$, $p_1 + \cdots + p_{m-1} \geqslant q_1 + \cdots + q_{m-1}$. Now

$$\begin{aligned}\mathbf{s}[i,p] - \mathbf{s}[i,q] &= \sum_{j=1}^{m-1} s_j(p_j - q_j) \\ &= \sum_{j=1}^{m-1} s_j((p_1 + \cdots + p_j) - (q_1 + \cdots + q_j)) - s_j((p_1 + \cdots + p_{j-1}) - (q_1 + \cdots + q_{j-1})) \\ &= \sum_{j=1}^{m-1} (s_j - s_{j+1})((p_1 + \cdots + p_j) - (q_1 + \cdots + q_j)) \geqslant 0.\end{aligned}$$

Here, the third equality uses that $s_m = 0$. The inequality uses that $s_j - s_{j+1} \geqslant 0$ since $\mathbf{s}$ is weakly decreasing. Hence, $\mathbf{s}[i,p] \geqslant \mathbf{s}[i,q]$, showing one direction of (a). We can use the same calculation to show (b): the last sum in the calculation is strictly positive because $(s_j - s_{j+1})$ is positive by the assumption that $\mathbf{s}$ is strictly decreasing, and because $p \succ_i^{\mathrm{SD}} q$, at least one of the terms of the sum is strictly positive. For the converse direction of (a), assume $\mathbf{s}[i,p] \geqslant \mathbf{s}[i,q]$ for all scoring vectors $\mathbf{s}$ with $s_m = 0$. This applies in particular to the scoring vectors $\mathbf{s} = (1,0,\dots,0), (1,1,0,\dots,0), \dots, (1,1,\dots,1,0)$. Applying $\mathbf{s}[i,p] \geqslant \mathbf{s}[i,q]$ to each of these vectors gives $p_1 \geqslant q_1$, $p_1 + p_2 \geqslant q_1 + q_2$, $\dots$, $p_1 + \cdots + p_{m-1} \geqslant q_1 + \cdots + q_{m-1}$, that is, $p \succcurlyeq_i^{\mathrm{SD}} q$. $\square$

Thus, p is SD-preferred to q if p gives a higher expected score than q under all scoring vectors.

3. Computation and basic properties

In this section, we look at elementary properties of the family of rules we have defined. We will note that some of the rules are familiar from the probabilistic context. We also study the computational complexity of finding an optimal distribution.

3.1. Utilitarianism

From a utilitarian perspective, it is never worth it to spend part of the budget on alternatives whose total $\mathbf{s}$-score is not maximal: shifting that spending to an $\mathbf{s}$-maximal alternative increases utilitarian welfare. Thus, up to ties, $\mathbf{s}$-utilitarianism never mixes and spends all resources on an $\mathbf{s}$-winner. Formally, $\mathbf{s}$-utilitarianism selects those distributions p for which $p_{x_j} > 0$ only if the score $\sum_{i \in N} \mathbf{s}[i,j]$ is maximum.

Since the behavior of $\mathbf{s}$-utilitarianism is familiar from work on scoring rules in voting [see, e.g, Zwicker, 2016], we will not study it in much detail.

3.2. Egalitarianism

Plurality-egalitarianism is easy to understand: it returns the uniform distribution over all alternatives that are ranked top by at least one voter. In the probabilistic context, this rule coincides with the *egalitarian simultaneous reservation* rule [Aziz and Stursberg, 2014], designed more generally for preferences with possible indifferences. For other scoring vectors, $\mathbf{s}$-egalitarianism is less simple, and it need not return a uniform distribution (see the example of Section 1). However, one can easily evaluate $\mathbf{s}$-egalitarianism using linear programming:

$$\text{maximize } t^* \text{ s.t.} \quad \begin{array}{l} \sum_{j=1}^m \mathbf{s}[i,j] \cdot p_j \geqslant t^* \text{ for } i \in N \\ \sum_{j=1}^m p_j = 1, \text{ and } p_j \geqslant 0 \text{ for } x_j \in X \end{array}$$

Algorithm 1 Computing an $\mathbf{s}$-leximin distribution

Set $N' \leftarrow \emptyset$. For $i \in N$, we will set t_i once i is added to N'.

while $N' \neq N$ **do**

Using linear programming, find the maximum value t^* such that there is a distribution $(p_1, \dots, p_m)$ satisfying

$$\begin{array}{ll} \sum_{j=1}^m \mathbf{s}[i,j]p_j \geqslant t^* & \text{for } i \in N \setminus N' \\ \sum_{j=1}^m \mathbf{s}[i,j]p_j = t_i & \text{for } i \in N' \\ \sum_{j=1}^m p_j = 1 & \\ p_1, \dots, p_m \geqslant 0 & \end{array}$$

for each $i' \in N \setminus N'$ **do**

Using linear programming, find the maximum ε such that there is a distribution $(p_1, \dots, p_m)$ satisfying

$$\begin{array}{ll} \sum_{j=1}^m \mathbf{s}[i',j]p_j \geqslant t^* + \varepsilon & \\ \sum_{j=1}^m \mathbf{s}[i,j]p_j \geqslant t^* & \text{for } i \in N \setminus N' \\ \sum_{j=1}^m \mathbf{s}[i,j]p_j = t_i & \text{for } i \in N' \\ \sum_{j=1}^m p_j = 1 & \\ p_1, \dots, p_m \geqslant 0 & \end{array}$$

If $\varepsilon = 0$, add i' to N' and set $t_{i'} \leftarrow t^*$.

end for

end while

return the solution $(p_1, \dots, p_m)$ of the last linear program solved

Now, $\mathbf{s}$-egalitarianism is not very decisive, in the sense that many different distributions may maximize the $\mathbf{s}$-egalitarian objective. For example, when $P = (abcd, acbd, bdac)$, and $\mathbf{s} = (1, 1, 0, 0)$, it selects all distributions of the form

$$p \cdot a + q \cdot b + (\tfrac{1}{2} - p) \cdot c + (\tfrac{1}{2} - q) \cdot d$$

where $0 \leqslant p, q \leqslant \frac{1}{2}$ and $\frac{1}{2} \leqslant p + q \leqslant 1$. A standard way of making egalitarianism more decisive and more efficient is by using leximin instead. In the above example, $\mathbf{s}$-leximin uniquely selects $\frac{1}{2}a + \frac{1}{2}b$.

It is still possible to evaluate $\mathbf{s}$-leximin in polynomial time, by solving $O(n^2)$ linear programs successively. Our algorithm uses the convexity of the set $\Delta(X)$ of all distributions, which allows it to greedily fix the identity of the agent who is worst off in the current iteration. The algorithm is a special case of a scheme discussed by Nace and Orlin [2007], and is similar to algorithms used by Kurokawa et al. [2018].

Theorem 1. *For every* $\mathbf{s}$*, one can compute a distribution selected by* $\mathbf{s}$*-leximin in polynomial time.*

Proof. The algorithm is specified as Algorithm 1. It requires solving at most $n(n+1)/2$ linear programs.

First, we will prove by induction on $k \geqslant 1$ that at the start of the kth iteration of the while-loop, it is the case that every distribution p selected by $\mathbf{s}$-leximin satisfies $\mathbf{s}[i,p] = t_i$ for every $i \in N$. Second, we show that in each iteration, at least one agent gets added to N'. From the second claim it follows that the while-loop terminates after at most n iterations. From the first claim, it follows that upon termination, we know the values $\mathbf{s}[i,p]$ for every $i \in N$ that must be attained by an $\mathbf{s}$-leximin distribution. We can then compute one such distribution by linear programming (or as stated in the algorithm, just reuse the most recent optimal solution we have computed because it satisfies the exact same constraints).

The inductive hypothesis holds vacuously at the start of iteration $k = 1$. Let $k \geqslant 1$. Denote by N'_k and N'_{k+1} the sets N' at the start of the kth and the $(k+1)$th iteration. Note $N'_k \subseteq N'_{k+1}$. Let p be a distribution selected by $\mathbf{s}$-leximin. From the inductive hypothesis, $\mathbf{s}[i, p] = t_i$ for all $i \in N'_k$. Letting t^* be the optimum value of the first linear program solved in the kth iteration, we know that $\mathbf{s}[i, p] \geqslant t^*$ for all $i \in N \setminus N'_k$. Let $i' \in N'_{k+1} \setminus N'_k$ be an agent who is added to N' in the kth iteration. Note that the algorithm sets $t_{i'} = t^*$. We claim that $\mathbf{s}[i', p] = t^*$. We already know $\mathbf{s}[i', p] \geqslant t^*$. For a contradiction suppose $\mathbf{s}[i', p] > t^*$. Then in the iteration of the for-loop in which i' was added to N', the distribution p is a feasible solution to the linear program solved in that iteration. Because the solution has a strictly positive objective value, this is a contradiction to i' being added then. Hence $\mathbf{s}[i', p] = t^*$. Since our choice of p among distributions selected by $\mathbf{s}$-leximin was arbitrary, this shows that the inductive hypothesis holds at the start of iteration $k + 1$.

It remains to show that at least one agent is added to N' at each iteration. Suppose this is not the case for the kth iteration. Then the linear programs solved in each for-loop iteration have solutions with positive objective value. For each $i' \in N \setminus N'_k$, let $p_{(i')}$ be the distribution found in its for-loop; note $\mathbf{s}[i', p_{(i')}] > t^*$. Let p' be the average of all these distributions, i.e., $p' = \sum_{i' \in N \setminus N'_k} \frac{1}{|N \setminus N'_k|} p_{(i')}$. Then we have $\mathbf{s}[i, p'] = t_i$ for all $i \in N'_k$, and $\mathbf{s}[i, p'] > t^*$ for all $i \in N \setminus N'_k$. This contradicts t^* having been the optimum value of the first linear program solved in the kth iteration of the while-loop.

It is clear that Algorithm 1 proceeds in polynomial time, except that we need to prove that the values t_i have polynomial size. This is shown by Nace and Orlin [2007, Theorem 1]. □

Let us illustrate the ideas of Algorithm 1 using an example.

Example 2. Let $P = (abcd, acbd, bdac)$, and $\mathbf{s} = (1, 1, 0, 0)$. We begin by finding an $\mathbf{s}$-egalitarian distribution by solving the linear program

$$\begin{aligned} \max \quad & t^* \\ \text{s.t.} \quad & p_a + p_b \geqslant t^* \\ & p_a + p_c \geqslant t^* \\ & p_b + p_d \geqslant t^* \\ & p_a + p_b + p_c + p_d = 1 \\ & p_a, p_b, p_c, p_d \geqslant 0 \end{aligned}$$

with optimal solution $t^* = \frac{1}{2}$. Thus, the best we can guarantee is that every voter receives an $\mathbf{s}$-score of at least $\frac{1}{2}$. We now wish to identify voters who receive an $\mathbf{s}$-score of *exactly* $\frac{1}{2}$ in *every* $\mathbf{s}$-egalitarian distribution. To do so, we solve the three linear programs

$$\begin{aligned} \max \quad & \varepsilon \\ \text{s.t.} \quad & p_a + p_b \geqslant \tfrac{1}{2} + \varepsilon \\ & p_a + p_c \geqslant \tfrac{1}{2} \\ & p_b + p_d \geqslant \tfrac{1}{2} \\ & p_a + p_b + p_c + p_d = 1 \\ & p_a, p_b, p_c, p_d \geqslant 0 \end{aligned} \qquad \begin{aligned} \max \quad & \varepsilon \\ \text{s.t.} \quad & p_a + p_b \geqslant \tfrac{1}{2} \\ & p_a + p_c \geqslant \tfrac{1}{2} + \varepsilon \\ & p_b + p_d \geqslant \tfrac{1}{2} \\ & p_a + p_b + p_c + p_d = 1 \\ & p_a, p_b, p_c, p_d \geqslant 0 \end{aligned} \qquad \begin{aligned} \max \quad & \varepsilon \\ \text{s.t.} \quad & p_a + p_b \geqslant \tfrac{1}{2} \\ & p_a + p_c \geqslant \tfrac{1}{2} \\ & p_b + p_d \geqslant \tfrac{1}{2} + \varepsilon \\ & p_a + p_b + p_c + p_d = 1 \\ & p_a, p_b, p_c, p_d \geqslant 0 \end{aligned}$$

whose optimal solutions are, respectively, $\varepsilon = 1$, $\varepsilon = 0$ and $\varepsilon = 0$. It follows that voters 2 and 3 receive $\mathbf{s}$-score $\frac{1}{2}$ in all $\mathbf{s}$-egalitarian distributions. Thus, Algorithm 1 now sets $N' = \{2, 3\}$ and $t_2 = t_3 = \frac{1}{2}$.

To obtain the **s**-leximin distribution, we now need to maximize the **s**-score of agent 1. For this, we solve the linear program

$$\begin{aligned} \max \quad & t^* \\ \text{s.t.} \quad & p_a + p_b \geqslant t^* \\ & p_a + p_c = \tfrac{1}{2} \\ & p_b + p_d = \tfrac{1}{2} \\ & p_a + p_b + p_c + p_d = 1 \\ & p_a, p_b, p_c, p_d \geqslant 0 \end{aligned}$$

with optimal solution $t^* = 1$. The algorithm stops and outputs the distribution $\frac{1}{2}a + \frac{1}{2}b$. $\square$

3.3. Nash product

The defining optimization problem

$$\begin{aligned} \max \ & \textstyle\sum_{i \in N} \log\left(\sum_{j=1}^{m} \mathbf{s}[i,j] \cdot p_j\right) \\ \text{s.t.} \ & \textstyle\sum_{j=1}^{m} p_j = 1, \text{ and } p_j \geqslant 0 \text{ for } x_j \in X \end{aligned}$$

of **s**-Nash is a convex program that can be efficiently solved using standard convex programming solvers.[2] However, writing down the optimal distribution precisely, in decimal expansion, is impossible, as there are instances where **s**-Nash uniquely returns a distribution with irrational fractions. For instance, for $P = (abc, acb, cab, cab)$ and $\mathbf{s} = (2, 1, 0)$, **s**-Nash uniquely returns $\frac{1+\sqrt{33}}{8}a + \frac{7-\sqrt{33}}{8}c$. Still, one can approximate optimum Nash welfare up to an additive ε in time polynomial in n, m, and $1/\varepsilon$, for example using the ellipsoid method [see Grötschel et al., 1993, Sec. 4.2]. Thus, all the usual decision problems associated with computing **s**-Nash are easy.

We can analyze **s**-Nash using the first-order KKT conditions of the convex program. Let us first state a version of the KKT theorem.

Theorem 2 (e.g., Beck, 2014, Theorem 11.16)**.** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable convex function, and let $g_1, \dots, g_r : \mathbb{R}^n \to \mathbb{R}$ and $h_1, \dots, h_s : \mathbb{R}^n \to \mathbb{R}$ be affine functions. Suppose that $\mathbf{x}^* \in \mathbb{R}^n$ is an optimal solution to the following problem:*

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_j(\mathbf{x}) = 0 \quad \text{for } j = 1, \dots, r \\ & h_k(\mathbf{x}) \leqslant 0 \quad \text{for } k = 1, \dots, s \end{aligned}$$

Then there exist multipliers $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ and $\mu_1, \dots, \mu_s \geqslant 0$ such that for all $i = 1, \dots, m$,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) + \sum_{j=1}^{r} \lambda_j \frac{\partial g_j}{\partial x_i}(\mathbf{x}^*) + \sum_{k=1}^{s} \mu_k \frac{\partial h_k}{\partial x_i}(\mathbf{x}^*) = 0$$

and complementary slackness holds, that is, for all $k = 1, \dots, s$, we have

$$\mu_k h_k(\mathbf{x}^*) = 0.$$

Analyzing the KKT condition of the Nash optimization problem yields the following important inequality, which we will repeatedly use to establish properties of the **s**-Nash rule.

[2] An alternative method to compute **s**-Nash is via a *proportional response dynamic* that is known to converge to a Nash distribution [Cover, 1984, Brandl et al., 2022].

Proposition 2. *Let p be a distribution selected by* $\mathbf{s}$*-Nash. Then for every $j = 1, \dots, m$,*

$$n \geqslant \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p]}, \quad \textit{and if } p_j > 0 \textit{ then equality holds.} \tag{$*$}$$

Proof. Let p^* be an optimal distribution selected by $\mathbf{s}$-Nash. Because the quantity $\mathbf{s}[i, p^*]$ will appear in denominators, let us first note that $\mathbf{s}[i, p^*] > 0$ for all $i \in N$, since $\mathbf{s}[i, p^*] = 0$ for some i would imply $\prod_{i \in N} \mathbf{s}[i, p^*] = 0$, and for the uniform distribution p_{uniform} we have $\mathbf{s}[i, p_{\text{uniform}}] > 0$ for all i, contradicting the optimality of p^*.

Apply Theorem 2 with objective $f(p) = -\sum_{i \in N} \log(\sum_{j=1}^m \mathbf{s}[i,j] \cdot p_j)$, a single equality constraint $g_1(p) = \sum_{j=1}^m p_j - 1$, and with m inequality constraints $h_k(p) = -p_k$. Note that

$$\frac{\partial f}{\partial x_j}(p) = -\sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p]}.$$

Since p^* is an optimal solution, Theorem 2 implies that there are $\lambda = \lambda_1 \in \mathbb{R}$ and $\mu_1, \dots, \mu_m \geqslant 0$ such that for each $j = 1, \dots, m$, we have

$$-\sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p^*]} + \lambda - \mu_j = 0 \iff \lambda - \mu_j = \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p^*]}.$$

We have $\mu_j \geqslant 0$, and by complementary slackness, if $p_j^* > 0$ then $\mu_j = 0$. Thus for every $j = 1, \dots, m$, we have

$$\lambda \geqslant \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p^*]}, \text{ with equality if } p_j^* > 0. \tag{1}$$

Multiplying both sides by p_j^* then gives $\lambda p_j^* = \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p^*]} p_j^*$ (no matter whether p_j^* is positive or zero). Summing over all j, thus

$$\lambda = \lambda(p_1^* + \dots + p_m^*) = \textstyle\sum_{j=1}^m \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p^*]} p_j^* = n,$$

since $\mathbf{s}[i, p^*] = \sum_{j=1}^m \mathbf{s}[i,j] \cdot p_j^*$ by definition. Plugging $\lambda = n$ into (1) gives the desired result. □

Using ($*$), we can characterize plurality-Nash [see also Moulin, 2003, Example 3.6]:

Theorem 3. *Plurality-Nash selects p with $p_j = \mathrm{pl}(x_j)/n$ for all j, where $\mathrm{pl}(x_j)$ is the number of voters placing x_j top.*

Proof. Let p be optimal for plurality-Nash. If some voter i puts x_j top then $p_j > 0$, or else $\mathbf{s}[i, p] = 0$ and the Nash product equals 0. By ($*$), we get $n = \sum_{i \in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p]} = \mathrm{pl}(x_j)/p_j$, and so $p_j = \mathrm{pl}(x_j)/n$. It follows that $p_j = 0$ whenever no voter places x_j top. □

Thus, we see that plurality-Nash is the same rule as *random dictatorship*, familiar from the probabilistic context.

The *veto-Nash* rule seems sensible when alternatives are nuisances, where each agent wants to minimize the amount spent on the worst option. In some sense, veto-Nash for nuisances is as relevant as plurality-Nash for goods, in the portioning context. Mathematically, veto-Nash is also well-behaved. While we do not provide a closed formula, the following result shows that an exact optimum for veto-Nash can be found in polynomial time (and that it is rational). It gives a collection of at most m different explicit rational distributions, and guarantees that a veto-Nash optimum is among them.

Theorem 4. *Let P be a profile, and let $\mathrm{vt}(x_j)$ be the number of voters placing x_j bottom. Relabel alternatives so that $\mathrm{vt}(x_1) \leqslant \cdots \leqslant \mathrm{vt}(x_m)$. If $\mathrm{vt}(x_j) = 0$ for some x_j, then veto-Nash selects all distributions over such alternatives. Otherwise, there is $k \in [m]$ with $(k-1)\mathrm{vt}(x_k) < \sum_{j=1}^{k} \mathrm{vt}(x_j)$, such that veto-Nash selects the distribution p with*

$$p_j = 1 - \frac{(k-1)\mathrm{vt}(x_j)}{\sum_{\ell=1}^{k} \mathrm{vt}(x_\ell)} \quad \textit{if } j \in [k], \textit{ and } p_j = 0 \textit{ otherwise.}$$

Proof. If $\mathrm{vt}(x_j) = 0$ for some x_j, then the best-possible Nash product of 1 can be achieved, and is achieved precisely by distributions whose support consists of never-vetoed alternatives.

Now suppose that $\mathrm{vt}(x_j) > 0$ for all x_j. Let p be a distribution selected by veto-Nash. We may assume that there is some $k \leqslant m$ such that $p_1, \dots, p_k$ are strictly positive and $p_{k+1} = \cdots = p_m = 0$. If p does not satisfy this, then there are j and ℓ with $j < \ell$ (and thus $\mathrm{vt}(x_j) \leqslant \mathrm{vt}(x_\ell)$) such that $p_j = 0$ and $p_\ell > 0$. We claim that the distribution p' obtained from q by swapping p_j and p_ℓ has a Nash welfare at least as high as p. This is because

$$\frac{\prod_{i\in N} \mathbf{s}[i,q]}{\prod_{i\in N} \mathbf{s}[i,p]} = \frac{\prod_{x_k\in X}(1-q_k)^{\mathrm{vt}(x_k)}}{\prod_{x_k\in X}(1-p_k)^{\mathrm{vt}(x_k)}} = \frac{(1-q_j)^{\mathrm{vt}(x_j)}(1-q_\ell)^{\mathrm{vt}(x_\ell)}}{(1-p_j)^{\mathrm{vt}(x_j)}(1-p_\ell)^{\mathrm{vt}(x_\ell)}} = \frac{(1-p_\ell)^{\mathrm{vt}(x_j)}}{(1-p_\ell)^{\mathrm{vt}(x_\ell)}} \geqslant 1$$

where the last inequality follows because $\mathrm{vt}(x_j) - \mathrm{vt}(x_\ell) \leqslant 0$. Thus we can shift probability mass in p until $p_1, \dots, p_k$ are strictly positive and $p_{k+1} = \cdots = p_m = 0$ for some k, without lowering Nash welfare, as desired. Note that $k \geqslant 2$, because otherwise $p_1 = 1$ and voters who veto x_1 get utility 0, leading to Nash welfare 0 (since $\mathrm{vt}(x_1) > 0$), contradicting optimality.

It follows that for $j = 1, \dots, k$, inequality $(*)$ of Proposition 2 holds with equality and can be written as

$$n = \sum_{i\in N} \frac{\mathbf{s}[i,j]}{\mathbf{s}[i,p]} = \sum_{\ell \in X\setminus\{j\}} \mathrm{vt}(x_\ell) \cdot \frac{1}{1-p_\ell}.$$

Adding $\mathrm{vt}(x_j)/(1-p_j)$ to both sides and rearranging, we get

$$\frac{\mathrm{vt}(x_j)}{1-p_j} = \sum_{\ell\in X} \frac{\mathrm{vt}(x_\ell)}{1-p_\ell} - n.$$

Note that the right-hand side does not depend on j. Writing T for the value on the right-hand side, it follows that $\mathrm{vt}(x_j) = T(1-p_j)$, and hence that $p_j = 1 - \frac{1}{T}\mathrm{vt}(x_j)$. Now we get

$$1 = \sum_{j=1}^{k} p_j = k - \frac{1}{T}\sum_{j=1}^{k} \mathrm{vt}(x_j) \quad \text{and hence} \quad T = \frac{\sum_{j=1}^{k} \mathrm{vt}(x_j)}{k-1}.$$

Plugging this into $p_j = 1 - \frac{1}{T}\mathrm{vt}(x_j)$, we arrive at the conclusion that

$$p_j = 1 - \frac{k-1}{\sum_{\ell=1}^{k} \mathrm{vt}(x_\ell)}\mathrm{vt}(x_j) \quad \text{for all } j = 1, \dots, k.$$

These values sum to 1, and are positive provided that $(k-1)\mathrm{vt}(x_k) < \sum_{j=1}^{k} \mathrm{vt}(x_j)$. Thus, we have shown that there is a Nash optimal distribution p of the form promised by the theorem. □

This gives a polynomial-time algorithm for computing veto-Nash exactly: if some alternatives are never vetoed, return any distribution over these. Otherwise iterate over all $k \in [m]$ satisfying the condition of the theorem and calculate the corresponding distribution, and return the one with highest Nash product.

Example 3. Assume 2, 3, 3 and 5 voters rank x_1, x_2, x_3 and x_4 last, respectively: $\mathrm{vt}(x_1) = 2$, $\mathrm{vt}(x_2) = \mathrm{vt}(x_3) = 3$, $\mathrm{vt}(x_4) = 5$. We have to identify the maximum integer k^* such that $(k^* - 1)\mathrm{vt}(x_{k^*}) < \sum_{j=1}^{k^*} \mathrm{vt}(x_j)$. From $\mathrm{vt}(x_2) < \mathrm{vt}(x_1) + \mathrm{vt}(x_2)$, $2\mathrm{vt}(x_3) < \mathrm{vt}(x_1) + \mathrm{vt}(x_2) + \mathrm{vt}(x_3)$ and $3\mathrm{vt}(x_4) > \mathrm{vt}(x_1) + \mathrm{vt}(x_2) + \mathrm{vt}(x_3) + \mathrm{vt}(x_4)$ we get $k^* = 3$.

Now, $k = 2$ and $k = 3$ satisfy the condition of Theorem 4. The distributions corresponding to $k = 2$ and $k = 3$ are respectively $p = \frac{3}{5}x_1 + \frac{2}{5}x_2$ and $p' = \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3$, one of which is optimal. The Nash product of p and p' is, respectively, $(\frac{2}{5})^2(\frac{3}{5})^3 \approx 0.035$ and $\left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^6 \approx 0.045$. As p' has higher Nash social welfare, it is optimal. □

Theorems 3 and 4 show that both plurality-Nash and veto-Nash are rational. Are there any other score vectors $\mathbf{s}$ such that $\mathbf{s}$-Nash is guaranteed to be rational? The answer is no: for every $\mathbf{s}$ other than plurality and veto, we can construct a profile where $\mathbf{s}$-Nash uniquely returns an irrational distribution. This result suggests that a convex programming solver is the best way of computing $\mathbf{s}$-Nash for $\mathbf{s}$ other than plurality and veto.

Theorem 5. *Let $m \geqslant 3$, and let $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{Q}^m$ be a score vector with $s_m = 0$ and normalized so that $s_1 = 1$. Unless $\mathbf{s} = (1, 0, \dots, 0)$ or $\mathbf{s} = (1, \dots, 1, 0)$, there exists a profile $P \in \mathcal{L}(X)^n$ for some $n \in \mathbb{N}$ such that $\mathbf{s}$-Nash returns a unique distribution p with $p \notin \mathbb{Q}^m$.*

Proof sketch. We construct four infinite families of examples, for different shapes of score vectors $\mathbf{s}$. Here, we give a detailed proof for the case $m = 3$. The other families require a more involved construction, but work using similar calculus. The details and a full proof appear in Appendix A.

Suppose $m = 3$, and let $\mathbf{s} = (1, \frac{r}{s}, 0)$, where $0 < \frac{r}{s} < 1$ and $\frac{r}{s}$ is in lowest terms. Let D be a large-enough integer. Consider the following profile: D voters with abc, one voter with bac, and one voter with bca. Let p be the distribution selected by $\mathbf{s}$-Nash. Note that b Pareto-dominates c, so that $p_c = 0$. Hence $p = (x, 1 - x, 0)$ for some x. One can show that $0 < x < 1$ if D is large enough (see the appendix). Now, the Nash product obtained by this distribution is $(x + \frac{r}{s}(1 - x))^D \cdot ((1 - x) + \frac{r}{s}x) \cdot (1 - x)$. By optimality, x must make the derivative d/dx vanish. After a calculation, canceling non-zero factors, this implies that

$$\left[(D + 2)(r - s)^2\right] \cdot x^2 + [-(r - s)((D + 3)r - 2(D + 1)s)] \cdot x + \left[r^2 - 2rs - Drs + Ds^2\right] = 0$$

This is a quadratic equation with integer coefficients. Solutions to the equation "$ax^2 + bx + c = 0$" involve the term $\sqrt{b^2 - 4ac}$; thus, they are rational if and only if $b^2 - 4ac$ is a perfect square. (We are reusing the variable names a, b, c here to be consistent with the standard quadratic formula.) In our case, the term under the square root simplifies to

$$(D + 1)^2 r^2 + 4(rs + s^2).$$

The first summand is a large perfect square, and the second summand does not depend on D. Since the distance between consecutive perfect squares is large (in the sense that $(z + 1)^2 - z^2 = 2z + 1 = \Theta(z)$), the discriminant cannot be a perfect square for large enough D. Hence, x is irrational. □

This irrationality result is in contrast to some other settings, notably Fisher markets, where maximizing the Nash product is guaranteed to yield a rational outcome [Vazirani, 2012].

4. Fairness

Usually, $\mathbf{s}$-utilitarianism spends 100% on a single alternative. Some agents might rank this alternative in a very low position, or even in last place. In some contexts, this might be seen as unfair and thus might rule out **s**-utilitarianism as a desirable rule. In this section, we formalize several notions of fairness, and show that **s**-egalitarianism satisfies individual fairness, and that **s**-Nash satisfies group fairness. Our axioms are inspired by axioms introduced in a model with approval votes developed by Bogomolnaia et al. [2005].

A minimal fairness axiom is *positive share* which requires that if voter i ranks x in last position, then $p_x < 1$. Hence, for every voter, a positive amount is spent on alternatives not ranked in last position. As suggested above, **s**-utilitarianism fails positive share for any **s**. However, provided that $s_m = 0$, positive share is satisfied by **s**-egalitarianism, **s**-leximin, and **s**-Nash. To see this, note that the uniform distribution has positive egalitarian and Nash welfare, whereas a distribution violating positive share has zero egalitarian and Nash welfare.

We can strengthen positive share to *individual fair share*, requiring that if voter i ranks x in last position, then $p_x \leqslant 1 - \frac{1}{n}$. Thus, for each voter, at least $\frac{1}{n}$ is spent on alternatives not ranked last. Note that the distribution identified by random dictatorship satisfies this condition and has egalitarian welfare at least $\frac{1}{n}$, normalizing $s_1 = 1$. Thus, the optimum **s**-egalitarian welfare is at least $\frac{1}{n}$, and hence **s**-egalitarianism and **s**-leximin satisfy individual fair share (recalling that $s_m = 0$). In Theorem 6 below, we show that **s**-Nash also satisfies it.

Consider $X = \{a, b\}$, with 9 voters ab and 1 voter ba. Then **s**-egalitarianism returns $\frac{1}{2}a + \frac{1}{2}b$. While this is individually fair, the group of 9 voters is underrepresented. If we desire fairness to groups, we need a stronger axiom. Let us say that a distribution p satisfies *group fair share* if whenever k out of n voters rank x last, then $p_x \leqslant 1 - \frac{k}{n}$, so at least $\frac{k}{n}$ is spent on alternatives other than x. This condition is failed by **s**-egalitarianism and **s**-leximin, but **s**-Nash satisfies it. In our example, **s**-Nash picks $\frac{9}{10}a + \frac{1}{10}b$ (which is the only distribution satisfying group fair share).

All the notions above focus on avoiding voters' last-ranked alternative. Despite working in an ordinal setting, using the SD-extension, we can define a group fairness notion that uses more than just the last-ranked alternative. An important underlying intuition is that each agent is "entitled" to $1/n$ of the budget, and this share should be spent in accordance to the agent's preferences. Similarly, a group $S \subseteq N$ of k agents could pool together and be entitled to k/n of the budget.

The intuitive notion of entitlement can be formalized using a core-style concept. A coalition $S \subseteq N$ of voters is supposed to be able to 'control' a fraction of $|S|/n$ of the entire budget. The notion of control is ambiguous since coalitions may overlap and each share of the budget is simultaneously 'controlled' by several coalitions. However, the entitlement of S is certainly violated under p if S can come up with a way of using only its entitlement $|S|/n$ which all members prefer to the way that p uses the entire budget.

Definition 2 (SD-core)**.** *A coalition $S \subseteq N$* SD-blocks *a distribution p if there exists a partial distribution z (called a* deviation*) with $\sum_{x \in X} z_x = |S|/n$ such that $z \succcurlyeq_i^{\mathrm{SD}} p$ for all $i \in S$, and $z \succ_j^{\mathrm{SD}} p$ for some $j \in S$. A distribution p is in the* SD-core *if no coalition SD-blocks p.*

The SD-core is stronger than group fair share (and hence also individual fair share and positive share).

Proposition 3. *If p is in the SD-core then p satisfies group fair share.*

Proof. Suppose that p fails group fair share. Thus, there is a coalition S of voters that rank x last but $p_x > 1 - |S|/n$. Then S can SD-block p: Write $\varepsilon = p_x - (1 - |S|/n) > 0$ and define a deviation z with $z_y = p_y + \varepsilon/(m-1)$ for all $y \in X \setminus \{x\}$, and $z_x = 0$. Then $\sum_{y \in X} z_y = z_x + \sum_{y \in X \setminus \{x\}} z_y = \sum_{y \in X \setminus \{x\}} (p_y + \varepsilon/(m-1)) = \varepsilon + \sum_{y \in X \setminus \{x\}} p_y = \varepsilon + (1 - p_x) = |S|/n$, so that z has the required total weight. It is easy to check that $z \succ_i^{\mathrm{SD}} p$ for all $i \in S$. Thus, p is not in the SD-core. □

For an example, take the profile with voters abc, acb, bca. Which distributions p are in the SD-core? First, singleton coalitions $\{i\}$ block p if $p_x > \frac{2}{3}$ for i's bottom alternative x, using $z = \frac{1}{3}y$ where y is i's top alternative. The coalition of abc and acb blocks all p with $p_a + p_b \leqslant \frac{2}{3}$ and $p_a + p_c \leqslant \frac{2}{3}$ (one inequality strict), using $z = \frac{2}{3}a$. All other distributions are in the SD-core.

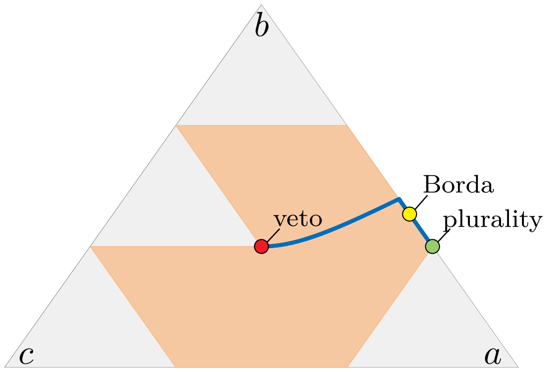

Figure 6: The SD-core of the profile (abc, acb, bca) within the simplex of all distributions. The shaded area shows the distributions that are in the SD-core. The thick blue line shows the output of $\mathbf{s}$-Nash for all $\mathbf{s} = (1, q, 0)$ with $q \in [0, 1]$. Plurality-Nash selects $\frac{2}{3}a + \frac{1}{3}b$, Borda-Nash selects $\frac{1}{\sqrt{3}}a + (1 - \frac{1}{\sqrt{3}})b \approx 0.58a + 0.42b$, and veto-Nash selects $\frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c$.

Figure 6 shows the simplex of all distributions, with the SD-core shaded (non-convex in this example).

Figure 6 shows the outputs of $\mathbf{s}$-Nash for all $\mathbf{s}$ as a blue line. The blue line is entirely contained in the SD-core. In fact, $\mathbf{s}$-Nash is always in the SD-core. We give a direct argument using equation $(*)$, which we derived in Proposition 2. The result can also be obtained via the theory of Lindahl equilibrium [Fain et al., 2016, Foley, 1970].

Theorem 6. *For any strictly decreasing* $\mathbf{s}$ *with* $s_m = 0$*, any distribution selected by* $\mathbf{s}$*-Nash is in the SD-core.*

Proof. Suppose p is selected by $\mathbf{s}$-Nash. For a contradiction, assume that $S \subseteq N$ is a blocking coalition of agents. Suppose they deviate using $(z_1, \dots, z_m) \in [0, 1]^m$ with $\sum_{j=1}^m z_j = |S|/n$, such that $z \succcurlyeq_i^{\mathrm{SD}} p$ for all $i \in S$, and $z \succ_j^{\mathrm{SD}} p$ for some $j \in S$. Now, $\mathbf{s}$ satisfies the conditions of Proposition 1(b), and thus we have $\mathbf{s}[i, z] \geqslant \mathbf{s}[i, p]$ for all $i \in S$, and $\mathbf{s}[j, z] > \mathbf{s}[j, p]$ for some $j \in S$. (This is where we use our definition of SD-comparisons between *partial* distributions.) Then

$$
\begin{aligned}
|S| = n \cdot \sum_{j=1}^m z_j = \sum_{j=1}^m n z_j & \\
&\overset{(*)}{\geqslant} \sum_{j=1}^m z_j \sum_{i \in N} \frac{\mathbf{s}[i, j]}{\mathbf{s}[i, p]} && \text{(from Proposition 2)} \\
&= \sum_{i \in N} \sum_{j=1}^m z_j \frac{\mathbf{s}[i, j]}{\mathbf{s}[i, p]} \\
&= \sum_{i \in N} \frac{\sum_{j=1}^m z_j \mathbf{s}[i, j]}{\mathbf{s}[i, p]} \\
&= \sum_{i \in N} \frac{\mathbf{s}[i, z]}{\mathbf{s}[i, p]} \\
&\geqslant \sum_{i \in S} \frac{\mathbf{s}[i, z]}{\mathbf{s}[i, p]} && \text{(since all summands are non-negative)} \\
&> \sum_{i \in S} 1 = |S|. && (S \text{ blocks})
\end{aligned}
$$

The last inequality follows because the sum contains $|S|$ terms, each of which is at least 1, and at least one of which is strictly larger than 1. This is a contradiction. □

Thus, the $\mathbf{s}$-Nash rules are particularly fair to groups. The SD-core can also be seen as a *proportionality* requirement: the common resource should be divided so that the share of an alternative is proportional to its support. For example, this is of interest in politics, to divide parliament seats among parties.

Theorem 6 only applies to strictly decreasing vectors $\mathbf{s}$, and it might fail otherwise. For example, veto-Nash is not always in the SD-core.

Example 4. Let $m = 4$, $\mathbf{s} = (1, 1, 1, 0)$ and $P = (abcd, abcd, abdc, abdc, acdb, acdb, bcda)$. Then $\mathbf{s}$-Nash uniquely selects $p = \frac{4}{7}a + \frac{1}{7}b + \frac{1}{7}c + \frac{1}{7}d$. But then the first 7 voters who all rank a top can deviate using $z = \frac{6}{7}a$, because $z \succ^{\text{SD}} p$ for each of those 7 voters.

On the other hand, one can check that plurality-Nash is always in the SD-core despite not satisfying the conditions of Theorem 6.

5. Efficiency

We study the efficiency of our rules. A weak version of Pareto efficiency for our context is known as *ex post efficiency* (with terminology taken from the probabilistic interpretation of social decision schemes) and requires that dominated alternatives must be assigned probability 0.

Definition 3 (ex post efficiency; e.g. Gibbard, 1977)**.** *Given a profile P, an alternative $y \in X$ is* dominated *if there is another alternative $x \in X$ such that $x \succcurlyeq_i y$ for all $i \in N$ and $x \succ_i y$ for some $i \in N$. A social decision scheme F is* ex post efficient *if for all profiles P and all distributions $p \in F(P)$, we have $p_y = 0$ for all $y \in X$ that are dominated in P.*

There are a variety of stronger definitions of the Pareto principle for social decision schemes [Brandt, 2017]. They do not just reason about dominated alternatives but about dominated distributions. A particularly natural definition uses the SD extension.

Definition 4 (SD-efficiency; e.g. Bogomolnaia and Moulin, 2001)**.** *Given a profile P, a distribution q* SD-dominates *p if $q \succcurlyeq_i^{\text{SD}} p$ for all $i \in N$, and $q \succ_j^{\text{SD}} p$ for some $j \in N$. A distribution p is* SD-efficient *if no distribution dominates it. A social decision scheme F is SD-efficient if for all profiles P, every distribution $p \in F(P)$ is SD-efficient.*

One can check that SD-efficiency implies ex post efficiency. Also, SD-core implies SD-efficiency, because a distribution is SD-dominated if and only if the grand coalition $S = N$ SD-blocks it. Thus, by Theorem 6, all $\mathbf{s}$-Nash rules are SD-efficient provided that $\mathbf{s}$ is strictly decreasing and $s_m = 0$.

More generally, which rules of our family are SD-efficient? Let us say that a welfare ordering $\geqslant_W$ is *weakly monotonic* if for every two utility vectors $\alpha, \beta \in \mathbb{R}^n_{\geqslant 0}$ with $\alpha_i > \beta_i$ for all $i \in N$, we have $\alpha >_W \beta$. We can show that any positional social decision scheme based on a strictly decreasing scoring vector and a weakly monotonic welfare ordering is SD-efficient. This result is slightly surprising because it only requires that $\geqslant_W$ is weakly monotonic. On first sight, we would require strict monotonicity (defined like weak monotonicity but with the premise "$\alpha_i \geqslant \beta_i$ for all $i \in N$ and $\alpha_i > \beta_i$ for some $i \in N$"). Because our result does not need strict monotonicity, it applies even to $\mathbf{s}$-egalitarianism, not just to $\mathbf{s}$-leximin.

Proposition 4. *If $\mathbf{s}$ is strictly decreasing with $s_m = 0$, and $\geqslant_W$ is weakly monotonic, then $F_{\mathbf{s},\geqslant_W}(P)$ is SD-efficient.*

Proof. Let us begin by noting that if $\succ$ is any linear order and p and q are distributions such that both $p \succcurlyeq^{\text{SD}} q$ and $q \succcurlyeq^{\text{SD}} p$, then $p = q$.[3] To see this, label alternatives so that $x_1 \succ x_2 \succ \cdots \succ x_m$.

[3] We thank Patrick Lederer for pointing this fact out to us.

Since we have an SD-indifference between p and q, the definition of SD implies that $p_1 = q_1$, $p_1 + p_2 = q_1 + q_2$, ..., $p_1 + \dots + p_m = q_1 + \dots + q_m$. Hence by induction $p_j = q_j$ for each $j = 1, \dots, m$, and so $p = q$.

Now, suppose for a contradiction that a distribution p SD-dominates q, where $q \in F_{\mathbf{s},\geqslant_W}(P)$. Thus, $p \succsim_i^{\mathrm{SD}} q$ for all $i \in N$. Because p SD-dominates q, we have $p \neq q$ and so by the above argument we in fact have $p \succ_i^{\mathrm{SD}} q$ for all $i \in N$. By Proposition 1(b), since $\mathbf{s}$ is strictly decreasing, this implies that $\mathbf{s}[i,p] > \mathbf{s}[i,q]$ for all $i \in N$. Since $\geqslant_W$ is weakly monotonic, we have that $\mathbf{s}[p] >_W \mathbf{s}[q]$, contradicting that $q \in F_{\mathbf{s},\geqslant_W}(P)$. □

As a corollary, $\mathbf{s}$-utilitarianism, $\mathbf{s}$-egalitarianism, $\mathbf{s}$-leximin, and $\mathbf{s}$-Nash are all SD-efficient provided that $\mathbf{s}$ is strictly decreasing and $s_m = 0$.

There are positional social decision schemes that violate the assumptions of Proposition 4 yet are still SD-efficient. For example, plurality-Nash and plurality-egalitarianism are both SD-efficient [Aziz and Stursberg, 2014] even though the plurality scoring vector is not strictly decreasing. However, Proposition 4 does not hold if we only assume that $\mathbf{s}$ is weakly decreasing. For example, any positional social decision scheme based on $\mathbf{s} = (1, 1, 0)$ cannot even be ex post efficient: consider the one-voter profile $P = (abc)$. The only ex post efficient distribution is the distribution p that places all weight on a. But the distribution q that places all weight on b induces the same $\mathbf{s}$-scores: $\mathbf{s}[1,p] = \mathbf{s}[1,q]$. Hence for any $\geqslant_W$, if $F_{(1,1,0),\geqslant_W}$ contains the distribution p, then it also contains q. Hence $F_{(1,1,0),\geqslant_W}$ is not ex post efficient. In particular, veto-Nash is not ex post efficient.

6. Incentive properties

We study incentive properties of our rules, including participation, strategyproofness, and monotonicity.

6.1. Participation

Typically, participation in a voting process is voluntary, and voters can decide whether they wish to abstain or to cast their ballot. Thus, it is desirable to be able to guarantee that participating is never detrimental to the voter. This is the motivation of the participation axiom, which we again phrase based on the SD-extension. To use participation axioms, we need to move to a *variable electorate* setting, i.e., allow social decision schemes to take as input profiles with varying numbers of voters.

Definition 5 (Weak SD-participation). *A social decision scheme F satisfies* weak SD-participation *if for any set N of voters, any profile P defined on N, and any $i \in N$, we do not have $q \succ_i^{\mathrm{SD}} p$ for any $p \in F(P)$ and $q \in F(P_{-i})$, where P_{-i} is the profile obtained from P by deleting voter i's report.*

In other words, if voter i abstains, the rule must yield a distribution that is no better under *some* utility function consistent with $\succ_i$. We can also define a stronger version, which requires abstaining to be (weakly) worse under *all* consistent utility functions.

Definition 6 (Strong SD-participation). *A social decision scheme F satisfies* strong SD-participation *if for any set N of voters, any profile P defined on N, and any $i \in N$, we have $p \succsim_i^{\mathrm{SD}} q$ for all $p \in F(P)$ and $q \in F(P_{-i})$, where P_{-i} is the profile obtained from P by deleting voter i's report.*

Both weak SD-participation and strong SD-participation were introduced by Brandl et al. [2015]. We will see that many positional social decision schemes satisfy weak SD-participation. The strong version, however, is satisfied by many fewer rules. Note that the specific classes of positional social decision schemes we have studied ($\mathbf{s}$-utilitarianism, $\mathbf{s}$-egalitarianism, $\mathbf{s}$-leximin, $\mathbf{s}$-Nash) all give rise to variable-electorate rules in an obvious way.

Proposition 5. *For any strictly decreasing scoring vector* $\mathbf{s}$ *with* $s_m = 0$*,* $\mathbf{s}$*-Nash satisfies weak SD-participation.*

Proof. Let $\mathbf{s}$ be strictly decreasing, and let F denote the $\mathbf{s}$-Nash rule. For a contradiction assume that there are $p \in F(P)$ and $q \in F(P_{-i})$ such that $q \succ_i^{\mathrm{SD}} p$, for some profile P and some voter $i \in N$. As $\mathbf{s}$ is strictly decreasing, by Proposition 1(b), we have that $\mathbf{s}[i, q] > \mathbf{s}[i, p] > 0$, where the last part holds because the Nash product of p must be positive. Because p is an outcome of $\mathbf{s}$-Nash at profile P, we have $\prod_{j\in N} \mathbf{s}[j, p] \geqslant \prod_{j\in N} \mathbf{s}[j, q]$. Combining these inequalities, and noting that $\prod_{j\in N} \mathbf{s}[j, p] > 0$, we find that

$$\prod_{j\in N\setminus\{i\}} \mathbf{s}[j, p] = \frac{\prod_{j\in N} \mathbf{s}[j, p]}{\mathbf{s}[i, p]} > \frac{\prod_{j\in N} \mathbf{s}[j, p]}{\mathbf{s}[i, q]} \geqslant \frac{\prod_{j\in N} \mathbf{s}[j, q]}{\mathbf{s}[i, q]} = \prod_{j\in N\setminus\{i\}} \mathbf{s}[j, q].$$

This contradicts that q is an outcome of $\mathbf{s}$-Nash on profile P_{-i}. ☐

Proposition 6. *For any strictly decreasing scoring vector* $\mathbf{s}$ *with* $s_m = 0$*,* $\mathbf{s}$*-leximin satisfies weak SD-participation.*

Proof. Let $\mathbf{s}$ be strictly decreasing, and let F denote the $\mathbf{s}$-leximin rule. For a contradiction assume that there are $p \in F(P)$ and $q \in F(P_{-i})$ such that $q \succ_i^{\mathrm{SD}} p$, for some profile P and some voter $i \in N$. As $\mathbf{s}$ is strictly decreasing, by Proposition 1(b), we have that $\mathbf{s}[i, q] > \mathbf{s}[i, p]$ for all $i \in N$. For a vector $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$, write $\alpha^\uparrow \in \mathbb{R}^k$ for the vector obtained from α by reordering its entries in non-decreasing order. Let $(x_1, \dots, x_n) = \mathbf{s}[q]^\uparrow$ and let $(y_1, \dots, y_n) = \mathbf{s}[p]^\uparrow$. Because p is an outcome of $\mathbf{s}$-leximin for the voter set N, we have

$$(y_1, \dots, y_n) \geqslant_L (x_1, \dots, x_n) \tag{2}$$

where $\geqslant_L$ denotes lexicographic comparison. Suppose $\mathbf{s}[i, q] = x_r$ and $\mathbf{s}[i, p] = y_s$. Since $\mathbf{s}[i, q] > \mathbf{s}[i, p]$, we have $x_r > y_s$. As q is the $\mathbf{s}$-leximin outcome for voter set $N \setminus \{i\}$, it follows that

$$(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n) \geqslant_L (y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_n) \tag{3}$$

Write y' for the vector obtained from $(y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_n)$ by inserting x_r in the rth position and write y'' for the vector obtained from y' by replacing x_r with y_s in the rth position. Then we have

$$\begin{aligned}
(x_1, \dots, x_n) &\geqslant_L y' && \text{(insert } x_r \text{ in } r\text{th position in both sides of (3))}\\
&>_L y'' && (y' \text{ and } y'' \text{ are equal before } r\text{th position, then } x_r > y_s)\\
&\geqslant_L (y'')^\uparrow && (\text{since } \alpha \geqslant_L \alpha^\uparrow \text{ for all } \alpha \in \mathbb{R}^n)\\
&= (y_1, \dots, y_n),
\end{aligned}$$

which is a contradiction to (2). ☐

On the other hand, $\mathbf{s}$-egalitarian rules may fail SD-participation.

Example 5. Let $m = 4$ and consider the profiles $P_1 = (abcd, cbad)$ and $P_2 = (abcd, cbad, cadb)$ which obtained from P_1 by adding a new voter $cadb$. At P_1, Borda-egalitarianism can select $0.5a + 0.5c$, and at P_2 it can select $0.4a + 0.2b + 0.4c$ which is SD-worse for the new voter. (Borda-leximin selects $0.5a + 0.5c$ at both profiles.)

Similarly, at P_1, veto-egalitarianism can select c and at P_2 it can select a, which is SD-worse for the new voter. (Veto-leximin selects b at P_1 and a at P_2.)

Strong SD-participation is much more difficult to attain. Brandl et al. [2015] show that for any strictly decreasing scoring vector $\mathbf{s}$, $\mathbf{s}$-utilitarianism satisfies strong SD-participation. However, this is not true for $\mathbf{s}$-Nash and $\mathbf{s}$-leximin. Examples 6 and 7 show respectively that Borda-leximin (and Borda-egalitarianism) and Borda-Nash do not satisfy strong SD-participation.

Example 6. Consider the profile where voter 1 has preferences $abcd$, voter 2 has preferences $cbda$ and voter 3 has preferences $cdba$. When only voters 1 and 2 participate, the unique outcome of Borda-egalitarianism is the distribution where b has probability 1 (giving both voters a Borda score of 2). If voter 3 participates, the unique outcome for Borda-egalitarianism is $0.4a + 0.6c$ (giving all three voters a Borda score of 1.8). But this distribution is not SD-better than the first according to voter 3's preferences. It follows that Borda-leximin and Borda-egalitarianism fail strong SD-participation.

Example 7. Consider the profile where voter 1 has preferences $abcdefgh$, voter 2 has preferences $fgchabde$ and voter 3 has preferences $fghcabde$. If only voters 1 and 2 participate, the (unique) outcome is the distribution where c has probability 1. If all three voters participate, we have a mixture between a and f (namely $\frac{19}{60}a + \frac{41}{60}f$), which is not SD-better for voter 3. It follows that Borda-Nash does not satisfy strong SD-participation.

6.2. Strategyproofness

Next, we briefly discuss strategyproofness of our rules, that is, robustness to strategic misrepresentation of voters' preferences. A social decision scheme is *resolute* if, for every profile P, it returns a *unique* distribution.[4]

Definition 7 (SD-strategyproofness)**.** *A resolute social decision scheme F is* SD-strategyproof *if for all profiles P, all voters $i \in N$, and all linear orders $\succ_i'$, we have that*

$$F(P) \succcurlyeq_i^{\text{SD}} F(P_{-i}, \succ_i'),$$

where $(P_{-i}, \succ_i')$ denotes the profile obtained from P by replacing i's preference report $\succ_i$ by $\succ_i'$.

One can check that plurality-Nash (that is, random dictatorship) is SD-strategyproof. In fact, a well-known result of Gibbard [1977] shows that random dictatorship is the only social decision scheme that is anonymous, ex post efficient, and SD-strategyproof. Hence, all other ex post efficient rules in our class, after making them resolute by tie-breaking, are manipulable.[5] (Outside of our class of rules, Barberà [1979] shows that *proportional Borda*, mentioned in the Introduction, is SD-strategyproof, but it is not ex post efficient.)

One can also define a weak version of SD-strategyproofness, that requires that a manipulation does not lead to an outcome that is SD-better for the manipulator [Postlewaite and Schmeidler, 1986, Bogomolnaia and Moulin, 2001], that is, that $F(P_{-i}, \succ_i') \not\succ_i^{\text{SD}} F(P)$, rather than requiring that it must be weakly SD-worse. These two notions are different because SD-preferences over distributions are not complete. We leave a characterization of which positional social decision schemes satisfy this weak version for future work, but many of these rules fail it (and so far we do not know a rule in our family that satisfies it, besides random dictatorship).

Example 8. Let $m = 3$ and consider the profiles $P_1 = (abc, abc, abc, bac)$ and $P_2 = (abc, abc, abc, bca)$.

[4] We need to assume resoluteness to avoid having to compare sets of distributions, and to be consistent with prior work: the characterization of strategyproof randomized rules by Gibbard [1977] requires it. All our rules can be made resolute by applying a tie-breaking mechanism based on a fixed priority relation between distributions.

[5] It remains to check that other positional social decision schemes that fail ex post efficiency also fail SD-strategyproofness, such as rules based on $\mathbf{s} = (1, 1, 0, 0)$. We leave this for future work.

- On P_1, Borda-Nash selects a and on P_2, Borda-Nash selects $0.5a + 0.5b$. Thus the fourth voter can manipulate from P_1 to P_2 and get a strictly SD-better outcome.
- On P_1, veto-Nash may select $0.5a + 0.5b$[6] and on P_2, veto-Nash selects b. Again the fourth voter can manipulate.
- On P_1, Borda-leximin selects $0.5a + 0.5b$ and on P_2, Borda-leximin selects $\frac{1}{3}a + \frac{2}{3}b$. Again, the fourth voter can manipulate.

6.3. Monotonicity

A social decision scheme is monotonic if whenever we move up an alternative in a voter's ranking, the share of that alternative in the output distribution does not decrease. Formally, given profiles P and P', we say that P' is an *x-improvement* of P if P' is obtained from P by moving up an alternative x in some voter's ranking, everything else being unchanged. A social decision scheme F is *monotonic* if for all profiles P and P' such that P' is an x-improvement of P, for all $p \in F(P)$, there exists $q \in F(P')$ such that $q_x \geqslant p_x$.[7]

Monotonicity is clearly satisfied by $\mathbf{s}$-utilitarianism. It is also satisfied by plurality-Nash (that is, random dictatorship), as well as plurality-egalitarianism [Aziz and Stursberg, 2014, Thm. 5]. However, other $\mathbf{s}$-Nash, $\mathbf{s}$-leximin and $\mathbf{s}$-egalitarian rules may fail it:

Example 9. Consider first Borda-Nash. Let $m = 3$, $\mathbf{s} = (2, 1, 0)$ and $P = (abc, abc, abc, acb, bac, cba)$, then $\mathbf{s}$-Nash selects an irrational distribution which rounds to $0.642a + 0.333b + 0.024c$. Let $P' = (abc, abc, abc, acb, bca, cba)$. P' is a c-improvement of P (the bac voter moves c up one place). For P', $\mathbf{s}$-Nash selects $0.5a + 0.5b$. Thus, c's share has strictly decreased.

Example 10. Consider now Borda-egalitarianism. Let $m = 3$, $\mathbf{s} = (2, 1, 0)$ and $P = (cba, abc)$. Then $\mathbf{s}$-egalitarianism selects the set of all distributions of the form $(\alpha, 1 - 2\alpha, \alpha)$; it contains in particular the deterministic distribution b. The profile $P' = (bca, abc)$ is a b-improvement of P. But on P', $\mathbf{s}$-egalitarianism selects only the distribution $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

In computational experiments, veto-Nash appears to satisfy monotonicity, but we did not prove this.

How bad is it if a rule fails monotonicity? It depends on the context. Monotonicity can be interpreted as fairness to alternatives: alternative x should get more if it performs better. However, rules like $\mathbf{s}$-Nash and $\mathbf{s}$-egalitarianism principally aim to be fair to *voters*. In Example 10, while b's share has decreased, the utility vector moves from $(1, 1)$ to $(\frac{4}{3}, \frac{4}{3})$ while the deterministic distribution b for P' would lead to utility vector $(2, 1)$.

7. Application to seat apportionment in party-list elections

Suppose that a number of seats (of a parliament or of the board of some organization) have to be filled. In *list-based elections*, the candidates are partitioned into *lists*, where a list is a linearly ordered set of candidates. In political elections, lists are typically associated with parties and called *party lists*. The election proceeds as follows:

1. each voter votes for one of the lists;
2. the score of each list is the fraction of votes it receives;
3. these fractional scores are then mapped into an integral distribution of seats;

[6] This depends on tie-breaking, since all distributions $xa + (1 - x)b$ are optimal for veto-Nash. It is easy to check that manipulations exist under other tie-breakings.

[7] Alternatively, one could require that $q_x \geqslant p_x$ for *all* $q \in F(P')$. This stronger version is still satisfied by $\mathbf{s}$-utilitarianism.

4. each list fills its assigned seats with candidates in the order of the list.

Step 3 is known as *apportionment*, and its formal study has a long history [Balinski and Young, 2001]. There is a variety of apportionment rules, including the two families of largest remainder and largest average rules, and the advantages and disadvantages of different rules are well understood. On the other hand, steps 1 and 2 are rarely critically examined. This is surprising because in settings where voters express preferences directly over candidates, social choice theorists commonly criticize systems in which voters only report their top choice. They argue that not enough information is transmitted to arrive at a high-quality social decision. For list-based elections, similar arguments suggest that we should elicit more detailed preferences over lists. Recently, Brill et al. [2020] explored this idea. They studied a model where voters are allowed to approve several lists instead of just one. In place of step 2, they then use a portioning method defined for approval voting. Instead of approval votes, it is a natural idea to allow inputs consisting of rankings over lists. We can then use a social decision scheme (i.e., a portioning method for rankings) to obtain a method to elect seats with this input format.

France elects its parliament (*Assemblée nationale*) using district-based elections, where each district elects one deputy using two-round plurality with runoff. Similar to first-past-the-post systems, this fails to provide proportional representation, and is biased towards the largest parties. To address this, there have been calls for some of the seats of the assembly to be elected via a list-based system [Cohendet et al., 2018]. As a case study, we apply our rules on a hypothetical profile that is roughly consistent with survey data from the French 2017 presidential and parliamentary elections. While such survey data reports preferences over candidates (party leaders), it seems reasonable to also interpret them as preferences over parties. The main 5 parties participating were *La France Insoumise* (LFI, left), *Parti Socialiste* (PS, centre-left), *La République en Marche / Modem* (LRM, centre-right), *Les Républicains* (LR, right), *Rassemblement National* (RN, extreme-right nationalist).

We believe that our profile is reasonably realistic. First, the fraction of first positions is consistent with the actual outcome of the presidential election,[8] except for PS whose unusually low score is thought to have been due to a form of strategic voting induced by plurality with runoff. (Polls show that many PS voters of the prior election chose to support another candidate in 2017 [Ipsos, 2017a].) Second, in accordance with standard assumptions, the profile is single-peaked along a left-right axis {LFI, PS, LRM, LR, RN}, except that we added a 9% fraction of "anti-system votes" ranking LFI–RN or RN–LFI first and second, and LRM last. (Polls show that some voters switched from one extreme to the other in the second round [Ipsos, 2017b].)

3 voters	LFI $\succ$ PS $\succ$ LRM $\succ$ LR $\succ$ RN
2 voters	PS $\succ$ LFI $\succ$ LRM $\succ$ LR $\succ$ RN
1 voter	PS $\succ$ LRM $\succ$ LFI $\succ$ LR $\succ$ RN
2 voters	LRM $\succ$ PS $\succ$ LR $\succ$ LFI $\succ$ RN
2 voters	LRM $\succ$ LR $\succ$ PS $\succ$ LFI $\succ$ RN
2 voters	LRM $\succ$ LR $\succ$ PS $\succ$ RN $\succ$ LFI
1 voter	LR $\succ$ LRM $\succ$ RN $\succ$ PS $\succ$ LFI
2 voters	LR $\succ$ LRM $\succ$ PS $\succ$ LFI $\succ$ RN
1 voter	LR $\succ$ RN $\succ$ LRM $\succ$ PS $\succ$ LFI
4 voters	RN $\succ$ LR $\succ$ LRM $\succ$ PS $\succ$ LFI
1 voter	RN $\succ$ LFI $\succ$ LR $\succ$ PS $\succ$ LRM
1 voter	LFI $\succ$ RN $\succ$ PS $\succ$ LR $\succ$ LRM

Table 1 shows the outcomes of several of our rules applied to this profile. We also consider proportional Borda (discussed in Section 1) for comparison. We did not include veto-Nash because

[8] The parties' actual first-round scores (after discounting blank and null votes) were 25% LRM, 22% RN, 20% LR, 20% LFI, and 7% PS (plus 7% in total for the other 6 candidates). In the same order, the fraction of votes in our profile with the respective parties on top are 27%, 23%, 18%, 18% and 14%.

it is not a very interesting rule in the context of party-list elections: since we seek to represent voters, we should pay more attention to candidates ranked near the top of votes.[9] Finally, for comparison, we include the actual seat distribution of the assembly, which gives many seats to LRM.

	LFI	PS	LRM	LR	RN	$\sum u_i$	$\prod u_i$	$\min u_i$
Borda-Nash	0.00	0.27	0.27	0.46	0.00	52.5	83.1	1.0
Borda-egalitarianism	0.12	0.38	0.00	0.37	0.13	45.8	9.5	2.0
Borda-utilitarianism	0.00	0.00	1.00	0.00	0.00	56.0	0.0	0.0
Plurality-Nash	0.18	0.14	0.27	0.18	0.23	44.3	4.6	1.8
Proportional Borda	0.15	0.22	0.25	0.24	0.14	46.5	13.1	1.7
Actual seat distribution	0.05	0.08	0.60	0.23	0.01	53.1	51.2	0.6

Table 1: The result of applying portioning rules to the list-based election.

Borda-Nash gives zero support to the two extreme parties, and gives moderate support to the dominant party LRM. The outcome of Borda-egalitarianism is slightly surprising: it gives zero support to the dominant party LRM, whose supporters are well enough represented by the two parties immediately to its left (PS) and to its right (LR). As expected, Borda-utilitarianism gives all its support to the dominant centrist party.[10] The rule that is most often used for party-list elections in the real world, Plurality-Nash, gives much more support to the extremes than the other rules, because those parties often appear in top position. None of the rules is clearly best. Intuitively, Borda-Nash looks like a good compromise, but extreme parties are not represented. This can be corrected either by taking a scoring vector between plurality and Borda, or an aggregation function between Nash and utilitarianism.

8. Conclusions

We have introduced a class of aggregation rules which can be used to split a budget or another continuous resource (such as time) between different uses. When combined with an apportionment step, the rules can also be used to select an outcome in a discrete space such as an allocation of seats to parties. While a sizeable literature has considered the portioning problem in a model with approval input, we have provided the first systematic study of portioning rules based on ordinal input given as rankings.

Portioning rules are formally equivalent to probabilistic social choice functions which have received significant attention in the literature. However, since their uses are quite different from portioning, many results from that literature are of limited interest for portioning (for example axioms such as Condorcet consistency). Indeed, rules that are deemed interesting for randomized voting are often not interesting for portioning. Informally, the discrepancy between randomized voting and portioning is that the final outcome in randomized voting is obtained by sampling

[9] In our profile, veto-Nash selects all distribution whose support contains only PS and LR, since those lists were not vetoed by any voter in the profile (see Theorem 4).

[10] Note that the 2017-2022 assembly (elected using district-based plurality with runoff) had more than 50% of its seats taken by LRM, and very few seats by the extreme parties (around 4% for LFI and 2% for RN).

	ind. fair share	SD-core	SD-efficient	poly time	strategyproof	monotonic
Plurality-Utilitarianism	–	–	✓	✓	–	✓
Borda-Utilitarianism	–	–	✓	✓	–	✓
Plurality-Egalitarianism	✓	–	✓	✓	–	✓
Borda-Egalitarianism	✓	–	✓	✓	–	–
Borda-Leximin	✓	–	✓	✓	–	–
Plurality-Nash	✓	✓	✓	✓	✓	✓
Borda-Nash	✓	✓	✓	approx.	–	–
Veto-Nash	✓	–	–	✓	–	open
Maximal Lotteries	–	–	✓	✓	–	–
Proportional Borda	✓	–	–	✓	✓	✓

Table 2: Summary of normative and computational results.

one winner according to the output distribution, while in portioning the final outcome is the distribution itself. This explains why *ex-post* notions make sense for randomized voting but are meaningless for portioning. For example, when there is an alternative that satisfies some quality criterion (like Condorcet) then selecting this alternative with probability 1 makes sense in randomized voting: admittedly, there will be unhappy voters in the end, but that is unavoidable since a single alternative will be selected *ex post*. In portioning, such an argument no longer makes sense, and selecting an alternative with probability 1 should be done only in cases where the agreement for an alternative is extremely strong; selecting a Condorcet winner with probability 1 would be extremely unfair to some voters (see our introductory example). Therefore, randomized voting rules that are designed so as to satisfy one of these properties (such as the *Maximal Lottery* rule) are unnecessarily unfair when applied to portioning. The reverse direction is less clear: our rules may be of interest to randomized voting. More generally, the connections and differences between randomization and splitting a common resource need to be discussed further.

Table 2 gives a summary of properties satisfied by some typical SDSs that belong to the family defined in this paper (the first seven), together with two other SDSs that are not part of the family (the last two); for these two, references to existing results as well as proofs of novel results are in Appendix B. As we discussed, plurality-egalitarianism was previously studied as egalitarian simultaneous reservation, and plurality-Nash is commonly known as random dictatorship.

Anyone wanting to implement positional social decision schemes will need to choose a scoring vector, and the 'right' choice is underdetermined from a theory perspective. This issue is similar to the choice of a positional scoring rule in deterministic voting. Based on the axioms we have considered, plurality-Nash (i.e. random dictatorship) is in a distinguished position. It is the only rule that satisfies all the axioms. (In particular it is provably the only one that satisfies ex post efficiency and SD-strategyproofness, see Section 6.2.) However, random dictatorship has a major drawback: it ignores voters' preferences except for their top alternative. This makes trade-offs impossible (as the rule is unable to distinguish between someone's second-best alternative and their worst alternative), and explains why in Section 7 the outcome of random dictatorship is biased towards the extremes. This drawback is difficult to formalize convincingly, and a successful axiomatic criticism of plurality-Nash would be interesting future work.

Using positional scores is certainly not the only way to do portioning based on ordinal profiles. Other voting rules or social welfare functions that are defined by maximizing scores may also be adapted towards this aim; further exploration of this topic, especially with a view towards fairness, is needed.

Finally, it is natural to enrich our setting by allowing ties in the input rankings, i.e., to allow voters to submit weak orders. One could certainly generalize positional scoring rules to this input, which would provide for interesting future work. For randomized social choice, weak orders have been considered [Aziz and Stursberg, 2014, Aziz et al., 2018a, Bogomolnaia et al., 2005], for example in the context of strategyproofness. Notably, random dictatorship is not well-defined anymore if voters do not have a unique top alternative, making the search for good strategyproof rules rather more difficult or impossible [Brandl et al., 2018].

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A. Proof of Theorem 5

We will construct, for each scoring vector $\mathbf{s}$ different from $(1, 0, \dots, 0)$ and $(1, \dots, 1, 0)$, a profile for which $\mathbf{s}$-Nash returns a distribution which is irrational. To establish irrationality of a distribution, we show that the probabilities in the output distribution are solutions to a quadratic equation which only has irrational solutions. We recall some elementary facts about square roots and quadratic equations:

- If $n \in \mathbb{N}$ is an integer, then $\sqrt{n} \in \mathbb{Q}$ if and only if $n = z^2$ for some $z \in \mathbb{N}$.
- Suppose $a, b, c \in \mathbb{Z}$, $a \neq 0$, and suppose $x \in \mathbb{R}$ satisfies $ax^2 + bx + c = 0$. Then

$$x = (2a)^{-1}(-b + \sqrt{b^2 - 4ac}) \quad \text{or} \quad x = (2a)^{-1}(-b - \sqrt{b^2 - 4ac}).$$

  Hence, $x \in \mathbb{Q}$ if and only if the integer $b^2 - 4ac \geqslant 0$ is a perfect square.
- If $a, b \in \mathbb{N} \setminus \{0\}$ and $a > b$, then $a^2 < a^2 + b < (a + 1)^2$, and thus $a^2 + b$ is not a perfect square.

We will also use the following two facts about an $\mathbf{s}$-Nash distribution p. Both facts use $s_m = 0$.

- Positive share: we have $\mathbf{s}[i, p] > 0$ for all $i \in N$, because otherwise the Nash product is 0 while a positive Nash product is possible (e.g. uniform distribution) contradicting optimality of p.
- $\mathbf{s}$-core: For any $S \subseteq N$, there does not exist a partial distribution z with $\sum_{x \in X} z_x = |S|/n$ such that $\mathbf{s}[i, z] > \mathbf{s}[i, p]$ for all $i \in S$. This is implicitly proven in the proof of Theorem 6 that $\mathbf{s}$-Nash satisfies SD-core.

We only consider rational vectors $\mathbf{s}$, and assume that they are non-increasing. We assume that $s_m = 0$. Without loss of generality (by rescaling), we may assume that $s_1 = 1$. We distinguish four cases:

1. $s_1 = \dots = s_k = 1$ and $s_{k+1} = \dots = s_m = 0$ for some $k \geqslant 2$ and $m \geqslant k + 2$;
2. $s_1 = 1$ and $0 < s_2 < 1$;
3. $s_1 = s_2 = 1$ and $0 < s_3 < 1$;
4. $s_1 = \dots = s_k = 1$ and $0 < s_{k+1} < 1$ for some $k \geqslant 3$.

In the latter three cases, we call the first non-1 entry $q \in \mathbb{Q}$ and write $q = r/s$ for $r, s \in \mathbb{N}$.

A.1. k-Approval

Suppose $\mathbf{s} = (1, \dots, 1, 0, 0)$, where $m = k + 2$ and $k \geqslant 2$. We extend the argument to $m > k + 2$ later; note that we are only interested in the case with at least two zeros at the end of the score vector, since otherwise $\mathbf{s}$ is veto.

Write $X = \{a_1, \dots, a_k, x, y\}$. Consider the following profile, which we can specify by just listing voters' k-approval sets.

- One voter approving $\{a_1, \dots, a_k\}$.
- For each $i \in [k]$, one voter approving $\{x, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_k\}$.
- For each $i \in [k]$, one voter approving $\{x, y, a_1, \dots, a_{i-1}, a_{i+2}, \dots, a_k\}$, with subscripts mod k.

In this profile, y is dominated by x, so in a Nash outcome, y gets probability 0. Further, all the a_i are symmetric, so by neutrality of Nash they all get the same probability. Hence, there exists some number $x \in [0,1]$ (abusing notation) such that the Nash outcome p satisfies

$$p_x = x, \quad p_y = 0, \quad p_{a_1} = \cdots = p_{a_k} = (1-x)/k.$$

Now, if $x = 1$, then the first voter approving $\{a_1, \dots, a_k\}$ gets utility 0, giving Nash product 0. So $x = 1$ is not optimal. If $x = 0$, then consider the coalition of the $2k$ voters approving x. If $x = 0$, they all obtain utility at most $(k-1)/k$ under p (some voters only get $(k-2)/k$). Note that this coalition can propose the partial distribution z with $\|z\| = (2k)/(2k+1)$ putting all its weight on alternative x. Under z, every member of the coalition obtains utility $(2k)/(2k+1)$. An easy calculation shows that, since $k \geqslant 2$,

$$\frac{2k}{2k+1} > \frac{k-1}{k}.$$

Hence, z is an $\mathbf{s}$-core deviation, contradicting optimality of p.

It follows that $0 < x < 1$. Now, the Nash product of p as a function of x is

$$\begin{aligned} f(x) &= (1-x) \cdot \left(x + \frac{k-1}{k}(1-x)\right)^k \cdot \left(x + \frac{k-2}{k}(1-x)\right)^k \\ &= (1-x) \cdot \left(\frac{k+x-1}{k}\right)^k \cdot \left(\frac{k+2x-2}{k}\right)^k \\ &= (1-x) \cdot k^{-2k} \cdot (k+x-1)^k \cdot (k+2x-2)^k. \end{aligned}$$

By optimality of p, and since $0 < x < 1$, we must have $\partial f(x)/\partial x = 0$. Now, by the product rule:

$$k^{2k} \cdot \partial f(x)/\partial x = -(k+x-1)^k \cdot (k+2x-2)^k + (1-x) \cdot \tfrac{\partial}{\partial x}\left[(k+x-1)^k \cdot (k+2x-2)^k\right].$$

By the product rule, we have

$$\begin{aligned} \tfrac{\partial}{\partial x}\left[(k+x-1)^k \cdot (k+2x-2)^k\right] &= k \cdot (k+x-1)^{k-1} \cdot (k+2x-2)^k \\ &\quad + (k+x-1)^k \cdot k \cdot (k+2x-2)^{k-1} \cdot 2. \end{aligned}$$

Inserting and factoring, we get

$$\begin{aligned} k^{2k} \cdot \partial f(x)/\partial x = (k+x-1)^{k-1} \cdot (k+2x-2)^{k-1} \cdot \Big[&-(k+x-1) \cdot (k+2x-2) \\ &+ (1-x) \cdot k \cdot (k+2x-2) \\ &+ (1-x) \cdot 2k \cdot (k+x-1)\Big] \end{aligned}$$

The first two factors are positive (since $k \geqslant 2$ and $x > 0$). Thus, for $\partial f(x)/\partial x = 0$ to be the case, the part in square bracket must be zero. Thus,

$$\begin{aligned} 0 &= -(k+x-1) \cdot (k+2x-2) + (1-x) \cdot k \cdot (k+2x-2) + (1-x) \cdot 2k \cdot (k+x-1) \\ &= (-4k-2)\,x^2 + (-3k^2+5k+4)\,x + (2k^2-k-2). \end{aligned}$$

Calculating "$b^2 - 4ac$", we get

$$\begin{aligned} (-3k^2+5k+4)^2 - 4 \cdot (2k^2-k-2) \cdot (-4k-2) &= k^2 \cdot (9k^2+2k+1) \\ &= k^2 \cdot \tfrac{1}{9} \cdot ((9k+1)^2 + 8) \end{aligned}$$

The square root of this quantity is rational if and only if $(9k+1)^2 + 8$ is a perfect square, but this is never true for $k \geqslant 2$. Thus, none of the solutions for x is rational.

To extend the argument to $m > k+2$, add extra alternatives to the profile constructed above which no voter approves. Then any Nash solution places 0 on those extra alternatives, and the argument as above establishes that Nash chooses an irrational distribution.

A.2. $\mathbf{s} = (1, q = \frac{r}{s}, 0, \dots, 0)$

We start by considering $\mathbf{s} = (1, q, 0)$ and later show how to extend the argument to all vectors $\mathbf{s}$ with $s_1 = 1$ and $0 < s_2 < 1$. Write $X = \{a, x, y\}$, and consider the following profile:

- c voters: a, x, y
- 1 voter: x, a, y
- 1 voter: x, y, a

Note that x dominates y, so y gets probability 0 in Nash-optimum.

Thus, for some $a \in [0, 1]$, the Nash optimum p satisfies $p_a = a$, $p_x = 1 - a$, and $p_y = 0$. Note that $a < 1$ by positive share for the last voter, and $a > 0$ for large c since p is in the $\mathbf{s}$-core.

The Nash product of p, as a function of a, is

$$\begin{aligned} f(a) &= \left(a + \frac{r}{s}(1-a)\right)^c \cdot \left(1 - a + \frac{r}{s}a\right) \cdot (1-a) \\ &= s^{-c-1} \cdot (sa + r(1-a))^c \cdot (s - sa + ra) \cdot (1-a). \end{aligned}$$

Since p is optimal, we must have $\partial f(a)/\partial a = 0$. Using the product rule twice, we get

$$\begin{aligned} s^{c+1} \cdot \partial f(a)/\partial a = (1-a) \cdot \Big[& c \cdot (sa + r(1-a))^{c-1} \cdot (s-r) \cdot (s - sa + ra) \\ & + (sa + r(1-a))^c \cdot (r - s) \Big] \\ & - (sa + r(1-a))^c \cdot (s - sa + ra) \end{aligned}$$

Since $\partial f(a)/\partial a = 0$, factoring out the positive factor $(sa + r(1-a))^{c-1}$, we obtain

$$\begin{aligned} 0 &= (1-a) \cdot c \cdot (s-r) \cdot (s - sa + ra) + (1-a) \cdot (sa + r(1-a)) \cdot (r-s) \\ &\quad - (sa + r(1-a)) \cdot (s - sa + ra) \\ &= a^2 \cdot ((r-s)^2(c+2)) + a \cdot \big(-(r-s)((c+3)r - 2s(c+1))\big) + ((r^2 - 2rs - crs + cs^2)) \end{aligned}$$

Calculating "$b^2 - 4ac$" of this quadratic equation, we get

$$\begin{aligned} &\big(-(r-s)((c+3)r - 2s(c+1))\big)^2 - 4((r-s)^2(c+2))((r^2 - 2rs - crs + cs^2)) \\ &= (r-s)^2((c+1)^2 r^2 + 4rs + 4s^2) \end{aligned}$$

The square root of this quantity is rational if and only if $(c+1)^2 r^2 + 4rs + 4s^2$ is a perfect square, which is not true for large c.

For $|X| > 3$ and a score vector $\mathbf{s} = (1, q, \dots, 0)$, take the profile constructed above and add extra alternatives $z_1, \dots, z_f$ in penultimate position, i.e.,

- c voters: $a, x, z_1, \dots, z_f, y$
- 1 voter: $x, a, z_1, \dots, z_f, y$
- 1 voter: $x, y, z_1, \dots, z_f, a$

In this profile, x dominates the extra alternatives, so the extra alternatives get probability 0 in the Nash outcome. Hence the argument above implies that Nash selects an irrational distribution.

A.3. $\mathbf{s} = (1, 1, q = \frac{r}{s}, 0, \dots, 0)$

We start by considering $\mathbf{s} = (1, 1, q, 0)$ and later show how to extend the argument to all vectors $\mathbf{s}$ with $s_1 = s_2 = 1$ and $0 < s_3 < 1$. Write $X = \{a, b, x, y\}$, and consider the following profile:

- $c/2$ voters: $x \sim a, b, y$
- $c/2$ voters: $x \sim b, a, y$
- 1 voter: $a \sim b, y, x$
- 1 voter: $a \sim b, x, y$

We have written "$\sim$" for an implicit indifference, because the score vector is 1 in both the first and second position.

Then y is dominated by a and b, so gets probability 0. Also, a and b are symmetric (in the sense that after permuting them, every permuted distribution has the same Nash product as before) so they get the same probability. The probability of x cannot be 1 (by positive share), and it cannot be 0 if c is large, since otherwise the c voters ranking x first have an $\mathbf{s}$-core deviation.

Writing x for the probability put on x, we see that the optimum p satisfies $p_x = x$, $p_y = 0$, $p_a = p_b = (1-x)/2$. Thus, the Nash product is

$$\begin{aligned} f(x) &= \left(x + \frac{1-x}{2} + \frac{r}{s}\frac{1-x}{2}\right)^c \cdot (1-x) \cdot \left(1 - x + \frac{r}{s}x\right) \\ &= 2^{-c} \cdot s^{-c-1} \cdot (2sx + s(1-x) + r(1-x))^c \cdot (1-x) \cdot (s - sx + rx) \\ &= 2^{-c} \cdot s^{-c-1} \cdot (s + r + (s-r)x)^c \cdot (1-x) \cdot (s + (r-s)x). \end{aligned}$$

Next, we have, using the product rule twice,

$$\begin{aligned} 2^c \cdot s^{c+1} \cdot \partial f(x)/\partial x = (1-x) \cdot \Big[&c \cdot (s + r + (s-r)x)^{c-1} \cdot (s-r) \cdot (s + (r-s)x) \\ &+ (s + r + (s-r)x)^c \cdot (r-s)\Big] \\ &- (s + r + (s-r)x)^c \cdot (s + (r-s)x). \end{aligned}$$

In optimum, we must have $\partial f(x)/\partial x = 0$. Thus, factoring out the positive factor $(s+r+(s-r)x)^{c-1}$, this implies

$$\begin{aligned} 0 &= (1-x) \cdot \Big[c \cdot (s-r) \cdot (s + (r-s)x) + (s + r + (s-r)x) \cdot (r-s)\Big] \\ &\qquad - (s + r + (s-r)x) \cdot (s + (r-s)x) \\ &= x^2 \cdot \big(-(c+2)(r-s)^2\big) \\ &\quad + x \cdot \big((r-s)((c+3)r - 2sc)\big) \\ &\quad + \big(-r^2 + (c+1)rs - (c-2)s^2\big). \end{aligned}$$

Calculating "$b^2 - 4ac$" of this quadratic equation, we get

$$\begin{aligned} &\big((r-s)((c+3)r - 2sc)\big)^2 - 4 \cdot \big(-(c+2)(r-s)^2\big) \cdot \big(-r^2 + (c+1)rs - (c-2)s^2\big) \\ &= (r-s)^2 \left((c+1)^2 r^2 + 8rs + 16s^2\right) \end{aligned}$$

The square root of this quantity is rational if and only if $(c+1)^2 r^2 + 8rs + 16s^2$ is a perfect square, and this is not true for large c.

For $|X| > 4$ and a score vector $\mathbf{s} = (1, 1, q, \dots, 0)$, take the profile constructed above and add extra alternatives $z_1, \dots, z_f$ in penultimate position, i.e.,

- $c/2$ voters: $x \sim a, b, z_1, \dots, z_f, y$
- $c/2$ voters: $x \sim b, a, z_1, \dots, z_f, y$
- 1 voter: $a \sim b, y, z_1, \dots, z_f, x$
- 1 voter: $a \sim b, x, z_1, \dots, z_f, y$

In this profile, a dominates the extra alternatives, so the extra alternatives get probability 0 in the Nash outcome. Hence the argument above implies that Nash selects an irrational distribution.

A.4. $\mathbf{s} = (1, 1, \dots, 1, q = \frac{r}{s}, 0, \dots, 0)$

Like in the other cases, we explicitly handle the case where $\mathbf{s} = (1, 1, \dots, 1, q = \frac{r}{s}, 0)$. The argument can be extended to all $\mathbf{s}$ with $s_1 = s_2 = \cdots = s_k = 1$ and $0 < s_{k+1} < 1$ by adding extra alternatives to the constructed profile in the penultimate position, noting that alternative b_1 dominates the extra alternatives.

Write $X = \{a, b_1, \dots, b_{k-1}, x, y\}$, and consider the following profile:

- one voter: $b_1, \dots, b_{k-1}, x, y, a$
- one voter: $b_1, \dots, b_{k-1}, x, a, y$
- $c/(k-1)$ voters: $a, b_2, \dots, b_{k-1}, x, b_1, y$
- for $i = 2, \dots, k-1$, $c/(k-1)$ voters: $a, b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{k-1}, y, b_i, x$

Note that b_1 dominates y and that b_2 dominates x, so both x and y get probability 0. Further, once we restrict attention to distributions that put 0 on both x and y, we see that $b_1, \dots, b_{k-1}$ are symmetric, in the sense that the Nash product does not change when we permute these alternatives in input and output. Thus, they all get the same probability. Alternative a cannot get probability 1 by positive share, and it cannot get probability 0 or else the bottom c voters have an $\mathbf{s}$-core deviation. Thus, the Nash product as a function of the probability put on a is

$$\begin{aligned} f(a) &= \left(a + \frac{k-2}{k-1}(1-a) + \frac{1}{k-1}\frac{r}{s}(1-a)\right)^c \cdot (1-a) \cdot \left(1 - a + a\frac{r}{s}\right) \\ &= (k-1)^{-c} \cdot s^{-c-1} \cdot (s(k-1)a + s(k-2)(1-a) + r(1-a))^c \cdot (1-a) \cdot (s - sa + ra) \end{aligned}$$

Applying the product rule twice, we get

$$\begin{aligned} &(k-1)^c \cdot s^{c+1} \cdot \partial f(a)/\partial a \\ &= (1-a) \cdot \Big[c \cdot (s(k-1) - s(k-2) - r) \cdot (s(k-1)a + s(k-2)(1-a) + r(1-a))^{c-1} \cdot (s - sa + ra) \\ &\qquad + (s(k-1)a + s(k-2)(1-a) + r(1-a))^c \cdot (r - s)\Big] \\ &\quad - (s(k-1)a + s(k-2)(1-a) + r(1-a))^c \cdot (s - sa + ra) \end{aligned}$$

In optimum, we must have $\partial f(a)/\partial a = 0$. Thus, factoring out the positive factor $(s(k-1)a + s(k-2)(1-a) + r(1-a))^{c-1}$, this implies

$$\begin{aligned} 0 &= (1-a) \cdot \Big[c \cdot (s(k-1) - s(k-2) - r) \cdot (s - sa + ra) \\ &\qquad + (s(k-1)a + s(k-2)(1-a) + r(1-a)) \cdot (r - s)\Big] \\ &\quad - (s(k-1)a + s(k-2)(1-a) + r(1-a)) \cdot (s - sa + ra) \\ &= a^2 \cdot ((c+2)(r-s)^2) \\ &\quad + a \cdot ((r-s)((c+3)r - 2s(c+3-k))) \\ &\quad + (r^2 + (-4 - c + k)rs + (4 + c - 2k)s^2). \end{aligned}$$

Calculating the discriminant and simplifying, we get

$$(r-s)^2((c+1)^2r^2+4(k-1)rs+4(k-1)^2s^2)$$

The square root of this quantity is rational if and only if $((c+1)^2r^2+4(k-1)rs+4(k-1)^2s^2)$ is a perfect square, which is not the case for large c.

B. Additional details for Table 2

Maximal lotteries

On profile $(a \succ b, a \succ b, b \succ a)$, the maximal lottery is the deterministic distribution a, henceforth the maximal lottery rule violates individual fair share, and thus also the SD-core. The violation of monotonicity is known from Fishburn [1984]. Maximal lotteries can be computed by linear programming, and thus in polynomial time. For the satisfaction of SD-efficiency and the violation of strategyproofness see Aziz et al. [2018d].

Proportional Borda

Polynomial-time computability and monotonicity are straightforward. For strategyproofness see the discussion by Barberà [1979]. The violation of SD-core and ex-post efficiency can be seen on the one-voter profile $(a \succ b \succ c)$, which outputs $\frac{2}{3}a + \frac{1}{3}b$. For the satisfaction of individual fair share, we give a proof by case analysis. Let pb_x be the proportional Borda score of x for a given profile.

1. $m \geqslant 4$ and $n \geqslant 2$. If x is ranked last by at least one voter then its Borda score is at most $(n-1)(m-1)$, hence $pb_x \leqslant \frac{(n-1)(m-1)}{\frac{nm(m-1)}{2}} = \frac{2(n-1)}{nm} < \frac{2}{m} \leqslant \frac{1}{2} \leqslant 1 - \frac{1}{n}$.
2. $m \geqslant 3$ and $n \geqslant 3$. If x is ranked last by at least one voter then $pb_x \leqslant \frac{2(n-1)}{3n} < \frac{2}{3} \leqslant 1 - \frac{1}{n}$.
3. $m = 3$ and $n = 2$. If x is ranked last by at least one voter then $pb_x \leqslant \frac{0+2}{6} = \frac{1}{3} < 1 - \frac{1}{n}$.
4. $m = 2$. If x is ranked last by at least one voter then $pb_x \leqslant \frac{n-1}{n} = 1 - \frac{1}{n}$.
5. $n = 1$. If the voter ranks x last then $pb_x = 0 = 1 - \frac{1}{n}$.