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The Expanding Approvals Rule: Improving Proportional Representation and Monotonicity

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Abstract Proportional representation (PR) is often discussed in voting settings as a major desideratum. For the past century or so, it is common both in practice and in the academic literature to look towards the single transferable vote (STV) rule as the solution for achieving PR. Some of the most prominent electoral reform movements around the globe are pushing for the adoption of STV. It has been termed a major open problem to design a voting rule that satisfies the same PR properties as STV and better monotonicity properties. In this paper, we first present a taxonomy of proportional representation axioms for general weak order preferences, some of which generalise and strengthen previously introduced concepts. We then present a rule called the Expanding Approvals Rule (EAR) that satisfies properties stronger than the central PR axiom satisfied by STV, can handle indifferences in a convenient and computationally efficient manner, and also satisfies better candidate monotonicity properties. In view of this, our proposed rule seems to be a compelling solution for achieving proportional representation in voting settings.

Keywords committee selection · multiwinner voting · proportional representation · single transferable vote.

JEL Classification: C70 · D61 · D71

1 Introduction

> Of all modes in which a national representation can possibly be constituted, this one [STV] affords the best security for the intellectual qualifications desirable in the representatives—John Stuart Mill (Considerations on Representative Government, 1861).

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> A major unsolved problem is whether there exist rules that retain the important political features of STV and are also more monotonic—Woodall (1997).

We consider a well-studied (single-district) voting setting in which n voters express ordinal preferences over m candidates and based on the preferences $k \leq m$ candidates are selected.[1] The candidates may or may not be from particular parties but voters express preferences directly over individual candidates.[2] This kind of voting problem is not only encountered in parliamentary elections but also when forming any kind of representative body. When making such a selection by a voting rule, a desirable requirement is that of proportional representation. Proportional representation stipulates that voters should get representation in a committee or parliament according to the strengths of their numbers. It is widely accepted that proportional representation is the fairest way to reflect the diversity of opinions among the voters.[3]

For the last 120 years or so, the most widely used and accepted way to achieve PR in a single-district setting is via single transferable vote (STV) (Black, 1958; Tideman and Richardson, 2000) and its several variants. In fact STV is used for elections in several countries including Australia, Ireland, India, and Pakistan. It is also used to select representative committees in hundreds of settings including professional organisations, scientific organisations, political parties, school groups, and university student councils all over the globe.[4]

The reason for the widespread adoption of STV is partly due to the fact that it has been promoted to satisfy proportional representation axioms. In particular, STV satisfies a key PR axiom called *Proportionality for Solid Coalitions* (PSC) (Tideman and Richardson, 2000; Woodall, 1994). Tideman (1995) argues that "*It is the fact that STV satisfies PSC that justifies describing STV as a system of proportional representation.*" Woodall (1997) calls the property the "*essential feature of STV, which makes it a system of proportional representation*." Dummett (1984) motivated PSC as a minority not requiring to coordinate its report and that it should deserve some high preferred candidates to be selected as long as enough voters are 'solidly committed to such candidates.' PSC captures the idea that as long as voters have the same top candidates (possibly in different ordering), they do not need to coordinate their preferences to get a justified number of such candidates selected. In this sense, PSC is also similar in spirit to the idea that if a clone of a candidate is introduced, then it should not affect the selection of the candidate. Voters from the same party not having

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[1] Here single-district means that the winning candidate set is expected to represent all voters, rather than having the voters ex-ante partitioned into districts with each district having their own winning candidate(s). Note that in practice this 'single-district' is often an amalgamation of multiple lower-government level districts.

[2] The setting is referred to as a preferential voting system. It is more general and flexible than settings in which voters vote for their respective parties and then the number of seats apportioned to the parties is proportional to the number of votes received by the party (Pukelsheim, 2014).

[3] Proportional representation may be the fairest way for representation but it also allows for an extreme group to have some representation at least when the group is large enough. PR also need not be the most effective approach to a stable government. Black (1958) wrote that "It [PR] makes it difficult to form a cabinet which can command a parliamentary majority and so makes for weak government."

[4] Notable uses of STV include Oscar nominations, internal elections of the British Liberal Democrats, and selection of Oxford Union, Cambridge Union, and Harvard/Radcliffe Undergraduate Councils.

to coordinate their reports as to maximise the number of winners from their own party can be viewed as a weak form of group-strategyproofness. PSC can also be seen as a voter's vote not being wasted due to lack of coordination with like-minded voters. PSC has been referred to as "*a sine qua non for a fair election rule*" by Woodall (1994).[5]

Although STV is not necessarily the only rule satisfying PR properties, it is at times synonymous with proportional representation. The outcome of STV can also be computed efficiently which makes it suitable for large scale elections.[6] Another reason for the adoption of STV is historical. Key figures proposed ideas related to STV or pushed for the adoption of STV. The ideas behind STV can be attributed to several thinkers including C. Andrae, T. Hare, H. R. Droop, and T. W. Hill. For a detailed history of the development of STV family of rules, please see the article by Tideman (1995). In a booklet, Aiyar (1930) explains the rationale behind different components of the STV rule. STV was supported by influential intellectuals such as John Stuart Mill who placed STV "*among the greatest improvements yet made in the theory and practice of government*." Bowler and Grofman (2000) note the British influence on the spread of STV among countries with historical association with Great Britain.

With a historical, normative, and computational motivation behind it, STV has become the 'go to' rule for PR and has strong support.[7] It is also vigorously promoted by prominent electoral reform movements across the globe including the Proportional Representation Society of Australia (`http://www.prsa.org.au`) and the Electoral Reform Society (`https://www.electoral-reform.org.uk`).

Despite the central position of STV, it is not without some flaws. It is well-understood that it violates basic monotonicity properties even when selecting a single candidate (see, e.g., Doron and Kronick, 1977; Zwicker, 2016). Increasing the ranking of the winning candidate may result in the candidate not getting selected. STV is also typically defined for strict preferences which limits its ability to tackle instances where voters have general weak preferences. There are several settings where voters may be indifferent between two candidates because the candidates have the same characteristics that the voter cares about. It could also be that the voter does not have the cognitive power or time to distinguish between two candidates and does not wish to break ties arbitrarily. It is not clearly resolved in the literature how STV can be extended to handle weak orders without compromising on its computational efficiency

[5] There are two PSC axioms that differ in only whether the Hare quota is used or whether the Droop quota is used. The one with respect to the Droop quota has also been referred to as DPC (Droop's proportionality criterion) (Woodall, 1994). Woodall (1994) went as far as saying that "I assume that no member of the Electoral Reform Society will be satisfied with anything that does not satisfy DPC."

[6] Although it is easy to compute one outcome of STV, checking whether a certain set is a possible outcome of STV is NP-complete (Conitzer et al., 2009).

[7] One notable exception was philosopher Michael Dummett who was a stringent critic of STV. He proposed a rival PR method called the Quota Borda System (QBS) and pushed its case (Dummett, 1984, 1997). However, even he agreed that in terms of achieving PR, "[STV] guarantees representation for minorities to the greatest degree to which any possible electoral system is capable of doing" (Dummett, 1997)[page 137].

or some of the desirable axiomatic properties it satisfies.[8] The backdrop of this paper is that improving upon STV in terms of both PR as well as monotonicity has been posed as a major challenge (Woodall, 1997).

Contributions We propose a new voting rule called *Expanding Approvals Rule (EAR)* that has several advantages. (1) It satisfies an axiom called Generalised PSC that is stronger than PSC. (2) It satisfies some natural monotonicity criteria that are not satisfied by STV. (3) It is defined on general weak preferences rather than just for strict preferences and hence constitutes a flexible and general rule that finds a suitable outcome in polynomial time for both strict and dichotomous preferences. Efficient computation of a rule is an important concern when we deal with large-scale elections.

Our work also helps understand the specifications under which different variants of STV satisfy different PR axioms. Apart from understanding how far STV and EAR satisfy PR axioms, one of the conceptual contributions of this paper is to define a taxonomy of PR axioms based on PSC and identify their relations with each other. In particular, we propose a new axiom for weak preferences called *Generalised PSC* that simultaneously generalises PSC (for strict preferences) and *Proportional Justified Representation (PJR)* (for dichotomous preferences).

Outline In Section 2, we present our model and review the relevant PR and monotonicity axioms studied in the literature. In Section 3, we extend these PR axioms to the more general setting where voters may have weak preferences over candidates, and establish connections between these new axioms and other known PR axioms. Sections 4 and 5 present our main contributions: we define a new voting rule called *Expanding Approvals Rule* (EAR), and analyse its properties with respect to the PR and monotonicity axioms. Section 6 provides a formal study of the prominent, and widely-used, family of STV rules – serving as a useful benchmark to consider the merits of EAR. Section 7 briefly reviews some other PR-motivated voting rules, and then Section 8 concludes the paper.

2 Model and Axioms

In this section, we lay the groundwork of the paper by first defining the model and then formalising the central axioms by which proportional representation rules are judged.

2.1 Model

We consider the standard social choice setting with a set of voters $N = \{1, \dots, n\}$, a set of candidates $C = \{c_1, \dots, c_m\}$ and a preference profile $\succsim = (\succsim_1, \dots, \succsim_n)$ such that

[8] Hill (2001) and Meek (1994) propose one way to handle indifferences but it leads to an algorithm that may take time $O(m!)$.

each $\succsim_i$ is a complete and transitive relation over C. Based on the preference profile, the goal is to select a committee $W \subset C$ of predetermined size k. Since our new rule is defined over weak preferences rather than only strict preferences, we allow the voters to express weak preferences over candidates. We assume that no voter is completely indifferent between all candidates, i.e., for each $i \in N$ we require there to exist candidates $c, c' \in C$ such that $c \succsim_i c'$ holds but $c' \succsim_i c'$ does not.

We write $a \succsim_i b$ to denote that voter i values candidate a at least as much as candidate b and use $\succ_i$ for the strict part of $\succsim_i$, i.e., $a \succ_i b$ if and only if $a \succsim_i b$ but not $b \succsim_i a$. Finally, $\sim_i$ denotes i's indifference relation, i.e., $a \sim_i b$ if and only if both $a \succsim_i b$ and $b \succsim_i a$. The relation $\succsim_i$ results in (non-empty) equivalence classes $E_i^1, E_i^2, \ldots, E_i^{m_i}$ for some m_i such that $a \succ_i a'$ if and only if $a \in E_i^l$ and $a' \in E_i^{l'}$ for some $l < l'$. Often, we will use these equivalence classes to represent the preference relation of a voter as a preference list $i : E_i^1, E_i^2, \ldots, E_i^{m_i}$. If candidate c is in E_i^j, then we say it has *rank* j in voter i's preference. For example, we will denote the preferences $a \sim_i b \succ_i c$ by the list $i : \{a, b\}, \{c\}$. In this case, candidates a and b have rank 1 and candidate c has rank 2. Abusing this notation slightly, when we restrict our attention to strict preferences we will omit braces, i.e., the preference $a \succ_i b \succ_i c$ will be denoted by the comma-separated list $i : a, b, c$. If each equivalence is of size 1, then the preference will be a called *strict preference*. If for each voter, the number of equivalence classes is at most two, the preferences are referred to as *dichotomous preferences*. When the preferences of the voters are dichotomous, the voters can be seen as approving a subset of voters. In this case for each voter $i \in N$, the first equivalence class E_i^1 is also referred to as an *approval ballot*, and is denoted by $A_i \subseteq C$. Note that in this special case, where a voter i has dichotomous preferences, the approval set A_i contains all information about voter i's preference. The vector $A = (A_1, \ldots, A_n)$ is referred to as the *approval ballot profile*.

The model allows for voters to express preference lists that do not include some candidates. In that case, the candidates not included in the list will be assumed to form an additional and lowest equivalence class.

When stating results we will use the following convention. Results which apply to special instances where all voters have preferences in a restricted domain, i.e., strict or dichotomous, will have this restriction explicitly stated. When no restriction is stated, the result holds for the general case where voters have weak preferences.

2.2 PR under Strict Preferences

In order to understand the suitability of voting rules for proportional representation, we recap the central PR axiom from the literature called Proportionality for Solid Coalitions (PSC). It was first mentioned and popularised by Dummett (1984). Dummett's original definition relating to PR focuses on a setting where voters are assumed to have strict preferences. Accordingly, we only consider Dummett's PSC axiom and the underlying 'solid coalition' definition in this context.[9]

[9] Later we provide a generalised version which accounts for weak preferences in a more appropriate manner. This is necessary since in some cases Dummett's axiom applied out-of-context to a setting where

We now present Dummett's definition of a solid coalition when voters have strict preferences. This definition is central to Dummett's PSC axiom. Informally speaking, a set of voters, N', form a solid coalition for a set of candidates C' if every voter in N' agrees that every candidate in C' is strictly more preferred to every candidate outside of C'.

Definition 1 (Solid coalition) Suppose all the voters have strict preferences. A set of voters N' is a *solid coalition* for a set of candidates C' if every voter in N' strictly prefers every candidate in C' over of every candidate in $C \backslash C'$. That is, for all $i \in N'$ and for any $c' \in C'$

$$\forall c \in C \backslash C' \quad c' \succ_i c.$$

The candidates in C' are said to be *supported* by the voter set N', and conversely the voter set N' is said to be *supporting* the candidate set C'.

Importantly, the definition of a solid coalition does not require voters in the set N' to maintain the same order of strict preferences among candidates in C' nor $C \backslash C'$. The definition requires only that all candidates in C' are strictly preferred to those in $C \backslash C'$. Also, notice that a set of voters N' may be a solid coalition for multiple sets of candidates and that the entire set of voters N is trivially a solid coalition for the set of all candidates C.

Given the definition of a solid coalition, we can now present Dummett's PSC axiom. The axiom is parameterised by a value $q \in (n/(k+1), n/k]$. Informally, a committee W satisfies PSC if every solid coalition of voters is 'sufficiently represented' in the committee. In particular, for any solid coalition of voters N' with size at least ℓq (for some positive integer ℓ) supporting a candidate set C' it must be that at least ℓ candidates from C' are in the committee W if $|C'| \geq \ell$; and otherwise, all candidates C' must be in the committee. Varying the parameter q leads to a hierarchy of PSC axioms denoted by q-PSC. One interpretation of this q parameter is that it prescribes the minimal size of a set of voters who if they all had a common most preferred candidate must have this candidate contained in the winning committee for q-PSC to be satisfied.[10]

Definition 2 (q-PSC) Suppose all voters have strict preferences and let $q \in (n/(k+1), n/k]$. We say a committee W satisfies q-PSC if for every positive integer ℓ, and for every solid coalition N' supporting a candidate subset C' with size $|N'| \geq \ell q$, then the following holds

$$|W \cap C'| \geq \min\{\ell, |C'|\}.$$

If $q = n/k$, then we refer to the property as Hare-PSC. If $q = n/(k+1) + \epsilon$ for sufficiently small $\epsilon > 0$, then we refer to the property as Droop-PSC.[11] Formally, ϵ

voters have weak preferences can lead to the PSC axiom being trivially satisfied due to the non-existence of a (non-trivial) solid coalition.

[10] A reason for restricting the parameter q to the interval $(n/(k+1), n/k]$ is as follows. If $q \leq n/(k+1)$, then a q-PSC committee need not exist. If $q > n/k$, then a committee may satisfy q-PSC but also maintain counterintuitive properties. For example, it may be the case that a committee W satisfies q-PSC while containing candidate(s) who are unanimously less preferred than some candidate not in W.

[11] Droop PSC is also referred to as Droop's proportionality criterion (DPC). Technically speaking the Droop quota is $\lfloor n/(k+1) \rfloor + 1$. The exact value $n/(k+1)$ is referred to as the *Hagenbach-Bischoff* quota.

is required to be positive and small enough so that that for any $\ell \leq k$,

$$\lceil \ell \cdot q \rceil \leq \ell \frac{n}{k+1} + 1, \tag{1}$$

where $q = n/(k+1) + \epsilon$. The inequality (1) ensures that if there exists a solid coalition N' of size $|N'| > \ell \frac{n}{k+1}$ supporting a candidate set C', then any q-PSC committee contains at least ℓ candidates from C'. To see this, note that if $|N'| > \ell \frac{n}{k+1}$, then $|N'| > \lceil \ell q \rceil - 1$ and so $|N'| \geq \lceil \ell q \rceil \geq \ell q$ since $|N'|$ is integer valued. The requirement that ϵ is positive guarantees that a q-PSC committee always exists. We denote the *Hare quota* value, n/k, by q_H. Abusing notation slightly, we denote any quota value $n/k + \epsilon$ with $\epsilon > 0$ satisfying (1) by q_D and refer to it as the *Droop quota*.[12]

There are some reasons to prefer the Droop quota $n/(k+1) + \epsilon$ for sufficiently small $\epsilon > 0$. Firstly, for $k = 1$ the use of the Droop quota leads to q-PSC rules returning a candidate that is most preferred by more than half of the voters whenever such a candidate exists. Hare-PSC was stated as an essential property that a rule designed for PR should satisfy (Dummett, 1984). When preferences are strict and $k = 1$, Woodall (1997) refers to the restriction of Droop-PSC under these conditions as the *majority principle*. The majority principle requires that if a majority of voters form a solid coalition for a set of candidates C', then one of the candidates from C' must be selected as a winning candidate. [13]

Example 1 Suppose the candidate set is $C = \{c_1, c_2, c_3, c_4, e_1, e_2, e_3\}$ and consider the following profile of strict preferences with 9 voters.

$$\begin{array}{rl}
1: & c_1, c_2, c_3, c_4, ... \\
2: & c_4, c_1, c_2, c_3, ... \\
3: & c_2, c_3, c_4, c_1, ... \\
4: & e_1, e_2, e_3, ... \\
5: & e_1, e_2, e_3, ... \\
6: & e_1, e_2, e_3, ... \\
7: & e_1, e_2, e_3, ... \\
8: & e_1, e_2, e_3, ... \\
9: & e_1, e_2, e_3, ...
\end{array}$$

[12] For fixed n and k, any pair of Droop quotas q_1, q_2 are equivalent in terms of the q-PSC requirement. To see this, note that if $|N'| \geq \ell q_2$ then $|N'| > \ell \cdot n/(k+1)$ but then by (1) it must be that $|N'| \geq \ell q_1$.

[13] A related property for general k is the *fixed majority principle* (Debord, 1993) which requires that if a majority of voters strictly prefers each candidate in a k-sized set Y to each candidate in $C \backslash Y$, then Y should be the outcome. For $k = 1$, this property is slightly more general than the majority principle of Woodall (1997) since it allows indifference within the candidate sets Y and $C \backslash Y$, while Woodall's focus is on settings where voters have strict preferences. For $k = 1$, *fixed majority principle* is a strictly weaker concept than generalised Droop-PSC (to be presented in Definition 6). However, for $k > 1$, the *fixed majority principle* is incomparable with any of our PSC concepts and is not aligned with proportional representation. For discussion on other properties in multiwinner voting that are related to majority comparisons, we refer to the papers by Gehrlein (1985) and Aziz et al. (2017b).

The voters in set $N' = \{1,2,3\}$ form a solid coalition supporting the candidate set $C' = \{c_1, c_2, c_3, c_4\}$. The voters in set $N'' = \{4,5,6,7,8,9\}$ form a solid coalition supporting three candidate subsets $\{e_1\}$, $\{e_1, e_2\}$ and $\{e_1, e_2, e_3\}$. Furthermore, the entire voter set N forms a (trivial) solid coalition supporting the entire candidate set C. $\diamond$

One can also define a weak version of PSC, called weak PSC – note that this weak version is still defined only for strict preferences. In some works (see, e.g. page 612 of Elkind et al. (2017a) or Definition 2.9 of Faliszewski et al. (2017)), weak PSC has been attributed to the original definition of PSC as defined by Dummett.

Weak PSC can be viewed as a minimal PR axiom, and is an important axiom for the focus of this paper. We will later show that a prominent voting rule used in practice, STV, can fail even this weak version of PSC when formulated inappropriately (i.e., with discrete reweighting). We provide links between weak PSC and a number of popular other PR axioms. Furthermore, the relaxation of PSC to weak PSC has been shown to be a sufficient weakening of PSC to allow for the desired monotonicity properties suggested by Woodall (1997) (a discussion is provided within Section 7). In this paper, we do not wish to relax PSC in our pursuit to attain improved monotonicity properties, but nonetheless, the discussion of weak PSC is highly relevant for a taxonomy of PR axioms.

Weak PSC is understood similarly to PSC. The weakening of PSC to weak PSC is attained by only considering solid coalitions of voters, N', supporting a candidates set C' which are of a restricted size. In particular, weak PSC requires that for any solid coalition of voters N' with size at least ℓq (for some positive integer ℓ) supporting candidate set C' where $|C'| \leq \ell$, it must be that all candidates from C' are contained in the winning committee W.

Definition 3 (weak q-PSC) Suppose all voters have strict preferences and let $q \in (n/(k+1), n/k]$. A committee W satisfies weak q-PSC if for every positive integer ℓ, and for every solid coalition N' supporting a candidate subset $C' : |C'| \leq \ell$ with size $|N'| \geq \ell q$, then the following holds

$$|W \cap C'| \geq \min\{\ell, |C'|\} = |C'|,$$

i.e., $C' \subseteq W$.

For weak q-PSC, we restrict our attention to solid coalitions that support sets of candidates of size at most ℓ whereas in q-PSC we impose no such restriction. Note that q-PSC implies weak q-PSC but the reverse need not hold. Furthermore, the condition that $|C'| \leq \ell$ and $|W \cap C'| \geq \min\{\ell, |C'|\}$ is equivalent to $C' \subseteq W$. We also note that under strict preferences and $k = 1$, if a majority of the voters have the same most preferred candidate, then weak Droop-PSC implies that the candidate is selected. In particular, this implies that the majority principle is satisfied when weak-Droop PSC is satisfied.

We now present a lemma connecting (weak) q-PSC for different values of q.[14] The proof is omitted since it is implied by a stronger lemma (Lemma 2) proven in Section 3.

[14] The lemma is stated for $q \in (n/(k+1), n/k]$, however, more generally it holds for any positive q.

Lemma 1 *Suppose all voters have strict preferences and let $q \in (n/(k+1), n/k]$. If a committee W satisfies (weak) q-PSC, then, for any $q' > q$, the committee W also satisfies (weak) q'-PSC*

In Section 3, we generalise the PSC property to weak preferences which has not been done in the literature.

2.3 Candidate Monotonicity Axioms

PR captures the requirement that cohesive groups of voters should get sufficient representation. Another desirable property is *candidate monotonicity*. Informally speaking, candidate monotonicity requires that increased support for an otherwise-elected candidate should never cause this candidate to become unelected. To provide a formal definition we must introduce the notion of a candidate being reinforced. We say a candidate is *reinforced* if its relative position is improved while not changing the relative positions of all other candidates.

More formally, we say that candidate c is *reinforced* in preference $\succsim_i$ to obtain preference $\succsim_i'$ if

- for all $a, b \in C \setminus \{c\}$
$$a \succsim_i b \iff a \succsim_i' b$$
- there exists a $d \in C$ such that ($d \succsim_i c$ and $c \succ_i' d$) or ($d \succ_i c$ and $c \sim_i' d$).

When the first condition ($d \succsim_i c$ and $c \succ_i' d$) is satisfied in the second bullet point, candidate c is said to *cross over* candidate d, and otherwise, is said to *not cross over* candidate d.

We are now in a position to formalise some natural candidate monotonicity properties of voting rules. At the end of this section, we will briefly discuss the connections between these definitions and those presented elsewhere in the literature. The definitions we present below apply not just to strict preferences but also to weak preferences. Two of the definitions are based on the ranks of candidates as specified in the preliminaries.

Definition 4 (Candidate Monotonicity)

- *Candidate Monotonicity (CM)*: if a winning candidate c is reinforced by a single voter, then c remains a winning candidate.
- *Rank Respecting Candidate Monotonicity (RRCM)*: if a winning candidate c is reinforced by a single voter without changing the ranks of other winning candidates in each voter's preferences, then c remains a winning candidate.
- *Non-Crossing Rank Respecting Candidate Monotonicity (NCRRCM)*: if a winning candidate c is reinforced by a single voter without changing the ranks of other winning candidate in each voter's preference, and without ever crossing over another winning candidate, then c remains a winning candidate.

To illustrate the definition of a reinforcement and also highlight the distinction between the various candidate monotonicity axioms we provide the following example.

Example 2 Let $\succsim$ be a profile of voters' weak preferences over a candidate set $C = \{c_1, \ldots, c_7\}$, and let W be the winning committee under some voting rule. Suppose voter i has preferences

$$\succsim_i = \{c_1, c_2\}, \{c_3\}, \{c_4, c_5, c_6\}, \{c_7\},$$

and $c_4 \in W$.

We consider three different reinforcements of candidate c_4 by voter i to illustrate the monotonicity axioms:

$$\begin{aligned}
\succsim_i^1 &= \{c_1, c_2\}, \{c_3, c_4\}, \{c_5, c_6\}, \{c_7\} \\
\succsim_i^2 &= \{c_1, c_2\}, \{c_4\}, \{c_3\}, \{c_5, c_6\}, \{c_7\} \\
\succsim_i^3 &= \{c_1, c_2, c_4\}, \{c_3\}, \{c_5, c_6\}, \{c_7\}.
\end{aligned}$$

Notice that despite all of the preferences $\succsim_i^1, \succsim_i^2, \succsim_i^3$ being reinforcements of candidate c_4 they each differ qualitatively. In particular, $\succsim_i^1$ does not change the ranks of any candidate in $C \setminus \{c_4\}$, and candidate c_4 crosses candidates $\{c_5, c_6\}$ but does not cross any other candidate. While, in both $\succsim_i^2, \succsim_i^3$ candidate c_4 crosses candidates $\{c_3, c_5, c_6\}$ but the ranks of candidates only changes in $\succsim_i^2$, e.g., the ranks of all candidates in $C \setminus \{c_1, c_2, c_4\}$ are increased (lower equivalence class) in $\succsim_i^2$, and in $\succsim_i^3$ no candidate in $C \setminus \{c_4\}$ has their rank changed.

To illustrate the different monotonicity axioms, suppose that $W = \{c_4, c_3, c_7\}$, and let W^1, W^2, W^3 be the winning committee when candidate c_4 has been reinforced by voter i via the preference $\succsim_i^1, \succsim_i^2, \succsim_i^3$. The CM axiom requires that $c_4 \in W^1, W^2, W^3$ since candidate c_4 is a winning candidate in W and is reinforced in $\succsim_i^1, \succsim_i^2$, and $\succsim_i^3$. RRCM only requires that $c_4 \in W^1, W^3$ since the ranks of the other winning candidates in W, i.e., $\{c_3, c_7\}$, are unchanged in $\succsim_i^1$, and $\succsim_i^3$. NCRRCM only requires that $c_4 \in W^1$ since $\succsim_i^1$ is the only reinforcement out of $\succsim_i^1, \succsim_i^2, \succsim_i^3$ that does not have candidate c_4 crossing another winning candidate. $\diamond$

We observe the following relations between the properties.

Proposition 1 *The following relations hold.*

- CM $\implies$ RRCM $\implies$ NCRRCM.
- *Under* $k = 1$, RRCM, NCRRCM, *and* CM *are equivalent.*

Furthermore,

- *When all voters have strict preferences,* RRCM *and* NCRRCM *are equivalent.*
- *When all voters have dichotomous preferences,* RRCM *and* CM *are equivalent.*[15]

CM (Candidate Monotonicity)

$\downarrow$

RRCM (Rank Respecting Candidate Monotonicity)

$\downarrow$

NCRRCM (Non-Crossing Rank Respecting Candidate Monotonicity)

Fig. 1 Relations between candidate monotonicity axioms

Proposition 1 follows immediately from the relevant candidate monotonicity definitions. Note that if a rule fails CM for $k = 1$, then it also fails RRCM and NCRRCM. NCRRCM is an extremely weak property but, as will be shown in Section 6, the widely-used STV rule violates it even for $k = 1$ (see Proposition 10).

We now discuss the connections between the monotonicity axioms in Definition 4 and the existing literature. Monotonicity axioms were first formally introduced to the social choice literature by Arrow (1959, Condition 2) and have since continued to receive attention; see for example, Smith (1973); Fishburn (1982); Sanver and Zwicker (2012). In the setting where voters are assumed to have strict preferences, our CM definition is a natural extension of the monotonicity axioms considered in Arrow (1959); Sanver and Zwicker (2009); Smith (1973); Fishburn (1982); Sanver and Zwicker (2012), which only apply to the single-winner ($k = 1$) case.[16] More recently, monotonicity axioms have been considered in the multiwinner, or committee, voting setting and have also been considered for more general preference domains (Lackner and Skowron, 2018; Sánchez-Fernández and Fisteus, 2019; Elkind et al., 2017b). Lackner and Skowron (2018) and Sánchez-Fernández and Fisteus (2019) focus on the setting where voters are assumed to have dichotomous preferences and the voting rule is *irresolute*; that is, the rule outputs a set of 'winning' k-sized committees. Both works apply the same monotonicity axiom and this coincides with our CM axiom (after adapting the definition for a resolute voting rule).[17] Note that under dichotomous preferences CM and RRCM are equivalent (Propostion 1). Elkind et al. (2017b) focuses on the multiwinner setting where voters are assumed to have strict preferences and the voting rule is irresolute. The candidate monotonicity and non-crossing monotonicity axioms presented by the authors correspond to our CM

[15] Recall that it is assumed that no voter is completely indifferent between all candidates.

[16] Some of these papers consider *irresolute* voting rules whereby the rule selects a set of potential winners (of arbitrary size) rather than a single winner.

[17] Sánchez-Fernández and Fisteus (2019) also consider two other monotonicity axioms that involve increasing the number of voters, and allowing (simultaneous) reinforcements by groups of voters.

and NCRRCM axioms presented in Definition 4 (after adapting the definition for a resolute voting rule).[18]

3 PR under generalised preference relations

The notion of a solid coalition and PSC can be generalised to the case of weak preferences. In this section, we propose a new axiom called *generalised PSC* which not only generalises the PSC axiom from Section 2.2 (that is only defined for strict preferences) but also *Proportional Justified Representation* (PJR) a PR axiom that is only defined for dichotomous preferences.

We begin by defining a generalised solid coalition. Informally speaking, a set of voters N' forms a generalised solid coalition for a set of candidates C' if every voter in N' agrees that every candidate in C' is weakly preferred to every candidate outside of C'. Note that the point of distinction between a solid coalition (Definition 1) and a generalised solid coalition is simply in the requirement of each voter in a solid coalition must strictly prefer candidates in C' to those outside of C'.

Definition 5 (Generalised solid coalition) Suppose voters have weak preferences. A set of voters N' is a *generalised solid coalition* for a set of candidates C' if every voter in N' weakly prefers every candidate in C' at least as high as every candidate in $C \backslash C'$. That is, for all $i \in N'$ and for any $c' \in C'$

$$\forall c \in C \backslash C' \quad c' \succsim_i c.$$

The candidates in C' are said to be *supported* by the voter set N', and conversely the voter set N' is said to be *supporting* the candidate set C'.[19]

We note that under strict preferences, a generalised solid coalition is equivalent to a solid coalition (Definition 1).

Defining generalised q-PSC is more involved than q-PSC due to the presence of indifferences. However, the intuition is similar. We require that every generalised solid coalition of voters N' is sufficiently represented in the winning committee. But now due to indifferences, when attempting to represent a generalised solid coalition of voters N' supporting candidate set C', representation can be provided via a candidate $c'' \notin C'$ which some voter in N' finds indifferent to candidate(s) in C'. In particular, a committee W satisfies generalised PSC if for any generalised solid coalition of voters N' with size at least ℓq (for some positive integer ℓ) supporting candidate set C' it must be that there exists another set of candidates $C'' \subseteq W$ contained within the winning committee of size at least $\min\{\ell, |C'|\}$ such that for any candidate in C'' at least one voter in the solid coalition N' weakly prefers this candidate to their $|C'|$-th most preferred candidate.

[18] Elkind et al. (2017b) also consider a committee monotonicity axioms that involves comparing outcomes of voting rules as the size of the winning committee, k, increases. For resolute voting rules, Barberà and Coelho (2008) introduced a similar axiom referred to as *enlargement consistency*.

[19] This terminology of “supporting” and “supported” is also used in the context of strict preferences in Definition 1. It will be clear from context whether the intended meaning is with respect to the generalised solid coalition definition or the original solid coalition definition.

To assist in providing a formal definition we first introduce some notation. Let $c^{(i,j)}$ denote voter i's j-th most preferred candidate or one such candidate if indifferences are present. To attain such a candidate $c^{(i,j)}$ in the presence of indifferences the following procedure can be used: (1) break all ties in voter i's preferences temporarily to get an artificial strict order and (2) identify the j-th candidate $c^{(i,j)} := d$ in the artificial strict order.[20]

Definition 6 (Generalised q-PSC) Suppose voters have weak preferences and let $q \in (n/(k+1), n/k]$. A committee W satisfies *generalised q-PSC* if for every positive integer ℓ, and for all generalised solid coalitions N' supporting candidate subset C' with size $|N'| \geq \ell q$, there exists a set $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\}$ such that for all $c'' \in C''$

$$\exists i \in N' : \quad c'' \succsim_i c^{(i,|C'|)}.$$

Remark 1 The q-PSC axiom (Definition 2) introduced in Section 2.2 for strict preferences can be naturally extended to weak preferences by modifying the q-PSC definition to consider generalised solid coalitions rather than solid coalitions. An issue with this approach is that in general there need not exist a committee satisfying this axiom.

Similar to the weakening of q-PSC to weak q-PSC, we define the Generalised weak q-PSC axiom. This is a natural weakening of generalised q-PSC in which we require that the supported set of candidates C' is of size at most ℓ.

Definition 7 (Generalised weak q-PSC) Suppose voters have weak preferences and let $q \in (n/(k+1), n/k]$. A committee W satisfies weak *generalised q-PSC* if for every positive integer ℓ, and every generalised solid coalition N' supporting a candidate subset $C' : |C'| \leq \ell$ with size $|N'| \geq \ell q$, there exists a set $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\} = |C'|$ such that for all $c'' \in C''$

$$\exists i \in N' : \quad c'' \succsim_i c^{(i,|C'|)}.$$

The following example shows that generalised q-PSC is a weak property when solid coalitions equal, or just barely exceed, the quota q.

Example 3 Let $N = \{1,2,3,4\}$, $C = \{a,b,\ldots,j\}$, $k = 2$, and suppose voter 1 and 2's preferences are given as follows:

$$\begin{aligned} 1 : &\quad \{a,b,\ldots,g,\ h,\ j\},\ \{i\} \\ 2 : &\quad \{a\},\{b\},\ldots,\{h\},\ \{i\},\ \{j\} \end{aligned}$$

We consider generalised PSC with respect to the Hare quota; that is, $q_H = n/k = 2$. There is a generalised solid coalition $N' = \{1,2\}$ with $|N'| \geq q_H$ supporting candidate subset $C' = \{a\}$. The generalised q_H-PSC axiom requires the election of one ($\ell = 1$)

[20] We note that despite this procedure choosing an arbitrary candidate when indifferences are present, the definitions involving $c^{(i,j)}$ are independent of this candidate choice. This follows since the definitions utilise only the relative rank of candidate $c^{(i,j)}$ with respect to $\succsim_i$. Thus, our results and definitions are not simply attained by deriving a profile of strict preferences from a profile of weak preferences.

candidate into W who is weakly preferred to either voter 1's most preferred candidate, i.e., any candidate in $C\backslash\{i\}$ will suffice, or voter 2's most preferred candidate, i.e., candidate a. Since voter 1 is indifferent between all candidates in $C\backslash\{i\}$, electing candidate $j \in C$ will satisfy the axiom – this is despite candidate j being voter 2's least preferred candidate. ⋄

Remark 2 We note that our generalised PSC concept for weak preferences cannot be recovered by breaking ties to make the agents' preferences strict and then requiring that PSC be satisfied for these strict preferences. Consider the following example with 2 voters, candidate set $C = \{a, b, c, d\}$, a desired winning committee of size $k = 1$. Suppose the voters have weak preferences as follows.

$$\begin{aligned}1 :&\quad \{b\}, \{c\}, \{d\}, \{a\},\\ 2 :&\quad \{a, b\}, \{d\}, \{c\}.\end{aligned}$$

If we use our generalised PSC axiom one sees that voters 1 and 2 form a generalised solid coalition for candidate set $\{b\}$ and applying the Hare quota ($q_H = 2$) we would need $b \in W$ or $a \in W$. However, if instead we break ties, say alphabetically, we get the following strict preferences.

$$\begin{aligned}1 :&\quad \{b\}, \{c\}, \{d\}, \{a\},\\ 2 :&\quad \{a\}, \{b\}, \{d\}, \{c\}.\end{aligned}$$

With these new preferences, voters 1 and 2 no longer form a solid coalition for candidate set $\{b\}$. Instead, they only form a solid coalition for the trivial candidate set $C = \{a, b, c, d\}$. Applying the PSC axiom (the non-generalised version) to these strict preference with the Hare quota enforces no representation criterion on the winning set. In particular, a winning committee $W' = \{d\}$ satisfies Hare-PSC with respect to the induced strict preferences. Of course, such a committee W' does not satisfy generalised Hare-PSC with respect to the original weak preferences. Furthermore, this outcome is intuitively less representative of the original preferences.

We now show that if a committee W satisfies generalised (weak) q-PSC, then the committee also satisfies generalised (weak) q'-PSC for all $q' > q$.

Lemma 2 *Suppose all voters have strict preferences and let q, q' be real numbers such that $q < q'$. If a committee W satisfies generalised (weak) q-PSC, then W satisfies generalised (weak) q'-PSC.*

Proof. Let $q < q'$ and suppose that the committee W satisfies generalised (weak) q-PSC. We wish to show that (weak) generalised q'-PSC is also satisfied by W. To see this, notice that any generalised solid coalition N' requiring representation under generalised (weak) q'-PSC also requires at least as much representation under generalised (weak) q-PSC since $|N'| \geq \ell q'$ implies that $|N'| \geq \ell q$. □

Under strict preferences, generalised PSC and generalised weak PSC coincide with PSC and weak PSC, respectively. Generalising PSC and weak PSC to the case of weak preferences is important because it provides a useful link with PR properties defined on dichotomous preferences. Proportional Justified Representation (PJR) (Sánchez-Fernández et al., 2017b; Aziz and Huang, 2016) is a proportional representation property for dichotomous preferences (Laslier and Sanver, 2010; Aziz et al., 2017a), and is defined as follows.

Definition 8 (PJR) Suppose all voters have dichotomous preferences. A committee W with $|W| = k$ satisfies PJR for an approval ballot profile $\boldsymbol{A} = (A_1, \ldots, A_n)$ over a candidate set C if for every positive integer $\ell \leq k$ there does not exists a set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \frac{n}{k}$ such that

$$\Big|\bigcap_{i \in N^*} A_i\Big| \geq \ell \qquad \text{but} \qquad \Big|(\bigcup_{i \in N^*} A_i) \cap W\Big| < \ell.$$

Proposition 2 *Under dichotomous preferences, Generalised weak Hare-PSC implies PJR.*

Proof. For the sake of contradiction, let W be a committee of size k and suppose that generalised weak Hare-PSC holds but PJR does not. If PJR does not hold, then there must exist a set N^* of voters and a positive integer ℓ such that $|N^*| \geq \ell \frac{n}{k} = \ell q_H$ (where q_H is the Hare quota) and both

$$\Big|\bigcap_{i \in N^*} A_i\Big| \geq \ell \qquad \text{and} \qquad \Big|(\bigcup_{i \in N^*} A_i) \cap W\Big| < \ell. \tag{2}$$

Note that if $i \in N^*$, then it must be that i is not indifferent between all candidates (i.e. $A_i \neq \emptyset, C$), otherwise (2) cannot hold.[21]

Now it follows that N^* is a generalised solid coalition for each candidate subset $C' \subseteq \bigcap_{i \in N^*} A_i$ since every candidate in C' is weakly preferred to every candidate in C for all $i \in N^*$. Furthermore, $\big|\bigcap_{i \in N^*} A_i\big| \geq \ell$ and so we can select a subset C' with exactly ℓ candidates.

Thus, if generalised weak Hare-PSC holds then there exists a set $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\} = \ell$ such that for all $c'' \in C''$ there exists $i \in N^* \, : \, c'' \succsim c^{(i,\ell)}$. But note that for any voter $j \in N^*$ we have $c^{(j,\ell)} \in A_j$ and hence for this particular candidate c'' and voter $i \in N^*$ we have $c'' \in A_i$. It follows that $C'' \subseteq (\bigcup_{i \in N^*} A_i) \cap W$, and

$$\Big|(\bigcup_{i \in N^*} A_i) \cap W\Big| \geq |C''| \geq \ell,$$

which contradicts (2). □

Proposition 3 *Under dichotomous preferences, PJR implies generalised Hare-PSC.*

Proof. Suppose that for dichotomous preferences, a committee W of size k satisfies PJR. Then there exists no set of voters $N^* \subseteq N$ with $|N^*| \geq \ell \frac{n}{k}$ such that

$$\Big|\bigcap_{i \in N^*} A_i\Big| \geq \ell \qquad \text{and} \qquad \Big|(\bigcup_{i \in N^*} A_i) \cap W\Big| < \ell.$$

[21] Although our model assumes no voter is completely indifferent between all candidates (and hence rules out this case), we include it to highlight that our result holds without this assumption.

Equivalently, for every set of voters $N^* \subseteq N$ with $|N^*| \geq \ell\frac{n}{k}$, the following holds:

$$\Big|\bigcap_{i\in N^*} A_i\Big| \geq \ell \implies \Big|\big(\bigcup_{i\in N^*} A_i\big) \cap W\Big| \geq \ell.$$

We now prove that for all generalised solid coalitions N^* of size $|N^*| \geq \ell n/k = \ell q_H$ (where q_H is the Hare quota) supporting candidate subset C' there exists a set $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\}$ such that for all $c'' \in C''$

$$\exists i \in N^* : \quad c'' \succsim_i c^{(i,|C'|)}.$$

Consider a generalised solid coalition N^* of size $|N^*| \geq \ell n/k$ supporting candidate subset C'.

(i) Suppose there exists some voter $i \in N^*$ who has one of her least preferred candidates c in C'. In this case, each candidate in C' is at least as preferred for i as c. Hence, the condition of generalised Hare-PSC is trivially satisfied by any committee.

(ii) The other case is that for each $i \in N^*$ and each $c \in C'$, we have $c \in \max_{\succsim_i}(C)$.[22] Equivalently, for each $i \in N^*$ and each $c \in C'$, we have $c \in A_i$. Hence, $C' \subseteq \bigcap_{i\in N^*} A_i$. Since W satisfies PJR, it follows that $\big|\big(\bigcup_{i\in N^*} A_i\big) \cap W\big| \geq \ell$. In that case, we know that there exists a set $C'' = \big(\bigcup_{i\in N^*} A_i\big) \subseteq W$ of size at least $\min\{\ell, |C'|\}$ such that for all $c'' \in W$,

$$\exists i \in N^* : \quad c'' \succsim_i c^{(i,\ell)}.$$

Hence, the condition of generalised Hare-PSC is again satisfied.

This completes the proof. □

Corollary 1 *Suppose all voters have dichotomous preferences. The axioms: PJR, weak generalised Hare-PSC, and generalised Hare-PSC are equivalent.*

Since it is known that testing PJR is coNP-complete (Aziz and Huang, 2016; Aziz et al., 2018a), it follows that testing generalised q-PSC and generalised weak q-PSC is coNP-complete.

Corollary 2 *Testing generalised q-PSC and generalised weak q-PSC is coNP-complete even under dichotomous preferences.*

On the other hand, PSC and weak PSC can be tested efficiently (please see the appendix).

Figure 2 depicts the relations between the different PR axioms.

[22] Here $\max_{\succsim_i}(C)$ denote the equivalence class of (strictly) most preferred candidates in C with respect to $\succsim_i$.

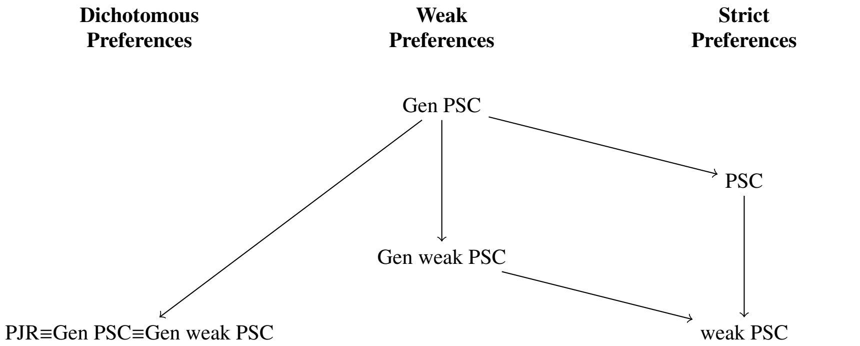

Fig. 2 Relations between properties. An arrow from (A) to (B) denotes that (A) implies (B). For any concept A, A with respect to Droop quota is stronger than A with respect to Hare quota. The equivalence in the dichotomous preference column holds only for the Hare quota.

4 Expanding Approvals Rule (EAR)

We now introduce our proposed rule called the Expanding Approvals Rule (EAR). It builds on the notion of approval voting (defined only for dichotomous preferences) with some important modifications which allow the approval voting method to be extended to general weak preferences, and satisfy monotonicity and general PR axioms.

To assist in the presentation of EAR we briefly review approval voting and then describe weighted j-approval voting which forms the basis of EAR.

Approval voting is defined for the setting where voters have dichotomous preferences over candidates, and typically works as follows. Each voter $i \in N$ submits an (unordered and unranked) approval ballot $A_i \subseteq C$ equal to their most preferred equivalence class of candidates. Each candidate $c \in C$ receives an approval score equal to the number of voters who approve of the candidate, i.e, $\sum_{i\in N\,:\,c\in A_i} 1$. The rule then selects the desired number of candidates achieving the highest approval scores (tie-breaking where necessary).

Weighted j-approval voting shares some similarities with approval voting but also involves some important differences. Weighted j-approval voting does not require voters to have dichotomous preferences, and introduces voter weights. Suppose each voter $i \in N$ has a general weak preference, denoted by $\succsim_i$, and a weight $w_i \in [0,1]$. We proceed as follows. For each voter $i \in N$ construct a j-approval ballot $A_i^{(j)} \subseteq C$ which includes all candidates in C that voter i weakly prefers to their j-th most preferred candidate. Recall the definition of candidate $c^{(i,j)}$ (used within Definition 6). We formally define the approval ballot as

$$A_i^{(j)} = \{c \in C \;:\; c \succsim_i c^{(i,j)}\}.$$

Each candidate $c \in C$ receives a weighted j-approval score equal to the weighted sum of voters who approve of the candidate, i.e.,

$$\sum_{\{i\in N\,:\,c\in A_i^{(j)}\}} w_i.$$

The rule then selects winning candidates via some method which considers the weighted j-approval score of candidates.

At a high level, EAR works as follows.

> An index j is initialised to 1. The voting weight of each voter is initially 1. We use a quota $q \in (n/(k+1), n/k]$. While k candidates have not been selected, we do the following. We perform weighted j-approval voting with respect to the voters' current voting weights. If there exists a candidate c with weighted j-approval score at least a quota q, we select such a candidate and reduce the voting weight of each voter whose j-approval ballot included this candidate.[23] If there exists no such candidate, then we increment j by one and repeat until k candidates have been selected.

Algorithm 1 Expanding Approvals Rule (EAR)

Input: $(N, C, \succsim, k)$ parameterised by quota $q \in (\frac{n}{k+1}, \frac{n}{k}]$. {$\succsim$ can contain weak preferences; if a voter i expresses her preferences over a subset $C' \subset C$, then $C \setminus C'$ is considered the last equivalence class of the voter.}

Output: $W \subseteq C$ such that $|W| = k$

Use some default strict priority ordering L over C.
$w_i \longleftarrow 1$ for each $i \in N$
$j \longleftarrow 1$
while $|W| < k$ **do**
  for $i \in N$ **do**
    $A_i^{(j)} \longleftarrow \{c \in C \ : \ c \succsim_i c^{(i,j)}\}$
  end for
  $C^* \longleftarrow \{c \in C \backslash W \ : \ \sum_{\{i \in N \,:\, c \in A_i^{(j)}\}} w_i \geq q\}$
  if $C^* = \emptyset$ **then**
    $j \longleftarrow j + 1$
  else
    Select the candidate c^* from C^* that has highest ranking wrt L.
    $N' \longleftarrow \{i \ : \ c^* \in A_i^{(j)}\}$
    Modify the weights of voters in N' so the total weight of voters in N', i.e., $\sum_{i \in N'} w_i$, decreases by exactly q. {Comment: Details of this reweighting process will be provided later.}
  end if
end while
return W

The rule is formally specified as Algorithm 1. It is well-defined for weak preferences. EAR is based on a combination of several natural ideas that have been used in the design of voting rules.

[23] Details of this reweighting process will be provided later.

(i) Candidates are selected in a sequential manner.
(ii) A candidate needs to have at least some quota of support q to be selected, i.e., a threshold requirement.
(iii) The voting weight of a voter is reduced if some of her voting weight has already been used to select some candidate.
(iv) We use weighted j-approval voting for varying j.

For EAR, the default value of q that we propose is

$$\bar{q} := \frac{n}{k+1} + \frac{1}{m+1}\left(\lfloor\frac{n}{k+1}\rfloor + 1 - \frac{n}{k+1}\right).$$

The reason for choosing this quota is that it constitutes a Droop quota, i.e., $\bar{q} = n/(k+1) + \epsilon$ for some $\epsilon > 0$ satisfying (1) and, at the same time, ϵ is large enough so that the algorithm is polynomial-time.

We also propose a default strict priority ordering, denoted by L in Algorithm 1, that is with respect to *rank maximality* of the preference profile $\succsim = (\succsim_i)_{i\in N}$. This priority ordering is applied in Line 12 (Algorithm 1) to choose among candidates 'eligible' for election, i.e., candidates in the set C^* who receive support at least q. We emphasise that this priority ordering is not used as a tie-breaking method for indifferences within voter preferences. We now define *rank maximality*. For any candidate a, its *corresponding rank vector* is $r(a) = (r_1(a), \dots, r_m(a))$ where $r_j(a)$ is the number of voters who have a in her j-th most preferred equivalence class. We compare rank vectors lexicographically. One rank vector $r = (r_1, \dots, r_m)$ is *better* than $r' = (r'_1, \dots, r'_m)$ if there exists $i = 1, \dots, m$ such that $r_j = r'_j$ for all $j < i$ and $r_i > r'_i$, or $r = r'$. A priority ordering L respects rank maximality if whenever two candidates c, c' with rank vectors r, r' such that r is better than r', c is higher in the priority ordering L than candidate c'.[24] This way of tie breaking is one but not the only way to ensure that EAR satisfies RRCM. One can also use a dynamic version of L that changes during the course of the algorithm. For example, in L, one could order candidates according to their current j-approval score.

Finally we propose uniform fractional reweighting as a means of implementing the reweighting in Step 14.[25] If the total support for c in the weighted j-approval election is T, then *uniform fractional reweighting* means that each voter $i \in N$ contributing support to c has their voting weight reduced to

$$w_i \longleftarrow w_i \times \frac{T - \bar{q}}{T}.$$

This ensures that exactly $\bar{q}$ weight is fractionally reduced and in a uniform manner.

In the following example, we demonstrate how EAR works when voters have strict preferences. An illustration of EAR when voters have weak preferences is more involved, and is left to Appendix B for the interested reader.

[24] If two candidates have equivalent rank vectors we can break ties using any pre-determined ordering of candidate labels to attain L, e.g., alphabetical. This ensures that the priority ordering L is complete.

[25] When introducing the family of STV rules in Section 6 we will discuss other possible reweighting schemes.

Example 4 [Illustration of EAR with strict preferences] Consider an instance with 9 voters $N = \{1, 2, \ldots, 9\}$, candidate set $C = \{c_1, c_2, c_3, d_1, e_1, \ldots, e_4\}$, and a desired winning committee of size $k = 3$. Note that the default quota is

$$\bar{q} = \frac{9}{4} + \frac{1}{9}\Big(\lfloor\frac{9}{4}\rfloor + 1 - \frac{9}{4}\Big) = 2\frac{1}{3}.$$

Suppose voter strict preferences are as follows.

$$\begin{aligned}
1:&\quad c_1, c_2, c_3, e_1, e_2, e_3, e_4, d_1\\
2:&\quad c_2, c_3, c_1, e_1, e_2, e_3, e_4, d_1\\
3:&\quad c_3, c_1, d_1, c_2, e_1, e_2, e_3, e_4\\
4:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1\\
5:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1\\
6:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1\\
7:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1\\
8:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1\\
9:&\quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1.
\end{aligned}$$

The priority ordering L over C, given via the rank maximality relation, is

$$e_1 \triangleright c_1 \triangleright c_3 \triangleright c_2 \triangleright e_2 \triangleright e_3 \triangleright d_1 \triangleright e_4,$$

where the relation $c \triangleright c'$ denotes that candidate c has a higher L priority than candidate c'.

To determine the EAR winning committee we proceed as follows. We initialise each voter with unit weight, i.e, $w_i = 1$ for all $i \in N$. We then compute the weighted 1-approval score of each candidate. As a vector with candidates ordered alphanumerically, i.e., $c_1, c_2, c_3, d_1, e_1, \ldots, e_4$, the weighted 1-approval scores of the candidates are

$$(1, 1, 1, 0, 6, 0, 0, 0).$$

Note that the weighted 1-approval score of a candidate here simply corresponds to the number of voters who have this candidate as their most preferred candidate. Candidate e_1 is the only candidate with weighted 1-approval score exceeding the quota $\bar{q} = 2\frac{1}{3}$ and so candidate e_1 is elected, and then removed from further consideration. The weight of voters supporting candidate e_1 in this 1-approval instance, i.e., $\{4, 5, \ldots, 9\}$, have their weight reduced such that

$$w_i = \frac{6 - 2\frac{1}{3}}{6} = \frac{11}{18} \qquad \text{for all } i \in \{4, 5, \ldots, 9\},$$

and all other voters maintain their unit voting weight. We now recompute the weighted 1-approval score of candidates who have not already been elected:

$$(1, 1, 1, 0, -, 0, 0, 0),$$

where the '–' symbol denotes the vector component corresponding to an already elected candidate (candidate e_1 in this instance).

No candidate has weighted 1-approval score exceeding the quota and so we move to the weighted 2-approval voting instance. As a vector, the weighted 2-approval scores of the candidates are

$$(2, 2, 2, 0, -, 6 \times \frac{11}{18}, 0, 0) = (2, 2, 2, 0, -, 3\frac{2}{3}, 0, 0).$$

Candidate e_2 is the only candidate with weighted 2-approval score exceeding the quota, and so we elect candidate e_2 and then remove e_2 from further consideration. The weight of voters supporting candidate e_2 in this 2-approval instance, i.e., $\{4, 5, \ldots, 9\}$, have their weight reduced such that

$$w_i = \frac{3\frac{2}{3} - 2\frac{1}{3}}{6} = \frac{4}{18} \qquad \text{for all } i \in \{4, 5, \ldots, 9\},$$

and all other voters maintain their previous voting weight. We now recompute the weighted 2-approval score of candidates:

$$(2, 2, 2, 0, -, -, 0, 0).$$

No candidate has weighted 2-approval score exceeding the quota, and so we move to the weighted 3-approval voting instance. As a vector, the weighted 3-approval scores of the candidates are

$$(3, 2, 3, 1, -, -, 6 \times \frac{4}{18}, 0) = (3, 2, 3, 1, -, -, 1\frac{1}{3}, 0).$$

Candidates c_1 and c_3 now both receive weighted 3-approval scores above the quota. Using the priority ordering L we elect candidate c_1 (since $c_1 \triangleright c_3$). Our winning committee

$$W_{EAR} = \{c_1, e_1, e_2\},$$

now contains the desired number of $k = 3$ candidates, and the EAR process is now complete. ◇

Remark 3 In case voters do not specify certain candidates in their list and do not wish that their vote weight to be used to approve such candidates, EAR can be suitably tweaked so as to allow this requirement. In this case, candidates are selected as long as a selected candidate can get approval weight $\bar{q}$. The required number of remaining candidates can be selected according to some criterion. Another way EAR can be varied is that instead of using L as the priority ordering, the candidate with the highest weighted support that is at least $\bar{q}$ is selected.

We point out the EAR outcome can be computed efficiently.

Proposition 4 *EAR runs in time $O(nm^2)$.*

The rank maximal vectors can be computed in $O(nm)$ and the ordering based on rank maximality, L, can be computed in time $O(nm^2)$. In each round, the smallest j is found for which there are some candidates in $C \setminus W$ that have an approval score of at least $\bar{q}$. The candidate c which is rank maximal is identified, i.e., highest L priority.

All voters who approved of c have their weight modified accordingly which takes linear time. Hence the whole algorithm takes time at most $O(nm^2)$.

A possible criticism of EAR is that the choice of quota, $\bar{q}$, as well as the uniform fractional reweighting makes it complicated enough to not be usable by hand or to be easily understood by the general public. However, it will be shown that without resorting to fractional reweighting even the prominent, and widely-used, PR rule STV fails to satisfy weak PSC (Proposition 8).

Since EAR is designed for proportional representation which is more meaningful for large k, EAR may not be the ideal rule for $k = 1$. Having said that, we mention the following connection with a single-winner rule from the literature.

Remark 4 For $k = 1$ and under strict preferences, EAR is equivalent to the Bucklin voting rule (see e.g., Brams and Sanver (2009)). For $k = 1$ and under strict preferences for all but a subset of equally least preferred candidates, applying the tweak in Remark 3 leads to the EAR being equivalent to the Fallback voting rule (Brams and Sanver, 2009).

Under dichotomous preferences and using Hare quota, EAR bears similarity to Phragmén's first method (also called Enström's method) described by Janson (2016) (page 59). However, the latter method when extended to strict preferences does not satisfy Hare-PSC. Although EAR has connections with previous rules, extending them to the case of multiple-winners and to handle dichotomous, strict and weak preferences simultaneously and satisfy desirable PR properties requires careful thought.

We observe some simple properties of the rule. It is anonymous (the names of the voters do not matter). It is also neutral if a uniform-random tie-breaking rule is used in Line 12 of Algorithm 1. Under strict preferences and when using EAR with the default quota ($\bar{q}$), if more than half the voters most prefer a candidate, then that candidate is selected. This is known as the majority principle.

EAR is defined with default quota $\bar{q}$. However, it is possible to consider variants of EAR with other quota values such as the Hare quota. We refer to the variant of EAR with the Hare quota as Hare-EAR. It is straightforward to show that this choice of quota can lead to different outcomes.

5 Proportional Representation and Candidate Monotonicity under EAR

5.1 Proportional Representation under EAR

We show that EAR satisfies the central PR axioms for general weak order preference profiles.

Proposition 5 *EAR satisfies generalised Droop-PSC.*

Proof. Let W be an outcome of the EAR and suppose for the sake of contradiction that generalised Droop-PSC is not satisfied. That is, there exists a positive integer ℓ and a generalised solid coalition N' such that $|N'| \geq \ell q_D$ (where q_D is the

Droop quota) supporting a candidate subset C' and for every set $C'' \subseteq W$ with $|C''| \geq \min\{\ell, |C'|\}$ there exists $c'' \in C''$ such that

$$\forall i \in N' \qquad c^{(i,|C'|)} \succ_i c''. \tag{3}$$

Let N' be one such generalised solid coalition supporting a candidate subset C' and let ℓ be the largest integer such that $|N'| \geq \ell q_D$.

Without loss of generality, assume that N' is chosen such that $|C'| \geq \ell$. This assumption can be made for the following reason. Suppose that the chosen generalised solid coalition N' supports a candidate subset $C' \ : \ |C'| < \ell$. Now let N'' be any subset of N' such that $\tilde{\ell} := |C'|$ is the largest integer r satisfying $|N''| \geq r \cdot q_D$. Since any subset of N' is also a generalised solid coalition for C', we infer that N'' is a solid coalition of voters supporting the candidate set C'. Finally, given that N' and C' satisfy the condition (3) for every set $C'' \subseteq W$ with $|C''| \geq \min\{\ell, |C'|\} = |C'|$, it is immediate that this same condition is satisfied for N'' and C' (i.e., replace ℓ with $\tilde{\ell}$ and N' with N'').

Let j^* be the smallest integer such that in the j^*-approval election each voter in N' supports all candidates in C'. We claim that the j^*-approval election must be reached. Suppose not, then it must be that $|W| = k$ at some earlier weighted j-approval election where $j < j^*$. But if $|W| = k$, then this implies that after reweighting

$$\sum_{i \in N} w_i = n - k\bar{q} = \frac{n}{k+1} - k\bar{\epsilon}, \tag{4}$$

where $\bar{\epsilon} = \frac{1}{m+1}\Big(\lfloor \frac{n}{k+1} \rfloor + 1 - \frac{n}{k_1}\Big)$. However, in every j'-approval election for $j' \leq j$ each voter $i \in N'$ only supports candidates weakly preferred to $c^{(i,|C'|)}$. Let $\hat{C}_i$ be the set of candidates which voter i finds weakly preferable to $c^{(i,|C'|)}$ and define $\hat{C} = \cup_{i \in N'} \hat{C}_i$. The total weight of voters in N' at the termination of the algorithm (i.e. at the end of the weighted j-approval election) is reduced by at most $|C''|\bar{q}$ where $C'' = W \cap \hat{C}$. But by assumption, $C'' \subseteq W$ and every candidate $c'' \in C''$ is weakly preferred to $c^{(i,|C'|)}$ for some voter $i \in N'$, and so it must be that $|C''| < \min\{\ell, |C'|\} = \ell$. It follows that

$$\sum_{i \in N'} w_i \geq \ell q_D - (\ell - 1)\bar{q} = \frac{n}{k+1} - (\ell - 1)\bar{\epsilon}, \tag{5}$$

which contradicts (4) since $N' \subseteq N$. Thus, we conclude that the j^*-approval election is indeed reached.

Now at the j^*-approval election each voter $i \in N'$ supports only the candidates in the set $\hat{C}_i$, excluding those already elected in an earlier approval election. Recall that $\hat{C} = \cup_{i \in N'} \hat{C}_i$ and let m^* be the number of candidates elected from $\hat{C}$ in earlier approval elections. Since $|N'| \geq \ell q_D$ implies that $|N'| \geq \ell \bar{q}$, it follows that the total weight of voters in N' when the j^*-approval election is reached is at least $(\ell - m^*)\bar{q}$. Since $C' \subseteq \hat{C}_i$ for all $i \in N'$, every unelected candidate in C' attains support at least $(\ell - m^*)\bar{q}$, note that there are at least $|C'| - m^* \geq \ell - m^*$ of these. In addition, each voter $i \in N'$ also supports the unelected candidates in $\hat{C}_i \backslash C'$. Thus, the EAR algorithm can only terminate if weight of voters in N' is reduced below $\bar{q}$ which can only occur if at least $(\ell - m^*)$ candidates from $\hat{C}$ are elected. It then follows that

$$|W \cap \hat{C}| = m^* + (\ell - m^*) = \ell.$$

By again defining $C'' = W \cap \hat{C}$ we attain a contradiction since $C'' \subseteq W$ and $|C''| \geq \ell$ but (3) does not hold since $C'' \subseteq \hat{C}$. □

Remark 5 Note that EAR satisfying PSC or generalised PSC does not depend on what priority tie-breaking is used (Step 1) or how the fractional reweighting is applied (Step 14).

Recalling Lemma 2 we have the following corollary.

Corollary 3 *EAR satisfies generalised Hare-PSC.*

Corollary 4 *Suppose all voters have strict preferences, EAR satisfies the majority principle.*

Proof. Suppose all voters have strict preferences, Droop-PSC implies the majority principle. □

We get the following corollary from Corollary 3.

Corollary 5 *Suppose all voters have dichotomous preferences, EAR satisfies PJR.*

Proof. We have observed that under dichotomous preferences, generalised Hare-PSC implies PJR. Furthermore, for general weak preferences EAR satisfies generalised Droop-PSC (Proposition 5) which implies that generalised Hare-PSC is satisfied (Lemma 2). □

Incidentally, the fact that there exists a polynomial-time algorithm to satisfy PJR for dichotomous preferences was the central result of two recent papers (Sánchez-Fernández et al., 2016; Brill et al., 2017). We have shown that EAR can in fact satisfy a property stronger than PJR that is defined with respect to the Droop quota. Since EAR satisfies generalised PSC, it implies that there exists a polynomial-time algorithm to compute a committee satisfying generalised PSC. Interestingly, we already observed that checking whether a given committee satisfies generalised PSC is coNP-complete.

5.2 Candidate Monotonicity under EAR

We will now show that EAR satisfies rank respecting candidate monotonicity (RRCM). In what follows we shall refer to the profile of all voter preferences (weak or strict) as simply the *profile*.

Proposition 6 *EAR satisfies rank respecting candidate monotonicity (RRCM).*

Proof. Consider a profile $\succsim$ with election outcome W and let $c_i \in W$. Now consider a modified profile $\succsim'$ in which candidate c_i has been reinforced relative to $\succsim$ without changing the ranks of other winning candidates in each voter's preference. Denote the election outcome under $\succsim'$ by W'. Since we use rank-maximality to define the order L, note that the relative position of c_i is at least as good under $\succsim'$ as it is under $\succsim$.

Let the order of candidates selected under $\succsim$ be $c_1, \ldots, c_i, \ldots, c_{|W|}$. In the modified profile $\succsim'$, let us trace the order of candidates selected. For the first candidate c_1, either it is selected first for exactly the same reason as it is selected first under $\succsim$ or, alternatively, now c_i is selected. If c_i is selected, then our claim has been proved. Otherwise, the same argument is used for candidates after c_1 until c_i is selected. □

This leads immediately to the following corollaries of the above proposition.

Corollary 6 *EAR satisfies non-crossing candidate monotonicity (NCRRCM).*

Corollary 7 *For $k = 1$, EAR satisfies candidate monotonicity (CM).*

Corollary 8 *If all the voters have dichotomous preferences, then EAR satisfies candidate monotonicity (CM).*

Proof. When all voters have dichotomous preferences RRCM is equivalent to CM (Proposition 1).[26] Thus, the corollary follows immediately from Proposition 6. □

On the other hand, EAR does not satisfy CM for $k > 1$.

Proposition 7 *EAR does not satisfy candidate monotonicity (CM).*

Proof. Let $N = \{1, 2, 3, 4\}$, $C = \{a, b, c, d, e, f\}$, $k = 2$, and let strict preferences be given by the following preference profile:

$$\begin{array}{rl} 1: & a, c, f, d, \ldots \\ 2: & d, b, f, a, c, \ldots \\ 3: & a, d, c, b, f, \ldots \\ 4: & f, e, c, d, \ldots, \end{array}$$

With this preference profile EAR will output the winning committee $W = \{a, f\}$. First, a is selected into W and each weight of each voter $i \in \{1, 3\}$ is reduced to $w_i = \frac{2-\bar{q}}{2}$ where $\bar{q}$ is the default quota. Since $\bar{q} = \frac{10}{7}$ we infer that $w_i = \frac{2}{7}$. Moving to the 2-approval election no candidate receives support of at least $\bar{q}$. Finally, in the 3-approval election candidate f receives support of $2\frac{2}{7}$ which exceeds the quota $\bar{q}$, and candidate c attains support $1 + 2(\frac{2}{7}) = \frac{11}{7}$. Thus, both candidates c and f exceed the quota; however, due to the default (rank maximal) priority ordering, f is elected into W.

Now consider a reinforcement of f by voter 1 (shift f from third to first rank), this is described by the following preference profile:

$$\begin{array}{rl} 1: & f, a, c, d, \ldots \\ 2: & d, b, f, a, c, \ldots \\ 3: & a, d, c, b, f, \ldots \\ 4: & c, e, f, d, \ldots, \end{array}$$

[26] Recall that we only consider non-degenerate preference profiles, i.e., we assume no voter is completely indifferent between all candidate.

With these preferences the winning committee is $W' = \{d, c\}$. In the 2-approval election both candidates a and d attain support of at least $\bar{q}$, however, due to the priority ordering L (rank maximality) candidate d is selected into W and each voter $i \in \{2, 3\}$ has their weight reduced to $w_i = \frac{2-\bar{q}}{2} = \frac{2}{7}$. In the 3-approval election both candidates c and f attain support of $2 + \frac{2}{7} \geq \bar{q}$. Furthermore, c and f are equally ranked with respect to rank maximality but applying alphabetical tie-breaking leads to c being elected into W'. □

Remark 6 The above proposition was proven for the default quota $\bar{q}$ however the same counter-example can be used to prove the statement for any other quota $q \in (\frac{n}{k+1}, \frac{n}{k}]$.

6 The Case of STV

In this section, we define the family of STV rules and provide a formal treatment of the properties of STV. Given the prominence of STV, and its wide-spread use in practice, this section provides a useful benchmark for comparing EAR and the results of the previous section.

The family of STV rules are defined for instances where voters submit strict preferences. The family is formalised as Algorithm 2. STV is a multi-round rule; every voter is initialised with a unit voting weight and then in each round either a candidate is selected as a winner or one candidate is eliminated from the set of potential winners. If a candidate is elected in a given round, then all voters strictly preferring this candidate to all other potential winners have their voting-weight reduced, i.e., reweighted. Depending on the quota q and the reweighting rule applied, one can obtain particular STV rules (see e.g., Aleskerov and Karpov, 2013). To distinguish between variants of STV, based on different quota values q, we denote the STV rule with quota q by q-STV. When the quota q is equal to the Hare or Droop quota we simply refer to the q-STV variant as Hare-STV or Droop-STV, respectively. In the literature, for single-winner election ($k = 1$) STV is commonly referred to as the *Alternative Vote (AV)* or *Instant-Runoff Voting (IRV)*.

As the paper proceeds, we will make a further distinction between variants of q-STV based on whether a discrete or fractional reweighting rule is applied in Line 11 (Algorithm 2).[27] Line 11 requires that after a set of N' voters, with weight sufficiently large ($\geq q$) and who plurality-support some candidate, is identified that the total weight of voters in N' is reduced by q. A *discrete reweighting* rule is applied when Line 11 is modified such that q is replaced with $\lceil q \rceil$. A *fractional reweighting* rule is applied when Line 11 is left unchanged. Note that a reduction in the total weight of voters in N', say T, (when using either reweighting rule) is attained by reducing individual voter weights w_i.[28] In this paper, for simplicity, we will focus on *uniform reweighting* procedures such for each voter $i \in N'$ their weight w_i is reduced

[27] Fractional reweighting in STV has been referred to as Gregory or 'senatorial' (see e.g., Janson, 2016; Tideman, 1995).

[28] Note that the discrete reweighting rule will reduce *individual* voter weights by fractions.

Algorithm 2 STV family of rules

Input: $(N, C, \succ, k)$, quota $q \in (\frac{n}{k+1}, \frac{n}{k}]$. {$\succ$ is a profile of strict preferences}
Output: $W \subseteq C$ such that $|W| = k$

```
1: W ⟵ ∅;
2: w_i ⟵ 1 for each i ∈ N
3: j ⟵ 1
4: C_temp ⟵ C
5: while |W| < k do
6:     if |W| + |C_temp| = k then
7:         return W ∪ C_temp
8:     end if
9:     if there is a candidate c with plurality support (i.e., the total weight of voters
       who have c as the most preferred candidate among candidates in C_temp) at least
       q then
10:        W ⟵ W ∪ {c}
11:        Let the set of voters supporting c be denoted by N′. Modify the weights of
           voters in N′ so the total weight of voters in N′ decreases by q.
12:        C_temp ⟵ C_temp\{c}
13:    else
14:        Remove a candidate with the lowest plurality support from the current pref-
           erence profile ≻
15:    end if
16: end while
17: return W
```

to $\min\{\frac{T-p}{T} w_i, 0\}$. In general, however, heterogeneous reductions in voter weights can be applied.[29]

Example 5 [Illustration of Droop-STV] For this example we consider the Droop-STV rule with *uniform fractional reweighting*. This reweighting method means that line 11 of Algorithm 2 is executed as follows: First, calculate the total weight of voters in N', i.e., $T = \sum_{i \in N'} w_i$, then the weight of each voter $i \in N'$ is updated from w_i to $w_i \times \frac{T-q_D}{T}$ where q_D is the Droop quota.

To illustrate this STV rule consider the following profile with 9 voters and suppose we wish to elect a committee of size $k = 3$

[29] Our results for STV will explicitly state the reweighting method where relevant. If no reference to the reweighting procedure is made, then the result holds for all reweighting methods.

$$\begin{array}{rl}
1: & c_1, c_2, c_3, e_1, e_2, e_3, e_4, d_1 \\
2: & c_2, c_3, c_1, e_1, e_2, e_3, e_4, d_1 \\
3: & c_3, c_1, d_1, c_2, e_1, e_2, e_3, e_4 \\
4: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
5: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
6: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
7: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
8: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
9: & e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1
\end{array}$$

In the first round e_1 is selected and the total weight of the voters in set $\{4, 5, 6, 7, 8, 9\}$ goes down by the Droop quota q_D, i.e., slightly more than 2.25. Candidate e_1 is then removed from the preference profile. In the second round, e_2 is selected and the total weight of the voters in the set $\{4, 5, 6, 7, 8, 9\}$ is now $6 - 2q_D$ (slightly less than 1.5). Candidate e_2 is then removed from the preference profile. After that since no candidate has plurality support, with respect to current weights, of at least the quota $q_D \approx 2.25$ the candidate with lowest plurality support is deleted. Note that candidate e_3 has plurality support of slightly less than 1.5, to be specific $6 - 2q_D$. The candidates c_1, c_2, c_3 have plurality support of just 1, while candidates d_1 and e_4 have the minimum plurality support of zero. Applying alphanumerical tie-breaking we remove candidate d_1. Again, no candidate attains the plurality support of at least the quota q_D, and so we remove candidate e_4. Yet once more, no candidate attains the plurality support and the minimum support of 1 is attained by candidates c_1, c_2, c_3. We delete candidate c_1 after applying alphanumerical tie-breaking. Still, no candidate has plurality support beyond q_D and so we delete candidate c_3 who attains the minimum plurality support of just 1, the plurality support of voter 3 is then given to candidate c_2 (recall that candidates c_1 and d_1 were already deleted). Finally, we have that candidate c_2 has plurality support of 3, from the voter set $\{1, 2, 3\}$, which exceeds the quota q_D. We elect candidate c_2, and the Droop-STV winning committee is $W = \{e_1, e_2, c_2\}$. $\diamond$

STV has been claimed to satisfy Proportionality for Solid Coalitions/Droop Proportionality Criterion (Dummett, 1984; Woodall, 1994). On the other hand, STV violates just about every natural monotonicity axiom that has been proposed in the literature.

In STV, voters are viewed as having an initial weight of one. When a candidate supported by a voter is selected, the voter's weight may decrease. As mentioned, STV can use fractional reweighting or discrete reweighting. We will show that fractional reweighting is crucial for guaranteeing the PR axioms such as PSC and even weak PSC. Incidentally, fractional reweighting is not necessarily introduced to achieve better PR but primarily to minimise the "stochastic aspect" of tie-breaking in STV (pp 32, Tideman, 1995). The following result shows that if STV resorts to discrete reweighting, then it does not even satisfy weak PSC.

Proposition 8 *Suppose all voters have strict preferences. Let $q \in (\frac{n}{k+1}, \frac{n}{k}]$, q-STV with discrete reweighting does not satisfy weak q'-PSC for any $q' \in (\frac{n}{k+1}, \frac{n}{k}]$.*

Proof. Let $N = \{1, 2, \ldots, 10\}$, $C = \{c_1, \ldots, c_8\}$, $k = 7$ and consider the following profile:

$$\begin{aligned}
1 &: \quad c_1, c_5, c_6, c_7, c_8, c_2, c_3, c_4 \\
2 &: \quad c_2, c_5, c_6, c_7, c_8, c_1, c_3, c_4 \\
3 &: \quad c_3, c_5, c_6, c_7, c_8, c_1, c_2, c_4 \\
4 &: \quad c_4, c_5, c_6, c_7, c_8, c_1, c_2, c_3 \\
5 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4 \\
6 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4 \\
7 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4 \\
8 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4 \\
9 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4 \\
10 &: \quad c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4
\end{aligned}$$

For fixed $q \in (\frac{n}{k+1}, \frac{n}{k}]$ we use q-STV with discrete uniform reweighting to select the candidates. Under discrete reweighting, the total weights of voters are modified by 2, since $2 = \lceil q \rceil$ for any $q \in (\frac{n}{k+1}, \frac{n}{k}]$.

Applying the q-STV rule, the winning committee is $W = \{c_1, c_2, \ldots, c_7\}$. First, c_5, c_6 and c_7 are selected. Each time we select these candidates, the total weight of voters in the set $N' = \{5, 6, 7, 8, 9, 10\}$ goes down by 2. Thus, the remaining four candidates are to be selected from c_1, c_2, c_3, c_4 and c_8. At this stage candidate c_8 has the lowest plurality support (equal to zero) and is removed from all preference profiles and the list of potentially elected candidates and hence $c_8 \notin W$. Now only 4 candidates remain both unelected and uneliminated and, since $k = 7$, we have that $W = \{c_1, c_2, \ldots, c_7\}$.

This committee W violates q'-PSC for any $q' \in (\frac{n}{k+1}, \frac{n}{k}]$. For any such q', weak q'-PSC requires that at least four candidates from $\{c_5, c_6, c_7, c_8\}$ be selected, since $|N'| \geq 4 \times q'$, but using discrete reweighting only three candidates are selected by q-STV. $\square$

The proof above has a similar argument as Example 1 in (Sánchez-Fernández et al., 2017a) that concerns an approval voting setting. Note that the proof of Proposition 8 does not rely on how the discrete reweighting is applied. For example, reducing individual voting weights in a non-uniform manner will not affect the proof.

Next we show that STV with uniform fractional reweighting satisfies PSC. The proof details are in the appendix.

Proposition 9 *Suppose all voters have strict preferences and let $q \in (\frac{n}{k+1}, \frac{n}{k}]$. q-STV with uniform fractional reweighting satisfies q-PSC.*

Below we provide an example of STV violating CM for $k = 1$. The example assumes the Droop quota; however, the same example violates CM for any q-STV rule with $q \in (\frac{n}{k+1}, \frac{n}{k}]$. Although it is well-known that STV violates CM, we give the argument for the sake of completeness.

Proposition 10 *For any $q \in (\frac{n}{k+1}, \frac{n}{k}]$, q-STV does not satisfy CM even for $k = 1$.*

Proof. Consider the following instance of 100 voters with strict preferences.

Total number of voters	Corresponding preferences
28 :	c, b, a
5 :	c, a, b
30 :	a, b, c
5 :	a, c, b
16 :	b, c, a
16 :	b, a, c.

We consider the single-winner election setting with the Droop quota; that is, $k = 1$ and $q_D = 50 + \epsilon$ for sufficiently small $\epsilon > 0$. The proof extends easily to other quota values.

Under the Droop-STV rule the outcome is $W_{STV} = \{a\}$. To see this, notice that the plurality support of the candidates a, b, c are $35, 32, 33$, respectively. Since no candidate receives plurality support $\geq q_D$ we remove the candidate with lowest plurality support, i.e, candidate b, and the 32 voters previously supporting candidate b now give their plurality support to their second preference. Thus, the updated plurality support of the two remaining candidates a and c are $35 + 16 = 51$ and $33 + 16 = 49$, and hence candidate a is elected.

Now to show a violation of CM we consider an instance where two voters originally with preferences c, a, b change their preferences to a, c, b, i.e., a reinforcement of the previously winning candidate a.[30] The new profile is shown below

Total number of voters	Corresponding preferences
28 :	c, b, a
3 :	c, a, b
30 :	a, b, c
7 :	a, c, b
16 :	b, c, a
16 :	b, a, c.

In this modified setting, the Droop-STV outcome is $W'_{STV} = \{b\}$ which is a violation of candidate monotonicity (CM). To see this, notice that plurality support of the candidates a, b, c are $37, 32, 31$, respectively. Since no candidate receives plurality support $\geq q_D$ we remove the candidate with lowest plurality support, i.e., candidate c, and the 31 voters previously supporting candidate c now give their plurality support to their second preference. Thus, the updated plurality support of the two remaining candidates a and b are $37 + 3 = 40$ and $32 + 28 = 60$, and hence candidate b is elected. □

30 Technically speaking, our definition of CM only considers reinforcements by a single voter. However, it is easy to see (by considering a sequence of single-voter reinforcements) that if a violation occurs when multiple voters reinforce the same winning candidate then there must be a violation of CM w.r.t. to a single voter's reinforcement.

As mentioned, STV is only defined for settings where voters have strict preferences and so cannot be said to satisfy the generalised PSC axioms. Naturally, one may ask whether arbitrarily breaking ties in a profile of weak order preferences to derive a profile of strict preferences and then applying STV suffices to satisfy the generalised PSC axiom. The following example shows that this is not the case.

Example 6 Suppose $N = \{1, 2, 3\}, C = \{a, b, c, d\}$ and voters have the following weak preferences.

$$\begin{aligned} 1 :& \quad \{b, c\}, \{a\}, \{d\} \\ 2 :& \quad \{c, d\}, \{a\}, \{b\}, \\ 3 :& \quad \{a\}, \{b\}, \{d\}, \{c\}. \end{aligned}$$

For a single-winner election, $k = 1$, generalised Droop-PSC requires that either candidate b, c, or d is elected, since voters $N' = \{1, 2\}$ form a solid coalition for candidate set $C' = \{c\}$ and $|N'| = 2 \geq q_D = \frac{3}{2}$.

Now suppose we break indifferences in the voters' preferences to induce strict preferences, and then apply Droop-STV. A possible strict preference profile attained, via alphabetical tie breaking, is

$$\begin{aligned} 1 :& \quad \{b\}, \{c\}, \{a\}, \{d\} \\ 2 :& \quad \{c\}, \{d\}, \{a\}, \{b\}, \\ 3 :& \quad \{a\}, \{b\}, \{d\}, \{c\}. \end{aligned}$$

If we apply Droop-STV, breaking ties where necessary for eliminating candidate in favour of alphabetical order, then the winning committee is $W = \{a\}$, which does not satisfy generalised Droop-PSC. This outcome occurs as follows. In the first iteration, no candidate attains plurality support $\geq q_D$ and so the candidate with minimal plurality support, i.e., candidate d is eliminated. In the second iteration, again no candidate attains sufficient plurality support and furthermore, all candidate have plurality support of one. Applying the tie breaking rule, we eliminate candidate c. Now in the third iteration, the remaining two candidates a and b have plurality support of 2 and 1, respectively. But since $q_D < 2$ we elect candidate a as the Droop-STV winner. $\diamond$

7 Other Rules

In the literature, several rules have been defined for PR purposes. We explain how EAR is better in its role at achieving PSC style PR axioms or has other relative merits.

7.1 Quota Borda System (QBS)

Dummett (1984) proposed a counterpart to STV called QPS (Quota Preference Score) or a more specific version QBS (Quota Borda System). The rule is well-defined for strict preferences and is designed to obtain a committee that satisfies Droop-PSC. It does so by examining the prefixes (of increasing sizes) of the strict preference lists of voters and checking whether there exists a corresponding solid coalition for a set of voters. If there is such a solid set of voters, then the appropriate number of candidates with the highest Borda count are selected.[31] We can partition the PSC demands among demands pertaining to rank from $j = 1$ to m. For any given j, we only consider candidates in the set of candidates involved in preferences of voters up till their first j positions, this candidate set is denoted by $C^{(j)}$ in Algorithm 3. In Algorithm 3, we present a formal description of QBS.

Algorithm 3 Quota Borda System (QBS)

Input: $(N, C, \succ, k)$ {$\succ$ is a profile of strict preferences}
Output: $W \subseteq C$ such that $|W| = k$

```
Set quota q as some value > n/(k + 1)
W ⟵ ∅
j ⟵ 1
while j < m do
    while there does not exist a solidly coalition supporting a set of candidates
    in C^j whose q-PSC demand is not met by W {Comment: For any given j, C^j
    is the set of candidates that feature in the top j positions in the preferences
    rankings of the voters} do
        j ⟵ j + 1
    end while
    Partition the set of voters into equivalence classes where each class forms a
    solid coalition supporting the same set of j candidates. {Comment: The next
    while loop is called the selection while loop in which the selection of candi-
    dates in a stage occurs.}
    while there exists an equivalence class of voters N′ ⊆ N whose q-PSC demand
    with respect to first j candidates is not met by W do
        Among the candidates in the first j positions that are supported by a solid
        coalition N′, select a candidate c ∉ W that has the highest Borda score.
        W ⟵ W ∪ {c}
    end while
end while
return W
```

Although Dummett did not show that the rule satisfies some axiom which STV does not, he argued that QPS satisfies the Droop proportionality criterion and is somewhat less “chaotic” than STV. Schulze (2002) argues that QBS is chaotic as well and his Example 3 implicitly shows that QBS in fact violates CM.[32] Tideman also feels that QBS is overly designed to satisfy Droop-PSC and is not robust enough to go beyond this criterion especially if voters in a solid coalition perturb their preferences.

[31] The description of the rule is somewhat informal and long in the original books of Dummett which may have lead to The Telegraph terming the rule as “a highly complex arrangement” (Telegraph, 2011).

[32] Geller (2002) wrongly claims that QBS satisfies the stronger axiom of CM.

EAR has some important advantages over QBS: (1) it can easily handle indifferences whereas QBS is not well-defined for indifferences. In particular, in order for QBS to be suitably generalised for indifferences and to still satisfy generalised PSC, it may become an exponential-time rule[33], (2) EAR can easily handle voters expressing partial lists by implicitly having a last indifference class whereas QBS cannot achieve this, (3) EAR rule satisfies a PR property called PJR for the case for dichotomous preferences. As said earlier, QBS is not well-defined for indifferences and even for dichotomous preferences, (4) EAR addresses a criticism of Tideman (2006): "*Suppose there are voters who would be members of a solid coalition except that they included an "extraneous" candidate, which is quickly eliminated among their top choices. These voters' nearly solid support for the coalition counts for nothing which seems to me inappropriate.*" We demonstrate the last flaw of QBS pointed out by Tideman in the explicit example below. In this example, EAR does not have this flaw. Understanding formally whether EAR, or other rules, satisfy Tideman's notion of 'robust' PSC is an interesting avenue for future work.

Example 7 Consider the profile with 9 voters and where $k = 3$.

$$
\begin{aligned}
1 &: \quad c_1, c_2, c_3, e_1, e_2, e_3, e_4, d_1 \\
2 &: \quad c_2, c_3, c_1, e_1, e_2, e_3, e_4, d_1 \\
3 &: \quad c_3, c_1, d_1, c_2, e_1, e_2, e_3, e_4 \\
4 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
5 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
6 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
7 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
8 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \\
9 &: \quad e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1
\end{aligned}
$$

In the example, $\{e_1, e_2, e_3\}$ is the outcome of QBS. Although PSC is not violated the outcome appears to be unfair to the voter subset $\{1, 2, 3\}$ because they almost have a solid coalition for the candidate set $C' = \{c_1, c_2, c_3\}$. In particular, if candidate d_1 was not present they would form a solid coalition for C' and a q-PSC outcome would necessarily include some candidate from C'. In contrast, it was shown in Example 4 that EAR does not have this possible 'flaw' and instead produces the outcome $\{c_1, e_1, e_2\}$. $\diamond$

7.2 Chamberlin-Courant and Monroe

There are other rules that have been proposed within the class of "fully proportional representation" rules such as Monroe (Monroe, 1995) and Chamberlin-Courant

[33] QBS checks for PSC requirements and adds suitable number of candidates to represent the corresponding solid coalition of voters. In order to work for generalised PSC, QBS will have to identify whether solid coalitions of voters and meet their requirement which means that it will need to solve the problem of testing generalised PSC which is coNP-complete.

(CC) (Chamberlin and Courant, 1983). Monroe (1995) used the term “fully proportional representation” to refer to PR-oriented rules that take into account the full preference list. Recently, variants of the rules called Greedy Monroe, and Greedy CC (Elkind et al., 2017a) have been discussed. However, these rules do not satisfy weak PSC (Elkind et al., 2017a). A reason for this is that voters are assumed to not care about how many of their highly preferred candidates are in the committee as long their most preferred candidate is present. Monroe and CC are also NP-hard to compute (Procaccia et al., 2008). On the other hand, Monroe and CC appear to have their merits when considering multi-district settings in which voters have their designated representative.

7.3 Phragmén’s First Method

Phragmén’s first method was first considered by Phragmén but not published or pursued by him (Janson, 2016). In the method, voters approve of most preferred candidates that have not yet been selected. The candidate with the highest weight of approval is selected. The total weight of the voters whose approved candidate was selected is reduced by the Hare quota if the total weight is more than the Hare quota. Otherwise, all such voters’ weights are set to zero.

Although Phragmén’s first method seems to have been defined primarily for dichotomous preferences, the same definition of the rule works for strict preferences. However, the rule when applied to strict preferences does not satisfy Hare-PSC. See the example below.

Example 8 Consider the profile with 9 voters and where $k = 3$.

$$\begin{aligned}
1 &: \quad c_1, c_2, c_3, \ldots \\
2 &: \quad c_2, c_3, c_1, \ldots \\
3 &: \quad c_3, c_1, c_3, \ldots \\
4 &: \quad e_1, c_1, c_2, c_3, \ldots \\
5 &: \quad e_2, c_1, c_2, c_3, \ldots \\
6 &: \quad e_3, c_1, c_2, c_3, \ldots \\
7 &: \quad e_4, c_1, c_2, c_3, \ldots \\
8 &: \quad e_5, c_1, c_2, c_3, \ldots \\
9 &: \quad e_6, c_1, c_2, c_3, \ldots
\end{aligned}$$

In the example, $\{e_1, e_2, e_3\}$ is a possible outcome of rule. When e_1 is selected, voter 4’s weight goes to zero. Then when e_2 is selected, voter 5’s weight goes to zero. Finally e_3 is selected. Hare-PSC requires that c_1 or c_2, or c_3 is selected. $\diamond$

The next example shows that Hare-EAR[34] and Phragmén’s first rule are not equivalent.

[34] Hare-EAR is simply EAR where, instead of using the default quota $\bar{q}$, the Hare quota $q_H = n/k$ is applied.

Example 9 Consider the profile with 4 voters, where $k = 2$ and with dichotomous preferences given as follows:

$$\begin{aligned} 1: &\quad \{b\},\ \{a,\ c\} \\ 2: &\quad \{a,\ c\},\ \{b\} \\ 3,4: &\quad \{a\},\ \{b,\ c\}. \end{aligned}$$

Note that in this example the Hare quota is $q_H = 2$. Now under Hare-EAR the winning committee is $W = \{a, c\}$. First, a is elected since her approval support is $3 \geq q_H$ and all other candidates have support less than the quota. Once candidate a is elected all of her supporting voters have their weights reduced to $w_i = \frac{1}{3}$. Now there is no candidate with support beyond the quota, q_H, and so we move to a 2-approval election. In this case the support of candidate b is $1 + \frac{2}{3} < q_H$ and the support of candidate c is $1 + 3(1/3) = 2 \geq q_H$, and so c is elected into W.

Under Phragmén's first rule we attain $W' = \{a, b\}$. First, candidate a is elected since she has maximal support, then all supporting voters have their weight reduced to $\frac{1}{3}$. Then the weighted supported of candidate b is 1 and the weight support of candidate c is $\frac{1}{3}$ – hence Phragmén's first rule elects b into the committee.

Thus, Hare-EAR and Phragmén's rule are not equivalent under dichotomous preferences. $\diamond$

7.4 Phragmén's Ordered Method

A compelling rule is Phragmén's Ordered Method that can even be generalised for weak orders. Under strict preferences, it satisfies weak Droop-PSC (Theorem 16.1 (ii), Janson, 2016). On the other hand, even under strict preferences, it does not satisfy Droop-PSC (page 51, Janson, 2016). If we are willing to forego PSC, then Phragmén's Ordered Method seems to be an exceptionally useful rule for strict preferences because unlike STV it satisfies both candidate monotonicity and committee monotonicity (Janson, 2016). Committee monotonicity requires that for any outcome W of size k, there is a possible outcome W' of size $k + 1$ such that $W' \supset W$.

7.5 Thiele's Ordered Method and other related rules

For strict preferences, Thiele's Ordered Method is based on identifying candidates that receive the largest share of voting weight. Each voter starts with an initial voting weight of one. The rule elects candidates sequentially. At each iteration, only each voter's most preferred candidate who has not already been selected is considered and, among these candidates, the candidate receiving the largest share of voting weight is elected until k candidates have been elected . At all iterations, if a voter has had their j most preferred candidates selected the voter has current weight $1/(j + 1)$. Janson (2016) presented an example (Example 13.15) that can be used to show that Thiele's Ordered Method does not satisfy weak Droop-PSC under strict preferences.

Thiele also developed three other methods for unordered preferences (i.e., dichotomous or approval-based preferences): *Thiele's Optimisation Method*, *Thiele's Addition Method*, and *Thiele's Elimination Method* (Janson, 2016). Of these three methods, Thiele's Addition Method has more commonly been used in practice and is often referred to as simply *Thiele's Method*. Thiele's Ordered Method (described in the paragraph above) is the ordered version of Thiele's Method. There is no known ordered versions of the other two methods (Janson, 2016).[35]

Thiele's Optimisation Method determines the winning committee by optimising a global objective function (see e.g, Kilgour (2010); Aziz et al. (2015)), and is commonly referred to as *Proportional Approval Voting (PAV)*. The PAV rule is defined as follows. Let H be a function defined on integers such that $H(p) = 0$ for $p = 0$ and $H(p) = \sum_{j=1}^{p} \frac{1}{j}$ otherwise. Let the PAV score of a committee W be $\sum_{i \in N} H(|W \cap A_i|)$. The PAV rule returns the committee with the maximum PAV score. PAV has been shown to satisfy desirable notions of proportional representation that are defined for dichotomous preferences. For example, it satisfies PJR (Sánchez-Fernández et al., 2017b). We note that EAR also satisfies Hare-PSC which is equivalent to PJR in the case of dichotomous preferences. However, optimising the PAV objective function is computationally demanding and so Thiele developed the other two methods (Thiele's Addition Method and Thiele's Elimination Method) which are less computationally demanding and offer an approximation of the PAV rule.

Thiele's Method (or Thiele's Addition Method) approximates the PAV rule by sequentially adding k candidates into the winning committee. Starting with a winning committee of no candidates, at each iteration the candidate c that maximises the PAV score of $W \cup \{c\}$ is added into the committee W, until W is of size k. Thiele's method is commonly referred to as the *Sequential Proportional Approval Voting (SeqPAV)* or *Reweighted Approval Voting (RAV)* rule (Aziz et al., 2017a).[36] Although PAV satisfies PJR, Aziz et al. (2017a) proved that SeqPAV does not even satisfy a property that is weaker than PJR.

Thiele's Elimination Method approximates the PAV rule by sequentially eliminating $|C| - k$ candidates to form the winning committee. Starting with a winning committee of all candidates, at each iteration the candidate c that maximises the PAV score of $W \backslash \{c\}$ is removed from the committee W, until W is of size k. Janson (2016) presents an example (Example 13.4) that shows Thiele's Elimination Method does not satisfy PJR.[37]

7.6 CPO-STV rules

A class of STV related rules is CPO-STV that was proposed by Tideman (2006). The rules try to achieve a PR-type objective while ensuring that for $k = 1$, a Condorcet

[35] However, Janson (2016) does note that one may possibly view the *Bottoms-up* method as an ordered version of Thiele's Elimination Method.

[36] A closely related method is called *Greedy Approval Voting (GreedyAV or GAV)* (Aziz et al., 2017a).

[37] In fact, the example shows that the method fails the even weaker concept of *Justified Representation* (Aziz et al., 2017a).

winner is returned if there exists a Condorcet winner.[38] One particular rule within this class is Schulze-STV (Schulze, 2011). All of these rules are only defined for strict preferences and hence do not satisfy generalised PSC. Furthermore, they all require enumeration of all possible committees and then finding pairwise comparisons between them. Thus, they are exponential-time rules and impractical for large elections. Tideman (1995) writes that CPO-STV is "*computationally tedious, and for an election with several winners and many candidates it may not be feasible.*" Tideman (2006) also considered whether CPO-STV rules satisfy Droop-PSC but was unable to prove that they satisfy PSC (page 282). In any case, having an exponential-time rule satisfying PSC may not be compelling because there exists a trivial exponential-time algorithm that satisfies PSC: enumerate committees, check whether they satisfy PSC, and then return one of them.

8 Conclusions

	STV	QBS	EAR
Generalised D-PSC / H-PSC	no	no	yes
Generalised weak D-PSC / H-PSC	no	no	yes
PJR	no	no	yes
D-PSC / H-PSC	yes	yes	yes
Weak D-PSC / H-PSC	yes	yes	yes
CM	no	no	no
CM for dichotomous preferences	no	no	yes
CM for $k = 1$	no	yes	yes
RRCM	no	no	yes
NCRRCM	no	yes	yes
polynomial-time	yes	yes	yes

Table 1 Properties satisfied by STV, QBS, and EAR. QBS and STV are the only rules from the literature that satisfy PSC and are computable in polynomial time. STV does not satisfy weak PSC if discrete reweighting is used.

In this paper, we undertook a formal study of proportional representation under weak preferences. The generalised PSC axiom we proposed generalises several well-studied PR axioms in the literature. We then devised a rule that satisfies the axiom. Since EAR has relative merits over STV and Dummett's QBS (two known rules that satisfy PSC), it appears to be a compelling solution for achieving PR via voting. At the very least, it appears to be another useful option in the toolbox of representative voting rules and deserves further consideration and study. The relative merits of STV, QBS, and EAR are summarised in Table 1.

EAR can be modified to also work for 'participatory budgeting' settings in which candidates may have a non-unit 'cost' and the goal is to select a maximal set of can-

[38] Since these rules are Condorcet-consistent, they are vulnerable to the no-show paradox (Moulin, 1988).

	Complexity of Computing	Complexity of Testing
PSC	in P	in P
Weak PSC	in P	in P
Generalised PSC	in P	coNP-complete
Generalised weak PSC	in P	coNP-complete

Table 2 Computational complexity of computing a committee satisfying a PR property and testing whether a given committee satisfies a property.

didates such that the total cost of candidates does not exceed a certain budget B (see e.g. Cabannes (2004); Rios et al. (2005); Goel et al. (2015); Fluschnik et al. (2019); Faliszewski and Talmon (2019)). In our original setting of committee voting, EAR with $q = n/k$ can be seen as follows. We initialise the weight of each voter as 1. Each candidate is viewed as having a unit cost and the budget is k. A candidate c is selected if including c does not exceed the budget (k) and if c has approval support of $1n/k$. When c is selected, the total weight of the voters supporting c is decreased by n/k times the unit cost. If no such candidate exists, then we increment j-approval to $j + 1$-approval. The process continues until no candidate can be included without exceeding the budget limit. We now explain how to modify EAR to work for non-unit costs and budgets. We again initialise the budget weight of each voter as 1. A candidate c is selected if including c does not exceed the budget B and if c has approval support of n/B times the cost of c. When c is selected, the total weight of the voters supporting c is decreased by n/B times the cost of c. If no such candidate exists, then we increment j-approval to $j+1$-approval. The process continues until no candidate can be included without exceeding the budget limit. Proportional representation axioms related to those studied in this paper have only recently been extended and explored in the participatory budgeting setting by Aziz et al. (2018b).

Our work also sheds light on the complexity of computing committees that satisfy PR axioms as well the the complexity of testing whether a given a committee satisfies a given PR axiom. We found that whereas a polynomial-time algorithm such as EAR finds a committee that satisfies generalised PSC, testing whether a given committee satisfies properties such as generalised PSC or generalised weak PSC is computationally hard. These findings are summarised in Table 2.

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A Complexity of Testing PSC

Proposition 11 *Suppose all voters have strict preferences, it can be tested in polynomial time whether a committee satisfies q-PSC.*

Proof. For each i from 1 to m one can look at prefixes of preference lists of sizes i. For these prefixes, we can check whether there exists a corresponding solid coalition. For such solid coalitions we can check whether the appropriate number of candidates to satisfy q-PSC are selected or not. □

The same idea can be used for weak PSC.

Proposition 12 *Suppose all voters have strict preferences, it can be tested in polynomial time whether a committee satisfies weak PSC.*

Proposition 13 *Suppose all voters have dichotomous preference, the problem of testing generalised q-PSC is coNP-complete.*

Proof. Under dichotomous preferences, generalised Hare-PSC is equivalent to PJR. Since Aziz and Huang (2016) and Aziz et al. (2018a) showed that testing PJR is coNP-complete, it follows that that testing Generalised Hare-PSC is coNP-complete as well. □

B Illustration of EAR with weak order preferences

Example 10 [Illustration of EAR with weak order preferences] Consider an instance with 6 voters $N = \{1, 2, \ldots, 6\}$, candidate set $C = \{a, b, \ldots, f\}$ and a desired winning committee of size $k = 2$. Note that

$$\bar{q} = \frac{6}{3} + \frac{1}{7}\left(\lfloor\frac{6}{3}\rfloor + 1 - \frac{6}{3}\right) = 2\frac{1}{7}.$$

Suppose voter preferences are as follows.

$$\begin{aligned}
1 &: \ \{a\}, \{b, f\}, \{c\}, \{d, e\},\\
2 &: \ \{a, b\}, \{f\}, \{c, d\}, \{e\},\\
3 &: \ \{c\}, \{b\}, \{a, d, e, f\},\\
4 &: \ \{d\}, \{e\}, \{b, c, f\}, \{a\},\\
5 &: \ \{e\}, \{b, f\}, \{a, c, d\},\\
6 &: \ \{f\}, \{b\}, \{c, d, e\}, \{a\}.
\end{aligned}$$

The priority ordering L over C, given via the rank maximality relation, is

$$a \triangleright b \triangleright f \triangleright e \triangleright c \triangleright d.$$

To determine the EAR winning committee we proceed as follows. We initialise each voter with unit weight, i.e., $w_i = 1$ for all $i \in N$. To compute the weighted 1-approval score of each candidate, we consider the 1-approval ballots $A_i^{(1)}$ of each voter[39], i.e.,

$$\begin{aligned}
A_1^{(1)} &: \ \{a\},\\
A_2^{(1)} &: \ \{a, b\},\\
A_3^{(1)} &: \ \{c\},\\
A_4^{(1)} &: \ \{d\},\\
A_5^{(1)} &: \ \{e\},\\
A_6^{(1)} &: \ \{f\}.
\end{aligned}$$

[39] Recall that $A_i^{(j)}$ is the subset of candidates which voter i weakly prefers to her j-th most preferred candidate.

From these 1-approval ballots, we compute the weighted 1-approval score for each candidate. As a vector (ordering candidates alphabetically), these weighted 1-approval scores are

$$(2, 1, 1, 1, 1, 1).$$

No candidate's weighted 1-approval score exceeds the quota $q = 2\frac{1}{7}$, we do not elect any candidate at this stage and instead proceed to consider the weighted 2-approval voting instance.

To compute the weighted 2-approval score of each candidate, we consider the 2-approval ballots $A_i^{(2)}$ of each voter, i.e.,

$$\begin{aligned} A_1^{(2)} &: \quad \{a, b, f\}, \\ A_2^{(2)} &: \quad \{a, b\}, \\ A_3^{(2)} &: \quad \{b, c\}, \\ A_4^{(2)} &: \quad \{d, e\}, \\ A_5^{(2)} &: \quad \{b, e, f\}, \\ A_6^{(2)} &: \quad \{b, f\}. \end{aligned}$$

From these 2-approval ballots, we compute the weighted 2-approval score for each candidate. As a vector, these weighted 2-approval scores are

$$(2, 5, 1, 1, 2, 3).$$

The set of candidates with weighted 2-approval score exceeding the quota is $\{b, f\}$. From these two candidates we elect the candidate with highest L priority, i.e., candidate b, into the winning committee, remove this candidate from future consideration, and reduce the weight of voters supporting this candidate in this 2-approval instance, i.e., $\{1, 2, 3, 5, 6\}$. In particular, we reduce the weights such that

$$w_i = \frac{5 - 2\frac{1}{7}}{5} = \frac{4}{7} \qquad \text{for all } i \in \{1, 2, 3, 5, 6\},$$

and voter 4's weight remains unchanged, i.e, $w_4 = 1$. We now recalculate the weighted 2-approval scores of candidates under the new weights. As a vector, these scores are

$$(2 \times \frac{4}{7}, -, \frac{4}{7}, 1, 1 + \frac{4}{7}, 3 \times \frac{4}{7}) = (1\frac{1}{7}, -, \frac{4}{7}, 1, 1\frac{4}{7}, 1\frac{5}{7}).$$

Under the new weights, no candidate exceeds the quota $q = 2\frac{1}{7}$. We do not elect any candidate and instead proceed to the weighted 3-approval instance.

To compute the weighted 3-approval score of each candidate, we consider the 3-approval ballots $A_i^{(3)}$ of each voter, i.e.,

$$\begin{aligned} A_1^{(3)} &: \quad \{a, b, f\}, \\ A_2^{(3)} &: \quad \{a, b, f\}, \\ A_3^{(3)} &: \quad \{a, b, c, d, e, f\}, \\ A_4^{(3)} &: \quad \{b, c, d, e, f\}, \\ A_5^{(3)} &: \quad \{b, e, f\}, \\ A_6^{(3)} &: \quad \{b, c, d, e, f\}. \end{aligned}$$

From these 3-approval ballots, we compute the weighted 3-approval score for each candidate. As a vector, these weighted 3-approval scores are

$$(3 \times \frac{4}{7}, -, 1 + 2 \times \frac{4}{7}, 1 + 2 \times \frac{4}{7}, 1 + 3 \times \frac{4}{7}, 1 + 5 \times \frac{4}{7}) = (1\frac{5}{7}, -, 2\frac{1}{7}, 2\frac{1}{7}, 2\frac{5}{7}, 3\frac{6}{7}).$$

The set of candidates with weighted 3-approval score (weakly) exceeding the quota is $\{c, d, e, f\}$. The candidate among these with highest priority under L is candidate f and so we elect candidate f into the winning committee. The winning committee now contains the desired number of candidate $k = 2$, and we conclude that the EAR committee is

$$W_{EAR} = \{b, f\}.$$

$\diamond$

C STV

Proof of Proposition 9

Proof. Let $q \in (\frac{n}{k+1}, \frac{n}{k}]$ and assume that all voters have strict preferences. Suppose for sake of contradiction that W is the STV outcome for some instance and q-PSC is not satisfied. That is, there exists a positive integer ℓ and a solid coalition $N' \subseteq N$ with $|N'| \geq \ell q$ supporting a set of candidates C' and $|W \cap C'| < \min\{\ell^*, |C'|\}$. Without loss of generality we assume that $|C'| \geq \ell$. If this were not the case, i.e., $|C'| < \ell$, one could simply consider a smaller subset of voters in N' and these voters will necessarily still form a solid coalition for all candidates in C'.

Thus, given that N' forms a solid coalition for the candidate set C' with $|C'| \geq \ell$ and $|N'| \geq \ell q$ we wish to derive a contradiction from the fact that

$$j := |W \cap C'| < \min\{\ell, |C'|\} = \ell.$$

The STV algorithm iteratively elects or eliminates a single candidate.[40] Let T be the number of the iteration which the STV algorithm runs through to output W, and let $(c_1, \dots, c_T)$ be a sequence of candidates such that in iteration $t \in \{1, \dots, T\}$ candidate c_t is either elected or eliminated. Note that $T \geq k$ and it need not be the case that $\cup_{t=1}^{T}\{c_t\} = C$. Furthermore, since $|W \cap C'| = j$ and $W \subseteq \cup_{t=1}^{T}\{c_t\}$, it must be the case that $| \cup_{t=1}^{T} \{c_t\} \cap C'| \geq j$.

First, we claim that $C' \subseteq \cup_{t=1}^{T}\{c_t\}$. If this were not the case, then in every iteration voters in N' support only candidates in C' and since precisely j candidates in C' were elected the remaining weight of voters in N' is $> (\ell - j)q$. But this is a contradiction; the STV algorithm cannot terminate when voters in N have remaining weight $\geq q$ since this would imply $< k$ candidates have been elected. Thus, we conclude that $C' \subseteq \cup_{t=1}^{T}\{c_t\}$. This implies that there exist a unique ordering of the candidates in C'

$$(c'_1, \dots, c'_{|C'|})$$

which maintains the ordering $(c_1, \dots, c_T)$, i.e., simply define the one-to-one mapping $\sigma : \{1, \dots, |C'|\} \to \{1, \dots, T\}$ such that $(c'_1, \dots, c'_{|C'|}) = (c_{\sigma(1)}, \dots, c_{\sigma(|C'|)})$.

Second, we claim that the j candidates $(c'_{|C'|-(j-1)}, c'_{|C'|-(j-2)}, \dots, c'_{|C'|})$ are all elected. Starting with iteration $t = \sigma(|C'|)$ suppose that candidate $c'_{|C'|}$ is not elected. This would then imply that at the start of iteration t voters in N' have weight $> (\ell - j)q > q$ – having only supported candidates in C' in earlier iterations and precisely j of these being elected. Furthermore, since $c'_{|C'|}$ is the only unelected and uneliminated candidate in C' all voters in N' support candidate $c'_{|C'|}$. Thus, the weighted-plurality support is $\geq q$ which contradicts the assumption that $c'_{|C'|}$ is not elected. We conclude that in fact candidate $c'_{|C'|}$ is elected and so in all earlier iterations $s < t$ the weight of voters in $> (\ell - j + 1)q$ and precisely $(j - 1)$ candidates from C' are elected in earlier iterations. Now consider iteration $t' = \sigma(|C'| = 1)$ and suppose that candidate $c'_{|C'|-1}$ is not elected. But at the start of this iteration each voter in N' supports with their plurality vote either candidate $c_{|C'|}$ or candidate $c_{|C'|-1}$ and in total have weight $\geq (\ell - j + 1)q \geq 2q$. Thus, at least one of the candidate must have weighted-plurality support $\geq q$ and so no candidate is eliminated in this iteration, and we conclude that it must be that candidate $c'_{|C'|-1}$ is elected. This argument can be repeated for all iterations $t'' = \sigma(|C'| - 2), \dots, \sigma(|C'| - (j - 1))$, and so we conclude that the j candidates $(c'_{|C'|-(j-1)}, c'_{|C'|-(j-2)}, \dots, c'_{|C'|})$ are all elected.

Finally, we argue that candidate $c_{|C'|-j}$ is also elected which contradicts the original assumption that $|W \cap C'| = j < \ell$. To see this, note that at the start of iteration $t^* = \sigma(|C'| - j)$ the total weight of voters in N' is $\geq \ell q$ and each voter in N' has a plurality vote for one of the $(j + 1)$ candidates in $\cup_{i=0}^{j}\{c'_{|C'|-i}\}$. Noting that $\ell \geq j + 1$, there exists at least one candidate in $\cup_{i=0}^{j}\{c'_{|C'|-i}\}$ who receives weighted-plurality support $\geq q$ and so no candidate is eliminated in this iteration. In particular, this implies that candidate $c_{|C'|-j}$ is elected. Combining this with the previous claims we see that $j + 1$ candidate are elected which contradicts the original assumption and completes the proof. □

[40] Technically speaking, if the number of elected and unelected, but also uneliminated, candidates is equal to k, i.e., $|C^*| = k$, then the set of all such candidates, C^*, are elected simultaneously in the same iteration. Within this proof and for simplicity of notation, we assume that if such situation has occurred the election of the candidates in C^* occurs in a sequential manner and supporting voter weights are reduced as per line 8-10 of Algorithm 2. Clearly, this assumption is without loss of generality.