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Approximate and Strategyproof Maximin Share Allocation of Chores with Ordinal Preferences

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Abstract We initiate the work on maximin share (MMS) fair allocation of m indivisible chores to n agents using only their ordinal preferences, from both algorithmic and mechanism design perspectives. The previous best-known approximation ratio using ordinal preferences is $2 - 1/n$ by Aziz et al. [AAAI 2017]. We improve this result by giving a deterministic 5/3-approximation algorithm that determines an allocation sequence of agents, according to which items are allocated one by one. By a tighter analysis, we show that for $n = 2$ and 3, our algorithm achieves better approximation ratios, and is actually optimal. We also consider the setting with strategic agents, where agents may misreport their preferences to manipulate the outcome. We first provide a strategyproof $O(\log(m/n))$-approximation consecutive picking algorithm, and then improve the approximation ratio to $O(\sqrt{\log n})$ by a randomized algorithm. Both algorithms only use the ordinal preferences of agents. Our results uncover some interesting contrasts between the approximation ratios achieved for chores versus goods.

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1 Introduction

Multi-agent resource allocation and fair division are major themes in mathematical economics [Brams and Taylor, 1996, Balinski and Young, 2001] and computer science [Bouveret et al., 2016]. In this work, we consider allocation algorithms to fairly assign m heterogeneous and indivisible chores to n agents with additive utilities. We take both algorithmic and mechanism design perspectives. Firstly, we investigate the extent to which fairness can be guaranteed when the algorithms can only use agents' ordinal preferences. There is a growing body of work on this issue [Anshelevich and Sekar, 2016b, Anshelevich, 2016, Anshelevich and Sekar, 2016a] where it is being explored how well ordinal information can help approximate objectives defined by cardinal valuations. Secondly, we take a mechanism design perspective to the problem when the preferences are agents' private information. We impose the requirement that the algorithm should be *strategyproof*, i.e., no agent should have an incentive to misreport her preference. Accordingly, we study how to approximately ensure fairness by designing strategyproof algorithms. This research falls under the umbrella of approximate mechanism design without money that has been popularized by Procaccia and Tennenholtz [2013].

The fairness concept we consider in this work is the intensively studied maximin share fairness. The *maximin fair share* (MMS) of an agent is the best utility she can guarantee if she is to partition the items into n bundles but receives the least preferred bundle, which was proposed by Budish [2011] as a fairness concept for allocating indivisible items. MMS fairness coincides with the classic concept of proportionality when the items are divisible. It was first proved by Kurokawa et al. [2018] that there may not exist an allocation such that every agent's utility is no worse than her MMS. As a result, significant effort has been spent on designing algorithms that compute approximate MMS allocations [Amanatidis et al., 2017, Kurokawa et al., 2018]. The state-of-the-art results are the $(3/4 + 1/12n)$- and $11/9$-approximate MMS fair allocation algorithms for goods and chores, respectively, designed by Garg and Taki [2021] and Huang and Lu [2021]. Recently, it is proved by Feige et al. [2021] that the approximation ratios cannot be better than $39/40$ for goods and $44/43$ for chores. For a more detailed literature view, please refer to Section 5.

On one hand, one agent's MMS is defined with respect to her cardinal preference, which places an exact numerical value on each item, and all the aforementioned works assume that the algorithm has full information of these cardinal values. Since cardinal values can sometimes be difficult to obtain, this has led researchers to study *ordinal algorithms* which only ask agents to rank the items in the order of their preferences, i.e. the ordinal preferences [Bouveret et al., 2010, Aziz et al., 2015]. A decision maker wants to know what the price of the missing information is by knowing only ordinal preferences.

Amanatidis et al. [2016] proved that with only ordinal information about the valuations, no algorithm can guarantee better than $\Omega(1/\log n)$-approximation for goods. Very recently, Halpern and Shah [2021] showed that there is an ordinal algorithm that guarantees $O(1/\log n)$-approximate MMS fairness for all agents. These works only focused on the allocation of goods, but there are many settings in which agents may have negative utilities such as when chores or tasks are to be allocated. In this work, we study to what extent MMS fairness can be guaranteed via ordinal preferences when the items are chores.

In the works discussed above, the focus has been on examining the existence or approximation of MMS allocations. In other words, the problem has been considered from an algorithmic point of view, but incentive compatibility has not been addressed. Strategic agents may have incentives to misreport their preferences to manipulate the final allocation in order to increase their own utilities. Thus, a natural question is whether it is possible to elicit truthful preferences and also guarantee approximate MMS fairness? Strategyproofness can be a demanding constraint especially when monetary transfers are not allowed. Amanatidis et al. [2016] were the first to embark on a study of strategyproof and approximately MMS fair algorithms. They designed a deterministic strategyproof ordinal algorithm which achieves $O(1/(m-n))$-approximation ratio when the items are goods. In this paper, we revisit strategyproof MMS allocation by considering the case of chores.

In a nutshell, we want to answer the following questions in this work.

> *When allocating indivisible chores, what approximation guarantee of maximin share fairness can be achieved using only ordinal preferences? Furthermore, how can we elicit agents' true preferences and still approximate maximin share fairness?*

1.1 Our results

Algorithmic Perspective. We first take an algorithmic perspective on fair allocation of indivisible chores to agents using ordinal preferences. With cardinal preferences, the best known result is the 11/9-approximate MMS algorithm in [Huang and Lu, 2021]. We note that the round-robin algorithm that uses only agents' ordinal preferences returns $2-1/n$ approximate MMS allocations [Aziz et al., 2017c]. In this work, we first improve this result by designing a simple periodic sequential allocation algorithm that ensures 5/3 approximation for all n. Interestingly, by refining our analyses and constructing examples for $n = 2, 3$, we show that no algorithm is able to achieve strictly better approximation ratios, i.e., our algorithm is actually optimal for these cases.

Our results depend on the following two ideas. Firstly, we reduce any chore allocation instance to a special one where all agents have the same ordinal preference for items, which is essentially the hardest situation for maximin share fair allocation. The technique has been used previously [Bouveret and Lemaître, 2016, Barman and Krishnamurthy, 2020, Huang and Lu, 2021]. Secondly, our algorithm falls under the umbrella of sequential allocating algorithms in which items are ordered in decreasing order of their costs and assigned to agents sequentially following the order. In particular, we consider allocation sequences that have a *pattern* and the sequence is obtained by repeating the pattern. We design a pattern with a length of roughly $1.5n$, and name our algorithm as the *Sesqui-Round Robin* Algorithm. While we prove that our algorithm is optimal for $n \leq 3$, we note that it is not optimal for $n = 4$ or larger (for a detailed discussion, please refer to Section 6). We leave exploring the optimal algorithm for arbitrary n as a future study.

	Goods		Chores	
	Upper	Lower	Lower	Upper
Ordinal	$1/H_n$ Amanatidis et al. [2016]	$1/2H_n$ Halpern and Shah [2020]	$4/3$ for $n = 2$ $7/5$ for $n = 3$ [Our work]	$4/3$ for $n = 2$ $7/5$ for $n = 3$ $5/3$ for $n \geq 4$ [Our work]
Cardinal	$39/40$ Feige et al. [2021]	$3/4 + 1/12n$ Garg and Taki [2020]	$44/43$ Feige et al. [2021]	$11/9$ Huang and Lu [2019]

Table 1: Lower and upper bounds on approximation of MMS fairness for allocating goods or chores using cardinal or ordinal preferences. Here $H_n = \Theta(\log n)$ is the n-th harmonic number and n is the number of agents.

Mechanism Design Perspective. We also take a mechanism design perspective for our problem when the agents may misreport their preferences to decrease costs. We design a deterministic sequential picking algorithm, ConsecutivePick, where each agent consecutively selects a number of items, and show that it is strategyproof. Roughly speaking, given an order of the agents, ConsecutivePick lets each agent i pick a_i items and leave, where $\sum_i a_i = m$. Amanatidis et al. [2016] proved that when the items are goods, the best ConsecutivePick algorithm can guarantee an approximation of $\lfloor (m - n + 2)/2 \rfloor$, and such an approximation can be easily achieved by letting each of the first $n - 1$ agents select one item and allocating all the remaining items to the last agent. Compared to their result, we show that by carefully deciding the a_i's, when items are chores, we are able to significantly improve the bound to $O(\log(m/n))$.[1] Moreover, we show that this approximation ratio is the best a ConsecutivePick algorithm can achieve. We further improve the approximation ratio by randomized algorithms. Particularly, we show that by randomly allocating each item but allowing each agent to reject a small set of "bad" items (i.e., with the largest cost) once, the resulting algorithm is strategyproof and achieves an approximation ratio of $O(\sqrt{\log n})$ in expectation.

Organization. We formally define our model and introduce necessary notations in Section 2. The algorithmic results for approximating MMS fairness using ordinal preferences are given in Section 3. We present strategyproof algorithms in Section 4 and a detailed literature review in Section 5. Finally, Section 6 concludes the paper with some discussions on the future work.

[1] In this paper we use $\log(\cdot)$ to denote $\log_2(\cdot)$.

2 Model and Preliminaries

In a fair allocation problem, N is a set of n agents, and M is a set of m indivisible items. The goal is to fairly distribute all the items to the agents. Different agents may have different preferences or utilities for the items and the preferences are captured by *valuation* functions: each agent i is associated with a function $v_i : 2^M \to \mathbb{R}$ that valuates any set of items.

MMS Fairness. Imagine that agent i gets the opportunity to partition all items into n bundles, but she is the last to choose a bundle. Then her best strategy is to partition the items such that the smallest value of a bundle is maximized. Let $\Pi(M)$ denote the set of all n-partitionings of M. Then the *maximin share (MMS)* of agent i is defined as

$$\mathsf{MMS}_i = \max_{(X_1,\dots,X_n)\in\Pi(M)} \min_{j\in N} v_i(X_j). \quad (1)$$

If agent i receives a bundle of items with value at least MMS_i, this allocation is called MMS fair to her.

In this work, it is assumed that items are chores, i.e., $v_i(S) \le 0$ for all $i \in N$ and $S \subseteq M$. Then each agent actually wants to receive as few items as possible. For ease of description, we ascribe a disutility or *cost* function $c_i = -v_i$ for each agent i. We further assume that the cost function of each agent i is additive. Accordingly, the cost function c_i can be represented by a cost vector $(c_{i1}, \dots, c_{im})$ where $c_{ij} = c_i(\{j\})$ is the cost of agent i for item j. Then for any $S \subseteq M$ we have $c_i(S) = \sum_{j\in S} c_{ij}$. Without loss of generality, we assume that $c_{ij} > 0$ for every $i \in N$ and $j \in M$, as otherwise we can allocate item j to agent i without increasing the cost of agent i. We refer $c = (c_1, \dots, c_n)$ as the *cardinal* preference profile. Agent i's maximin share can be equivalently defined as

$$\mathsf{MMS}_i = \min_{(X_1,\dots,X_n)\in\Pi(M)} \max_{j\in N} c_i(X_j), \quad (2)$$

and we have $\mathsf{MMS}_i > 0$ for every $i \in N$ by the assumption of positive costs.

Note that the maximin threshold defined in Equation 2 is exactly the opposite number of the threshold defined in Equation 1. Throughout the rest of our paper, we choose to use the second definition. For each agent i, we define a permutation over M, $\sigma_i : [m] \to M$, to denote agent i's *ranking* on the items such that $c_{i\sigma_i(1)} \ge \dots \ge c_{i\sigma_i(m)}$. Particularly, item $\sigma_i(1)$ is the least preferred item and $\sigma_i(m)$ is the most preferred. We refer to $\sigma = (\sigma_1, \dots, \sigma_n)$ as the *ordinal* preference profile. Let $x = (x_i)_{i\in N}$ be an *allocation*, where $x_i = (x_{ij})_{j\in M}$ and $x_{ij} \in \{0, 1\}$ indicates whether agent i gets item j under allocation x. A feasible allocation guarantees a partition of M, i.e., $\sum_{i\in N} x_{ij} = 1$ for any $j \in M$. We somewhat abuse the definition and let $X = (X_i)_{i\in N}$, $X_i = \{j \in M : x_{ij} = 1\}$ and $c_i(x) = c_i(x_i) = c_i(X_i)$. An allocation x is called an *MMS allocation* if $c_i(x_i) \le \mathsf{MMS}_i$ for every agent i and an *α-approximate MMS* (α-MMS) allocation if $c_i(x_i) \le \alpha \cdot \mathsf{MMS}_i$ for all agents i.

We first state the following simple observation about MMS, which implies that if an agent receives k items, then her cost is at most $k \cdot \mathsf{MMS}_i$.

Lemma 1 *For any agent i and any cost function c_i, we have*

- $\mathsf{MMS}_i \geq \frac{1}{n} \cdot c_i(M)$;
- $\mathsf{MMS}_i \geq c_{ij}$ *for any* $j \in M$.

Proof The first inequality is clear as for any partition of the items, the largest bundle has cost at least the average of total cost, i.e., $\frac{1}{n} \cdot c_i(M)$. For the second inequality, it suffices to show $\mathsf{MMS}_i \geq c_{i\sigma_i(1)}$. This is also clear since in any partition of the items, $\sigma_i(1)$ belongs to some bundle and thus the costliest bundle should have cost at least $c_{i\sigma_i(1)}$. □

By Lemma 1, it is easy to see that if $m \leq n$, any allocation that allocates at most one item to each agent is MMS fair. Thus throughout the rest of this paper, we assume $m > n$.

Ordinal Algorithm. An *ordinal algorithm* $\mathcal{A}$ takes the ordinal preferences σ of agents (instead of cardinal preferences c) as input, and computes an allocation $\mathcal{A}(\sigma)$. Note that the agents do have cardinal cost functions, according to which MMS_i's are defined. We call an ordinal algorithm α-approximate if for any cost functions c that are consistent with the ordinal preference σ, the allocation $\mathcal{A}(\sigma)$ given by the algorithm is an α-MMS allocation, i.e., $c_i(\mathcal{A}(\sigma)) \leq \alpha \cdot \mathsf{MMS}_i$ for all i. A randomized algorithm $\mathcal{A}$ returns a distribution over $\Pi(M)$ and is called α-approximate MMS if for any cost functions (consistent with the ordinal ranking) $c_1, \ldots, c_n$, $\mathbf{E}_{x \sim \mathcal{A}(\sigma)} \left[\max_{i \in N} \frac{c_i(x)}{\mathsf{MMS}_i} \right] \leq \alpha$.

Remark. Note that it is necessary and more interesting to define the approximation as the expectation of the maximum ratio over all agents. If the α-approximation is defined as for every agent i, $\mathbf{E}_{x \sim \mathcal{A}(\sigma)} c_i(x) \leq \alpha \cdot \mathsf{MMS}_i$, the problem becomes trivial as uniform-randomly allocating all items gives an exact MMS allocation.

Strategyproof Algorithm. In this work, we also study the strategic situation when the cost rankings σ_i are private information of agents. Each agent may misreport her true ranking to manipulate the allocation in order to minimize her own cost. We call an algorithm *strategyproof* if no agent can unilaterally misreport her ranking to reduce her cost. Formally, a deterministic algorithm $\mathcal{A}$ is called *strategyproof* if for every agent i, ranking σ_i and the ranking profile σ_{-i} of other agents,

$$c_i(\mathcal{A}(\sigma_i, \sigma_{-i})) \leq c_i(\mathcal{A}(\sigma'_i, \sigma_{-i})) \text{ holds for all } \sigma'_i.$$

We call a randomized algorithm $\mathcal{A}$ *strategyproof in expectation* if for every i, σ_i and σ_{-i},

$$\mathbf{E}_{x \sim \mathcal{A}(\sigma_i, \sigma_{-i})} c_i(x) \leq \mathbf{E}_{x \sim \mathcal{A}(\sigma'_i, \sigma_{-i})} c_i(x) \text{ holds for all } \sigma'_i.$$

3 Approximate Maximin Share

In this section, we consider the problem of computing an allocation of items that is approximately MMS based on the agents' ordinal rankings, and prove the results listed in Table 1.

3.1 Identical Ordinal Preference and Allocation Sequence

We first note that we can assume without loss of generality that all agents have *identical ordinal preference* (IDO). A chore allocation instance is called IDO if $\sigma_i(k) = \sigma_j(k)$ for agents i, j and index k. The original statement is proved for goods in [Bouveret and Lemaître, 2016] and [Barman and Krishnamurthy, 2020], which is then adapted to chores in [Huang and Lu, 2021].

Lemma 2 ([Huang and Lu, 2021]) *Suppose that there is an algorithm that runs in $T(n,m)$ time and returns an α-MMS allocation for all IDO instances. Then, there is an algorithm running in time $T(n,m)+O(nm\log m)$ outputing an α-MMS allocation for all instances that are not necessarily IDO.*

We provide some high-level ideas for this proof as follows, and a formal one can be found in [Huang and Lu, 2021]. For any instance $\mathcal{I}$ with parameters N, M, c, σ that is not IDO, we create a corresponding IDO instance $\mathcal{I}'$ on the same sets of agents N and items M, but with different cost functions. The costs are defined as $c'_{ij} = c_{i,\sigma_i(j)}$ for all $i \in N$ and $j \in M$. In other words, in $\mathcal{I}'$, item 1 is the most costly and m is the least costly to every agent. Consequently, the resulting instance is IDO; moreover, the MMS values do not change. Suppose we have an α-approximation algorithm for IDO instances $\mathcal{I}'$. Let $\pi_j \in N$ be the agent that receives item j in the allocation. Then we have a length-m sequence of "picking ordering" of agents $(\pi_m, \dots, \pi_1)$. Going back to $\mathcal{I}$, if we let agent π_j pick her favorite unselected item (with lowest cost) in M in the order of $j = m, m-1, \dots, 2, 1$, each agent's cost will not be higher than her cost in $\mathcal{I}'$ and thus the resulting allocation is also α-MMS.

Accordingly, in the following, it suffices to only focus on IDO instances, and assume items are ordered non-increasingly regarding their costs: for any agent $i \in N$, we have

$$c_{i1} \geq c_{i2} \geq \dots \geq c_{im}.$$

To simplify our statements, in this section, we assume that m is sufficiently larger than n. Note that this is also without loss of generality as we can append a sufficiently large number of items with cost 0 for everyone to M. The remaining part of this section focuses on the computation of an allocation sequence $\pi \in N^m$ (a length-m sequence of agents), where π_j is the agent that receives item j. Since an allocation algorithm is uniquely defined by an allocation sequence, we use the terms "allocation algorithm" and "allocation sequence" interchangeably.

Allocation Sequence. One of the most well-known allocation sequences is *round-robin*, where the sequence is defined as $[1, \ldots, n, 1, \ldots, n, \ldots]$. That is, for $j = 1, 2, \ldots, m$, we allocate item j to agent $((j-1) \mod n) + 1$, until all items are allocated. Observe that we can compactly represent the round-robin sequence as $\pi = [1, \ldots, n]^*$, which means that π is obtained by repeating the *pattern* $[1, \ldots, n]$ until the sequence has length m (and the last replica may not be complete). Like round-robin, in this paper we also focus on sequences with a certain pattern $p \in N^k$, for some $k \le m$. Formally speaking, the allocation sequence $\pi \in N^m$ with pattern $p \in N^k$ is obtained by repeating the pattern p until π has length m, and again the last replica may not be complete. We denote the full sequence as $\pi = p^*$, and call it a *periodic* allocation sequence.

Recall that the round-robin algorithm achieves a $(2 - \frac{1}{n})$ approximation ratio [Aziz et al., 2017c]. In the following, we improve this approximation via a carefully designed periodic allocation sequence.

3.2 Upper Bounds

In this section, we define the desired allocation sequences, and prove the approximation ratios (of MMS). We first show the following technical lemma, which will be useful later in the analysis.

Lemma 3 *Consider a sequence of items $S = \{j_1, j_2, \ldots, j_k\}$, ordered in non-increasing order of costs. Suppose an agent i receives two items $\{j_x, j_k\}$ from S, where $x \ge \frac{k}{2}$. Then we have $c_{i,j_x} + c_{i,j_k} \le \frac{2}{k} \cdot c_i(S)$.*

Proof For convenience, let $a = c_{i,j_x}$ and $b = c_{i,j_k}$, where $a \ge b$. We have

$$c_i(S) \ge x \cdot a + (k - x) \cdot b,$$

which implies

$$\frac{c_{i,j_x} + c_{i,j_k}}{c_i(S)} \le \frac{a+b}{x \cdot a + (k-x) \cdot b} = \frac{a+b}{k \cdot b + x \cdot (a-b)} \le \frac{a+b}{k \cdot b + \frac{k}{2} \cdot (a-b)} = \frac{2}{k},$$

where the second inequality follows from $x \ge \frac{k}{2}$. □

Next, we define a periodic allocation algorithm, called *Sesqui-Round Robin* (SesquiRR), where the length of the repeating pattern is roughly $1.5n$.

Sesqui-Round Robin (SesquiRR). Define the pattern of the periodic allocation sequence as

$$p = \left[1, 2, \ldots, n-1, n, n, n-1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\right].$$

For example, for $n = 2$, the full sequence is $\pi = [1, 2, 2]^*$; for $n = 3$, the sequence is $\pi = [1, 2, 3, 3, 2]^*$. Intuitively, within each pattern, (1) each agent from 1 to n is assigned an item and this part is the same with round-robin; (2) then each agent in the second half of $[n]$ is assigned one more item but according to the reverse order because they have an advantage in (1). The pseudocode is provided in Algorithm 1.

Algorithm 1: Sesqui-Round Robin Algorithm.

Input: IDO instance with $c_{i1} \geq c_{i2} \geq \ldots \geq c_{im}$ for all $i \in N$.
Initialize: $X_i = \emptyset$ for all $i \in N$.
Set $p = \left[1, 2, \ldots, n-1, n, n, n-1, \ldots, \lfloor \frac{n}{2} \rfloor + 1\right]$.
for $j = 1, 2, \ldots, m$ **do**
  $a = (j - 1 \mod |p|) + 1$ and $X_{p(a)} = X_{p(a)} \cup \{j\}$.
Output: Allocation $X = (X_1, \ldots, X_n)$.

Theorem 1 (Approximation Ordinal Algorithms) *Algorithm* SesquiRR *returns an allocation that is*

- $4/3$-*approximate MMS for* $n = 2$*;*
- $7/5$-*approximate MMS for* $n = 3$*;*
- $5/3$-*approximate MMS for any* $n \geq 4$.

We prove Theorem 1 by proving the following three lemmas.

Lemma 4 SesquiRR *is* $4/3$-*approximate MMS for* $n = 2$.

Proof For $n = 2$, SesquiRR has repeating pattern $[1, 2, 2]$. That is, we assign to agent 1 item set $X_1 = \{1, 4, 7, \ldots\} = \{3k + 1 \mid k \in \mathbb{N}_0\} \cap M$ and assign to agent 2 item set $X_2 = \{2, 3, 5, 6, 8, 9 \ldots\} = \{3k + 2, 3k + 3 \mid k \in \mathbb{N}_0\} \cap M$.[2]

Recall that items are indexed in descending order of costs. Let us first consider agent 1 and define $f := c_{11}/\mathsf{MMS}_1$. By the second statement in Lemma 1, we have $\mathsf{MMS}_1 \geq c_{11}$ and thus $f \in [0, 1]$. Note that after receiving item 1, agent 1 gets the last one out of every three consecutive items. Since $c_{1,3j-1} \geq c_{1,3j} \geq c_{1,3j+1}$ for all $j = 1, \ldots, \lfloor \frac{m-1}{3} \rfloor$, then

$$3 \cdot \sum_{j=1}^{\lfloor \frac{m-1}{3} \rfloor} c_{1,3j+1} \leq \sum_{j=1}^{\lfloor \frac{m-1}{3} \rfloor} (c_{1,3j-1} + c_{1,3j} + c_{1,3j+1}) = c_1(M) - c_{11}.$$

Thus

$$c_1(X_1) = c_{11} + \sum_{j=1}^{\lfloor \frac{m-1}{3} \rfloor} c_{1,3j+1} \leq f \cdot \mathsf{MMS}_1 + \frac{1}{3} \cdot (c_1(M) - c_{11}).$$

By the first statement in Lemma 1, we have $c_1(M) \leq 2 \cdot \mathsf{MMS}_1$ and thus

$$c_1(X_1) \leq \left(f + \frac{1}{3} \cdot (2 - f)\right) \cdot \mathsf{MMS}_1 = \frac{2}{3}(1 + f) \cdot \mathsf{MMS}_1 \leq \frac{4}{3} \cdot \mathsf{MMS}_1.$$

Next, we consider agent 2, who receives two items (of smallest costs) out of every three consecutive items, and $c_{2,3j-2} \geq c_{2,3j-1} \geq c_{2,3j}$ for all $j = 1, \ldots, \lfloor \frac{m}{3} \rfloor$, we have

$$c_2(X_2) \leq \frac{2}{3} \cdot c_2(M) \leq \frac{4}{3} \cdot \mathsf{MMS}_2,$$

where the inequality also comes from $c_2(M) \leq 2 \cdot \mathsf{MMS}_2$. □

[2] $\mathbb{N}_0$ represents the set of all non-negative integers $\{0, 1, 2, \ldots\}$.

Next, we consider the case when $n = 3$.

Lemma 5 SesquiRR *is* $7/5$*-approximate MMS for* $n = 3$*.*

Proof For $n = 3$, the allocation sequence has pattern $[1, 2, 3, 3, 2]$. In the following, we consider the three agents separately and the reasoning is similar to that of Lemma 4.

Agent 1. Let $c_{11} = f \cdot \mathsf{MMS}_1$, where $f \in [0, 1]$. Note that after receiving the first item, agent 1 receives one out of every 5 consecutive items. Hence

$$\begin{aligned} c_1(X_1) &\le f \cdot \mathsf{MMS}_1 + \frac{1}{5} \cdot (c_1(M) - c_{11}) \\ &\le \left(f + \frac{1}{5} \cdot (3 - f)\right) \cdot \mathsf{MMS}_1 \le \frac{7}{5} \cdot \mathsf{MMS}_1, \end{aligned}$$

where the second inequality holds due to $c_1(M) \le 3 \cdot \mathsf{MMS}_1$.

Agent 2. Let $c_{22} = f \cdot \mathsf{MMS}_2$, where $f \in [0, 1]$. Note that after receiving item 2, for every $t = 1, 2, \ldots$, among the 5 consecutive items

$$\{3 + 5(t - 1), 4 + 5(t - 1), \ldots, 7 + 5(t - 1)\},$$

agent 2 receives the third item $5 + 5(t - 1)$ and the last item $7 + 5(t - 1)$.

By Lemma 3, the total cost of items agent 2 receives after item 2 is at most $\frac{2}{5} \cdot \sum_{j=3}^{m} c_{2j}$. Hence we have

$$\begin{aligned} c_2(X_2) &\le f \cdot \mathsf{MMS}_2 + \frac{2}{5} \cdot (c_2(M) - c_{21} - c_{22}) \\ &\le \left(f + \frac{2}{5} \cdot (3 - 2f)\right) \cdot \mathsf{MMS}_2 \le \frac{7}{5} \cdot \mathsf{MMS}_2. \end{aligned}$$

Agent 3. Let $c_{33} + c_{34} = f \cdot \mathsf{MMS}_3$. Note that among the first four items $\{1, 2, 3, 4\}$, at least two of them must appear in the same bundle of the MMS allocation of agent 3. Hence we have $\mathsf{MMS}_3 \ge c_{33} + c_{34}$, which implies $f \in [0, 1]$. Also note that $c_{31} + c_{32} + c_{33} + c_{34} \ge 2 \cdot (c_{33} + c_{34}) = 2f \cdot \mathsf{MMS}_3$.

After receiving items 3 and 4, agent 3 receives two items (of smallest costs) out of every 5 consecutive items. Hence we have

$$\begin{aligned} c_3(X_3) &\le f \cdot \mathsf{MMS}_3 + \frac{2}{5} \cdot \left(c_3(M) - \sum_{j=1}^{4} c_{3j}\right) \\ &\le \left(f + \frac{2}{5} \cdot (3 - 2f)\right) \cdot \mathsf{MMS}_3 \le \frac{7}{5} \cdot \mathsf{MMS}_3. \end{aligned}$$

Combining three cases, all agents receive a bundle of cost at most $\frac{7}{5}$ times her MMS value, and the lemma follows. □

Finally, we show that the approximation ratio of SesquiRR is at most $\frac{5}{3}$ for any $n \ge 4$.

Lemma 6 SesquiRR *is 5/3-approximate MMS for* $n \geq 4$.

Proof Recall that for arbitrary n, the repeating pattern of the sequence is

$$\left[1, 2, \ldots, n-1, n, n, n-1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\right].$$

For convenience, we let $k = 2n - \lfloor \frac{n}{2} \rfloor$ be the length of the pattern. Note that we have $k = \frac{3n}{2}$ when n is even, and $k = \frac{3n+1}{2} > \frac{3n}{2}$ when n is odd. Fix any agent $i \in N$, we show that the set of items X_i agent i receives satisfies $c_i(X_i) \leq \frac{5}{3} \cdot \mathsf{MMS}_i$.

Case-1: $i \leq \lfloor \frac{n}{2} \rfloor$**.** The algorithm assigns to agent i the following items:

$$X_i = \{i, i+k, i+2k, \ldots\}.$$

Let $c_{ii} = f \cdot \mathsf{MMS}_i$, where $f \in [0, 1]$. Observe that after receiving item i, agent i gets the item with minimum cost out of every k items. Hence we have

$$\begin{aligned} c_i(X_i) &\leq f \cdot \mathsf{MMS}_i + \frac{1}{k} \cdot \sum_{j=i+1}^{m} c_{ij} \leq f \cdot \mathsf{MMS}_i + \frac{2}{3n} \cdot c_i(M) \\ &\leq f \cdot \mathsf{MMS}_i + \frac{2}{3n} \cdot n \cdot \mathsf{MMS}_i = \left(f + \frac{2}{3}\right) \cdot \mathsf{MMS}_i \leq \frac{5}{3} \cdot \mathsf{MMS}_i. \end{aligned}$$

Case-2: $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - \frac{k-2}{4}$**.** Note that $n - \frac{k-2}{4}$ is not necessarily an integer. In this case, agent i first receives item i, and then for every $t = 1, 2, \ldots$, among the k consecutive items

$$S_t = \{i + (t-1)k + 1, i + (t-1)k + 2, \ldots, i + t \cdot k\},$$

agent i receives item $i + (t-1)k + 2(n-i) + 1$ (the $(2(n-i)+1)$-th item in S_t) and item $i + t \cdot k$ (the last item in S_t). See Figure 1 for an illustration.

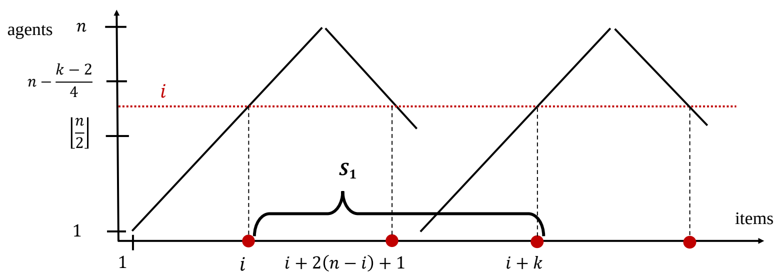

Fig. 1: Illustration of Case 2 in the proof of Lemma 6. The solid black lines represent the allocation of the first $2k$ items and the red points represent the items allocated to agent i.

Observe that for $i \leq n - \frac{k-2}{4}$,

$$2(n-i) + 1 \geq \frac{k-2}{2} + 1 = \frac{k}{2}.$$

Hence by Lemma 3, for every $t = 1, 2, \ldots$ we have

$$c_{i,i+(t-1)k+2(n-i)+1} + c_{i,i+t\cdot k} \leq \frac{2}{k} \cdot c_i(S_t).$$

As before, let $c_{ii} = f \cdot \mathsf{MMS}_i$, where $f \in [0,1]$. We have

$$\begin{aligned}
c_i(X_i) \leq & f \cdot \mathsf{MMS}_i + \frac{2}{k} \cdot \sum_{j=i+1}^{m} c_{ij} = f \cdot \mathsf{MMS}_i + \frac{2}{k} \cdot \left(c_i(M) - \sum_{j=1}^{i} c_{ij} \right) \\
\leq & f \cdot \mathsf{MMS}_i + \frac{2}{k} \cdot (n \cdot \mathsf{MMS}_i - i \cdot f \cdot \mathsf{MMS}_i) \\
= & \left(\frac{2n}{k} + (1 - \frac{2i}{k}) \cdot f \right) \cdot \mathsf{MMS}_i.
\end{aligned}$$

If $2i > k$ then we have

$$c_i(X_i) \leq \frac{2n}{k} \cdot \mathsf{MMS}_i \leq \frac{4}{3} \cdot \mathsf{MMS}_i.$$

Otherwise ($2i \leq k$), we have

$$c_i(X_i) \leq \left(\frac{2n}{k} + (1 - \frac{2i}{k}) \cdot f \right) \cdot \mathsf{MMS}_i \leq \left(1 + \frac{2(n-i)}{k} \right) \cdot \mathsf{MMS}_i.$$

For $k = 2n - \lfloor \frac{n}{2} \rfloor \geq \frac{3n}{2}$ and $i \geq \lfloor \frac{n}{2} \rfloor + 1 \geq \frac{n}{2}$, we have $\frac{n-i}{k} \leq \frac{1/2 \cdot n}{3/2 \cdot n} = \frac{1}{3}$, which implies

$$c_i(X_i) \leq \left(1 + \frac{2}{3} \right) \cdot \mathsf{MMS}_i = \frac{5}{3} \cdot \mathsf{MMS}_i.$$

Case-3: $i > n - \frac{k-2}{4}$**.** Note that agent i receives items

$$X_i = \{i, 2n - i + 1, i + k, 2n - i + 1 + k, i + 2k, 2n - i + 1 + 2k, \ldots\}.$$

In other words, agent i receives items i and $2n - i + 1$ first, and then for every $t = 1, 2, \ldots$, among the k consecutive items

$$S_t = \{2n - i + 2 + (t-1)k, 2n - i + 3 + (t-1)k, \ldots, 2n - i + 1 + t \cdot k\},$$

agent i receives item $i + t \cdot k$ (the $(k - 2(n-i) - 1)$-th item in S_t) and item $2n - i + 1 + t \cdot k$ (the last item in S_t). See Figure 2 for an illustration.

Since $i > n - \frac{k-2}{4}$, we have

$$k - 2(n-i) - 1 > k - 2 \times \frac{k-2}{4} - 1 = \frac{k}{2}.$$

Hence by Lemma 3, for every $t = 1, 2, \ldots$, the two items agent i receives from S_t have total cost at most $\frac{2}{k} \cdot c_i(S_t)$. Next, we bound the total cost $c_i(X_i)$ of agent i, taking into account the first two items agent i receives.

Let $c_{ii} = f_1 \cdot \mathsf{MMS}_i$ and $c_{i,2n-i+1} = f_2 \cdot \mathsf{MMS}_i$, where $1 \geq f_1 \geq f_2 \geq 0$.

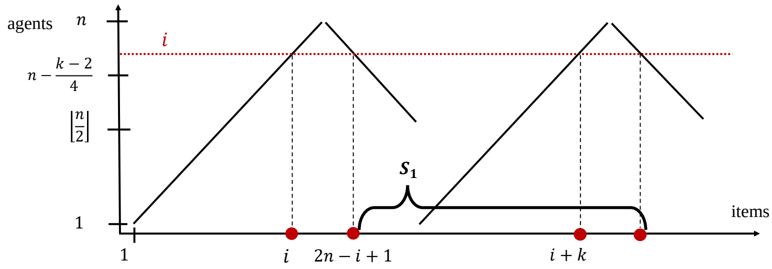

Fig. 2: Illustration of Case 3 in the proof of Lemma 6. The solid black lines represent the allocation of the first $2k$ items and the red points represent the items allocated to agent i.

Claim 2 *We have either $f_1 + f_2 \le 1$ or $f_2 \le \frac{1}{3}$.*

For continuity of presentation, we defer the proof of Claim 2 to the end of this subsection. By definition of f_1 and f_2 we have

$$
\begin{aligned}
c_i(X_i) &\le f_1 \cdot \mathsf{MMS}_i + f_2 \cdot \mathsf{MMS}_i + \frac{2}{k} \cdot \sum_{j=2n-i+2}^{m} c_{ij} \\
&\le (f_1 + f_2) \cdot \mathsf{MMS}_i + \frac{2}{k} \cdot \left(n \cdot \mathsf{MMS}_i - \sum_{j=1}^{2n-i+1} c_{ij} \right).
\end{aligned}
$$

Note that for all $j \le 2n - i + 1$, we have $c_{ij} \ge f_2 \cdot \mathsf{MMS}_i$; for all $j \le i$, we have $c_{ij} \ge f_1 \cdot \mathsf{MMS}_i$. Hence we have

$$
\sum_{j=1}^{2n-i+1} c_{ij} \ge \Big(i \cdot f_1 + (2n - 2i + 1) \cdot f_2 \Big) \cdot \mathsf{MMS}_i. \tag{3}
$$

Now, Inequality (3) implies

$$
\begin{aligned}
\frac{c_i(X_i)}{\mathsf{MMS}_i} &\le f_1 + f_2 + \frac{2}{k} \cdot \Big(n - i \cdot f_1 - (2n - 2i + 1) \cdot f_2 \Big) \\
&= \frac{2n}{k} + \frac{k - 2i}{k} \cdot f_1 + \frac{k - 2(2n - 2i + 1)}{k} \cdot f_2.
\end{aligned}
$$

Observe that the coefficient of f_2 is always positive since

$$
2n - 2i + 1 < 2n - 2(n - \frac{k-2}{4}) + 1 = \frac{k}{2}.
$$

If $2i \ge k$, then the coefficient of f_1 is non-positive, and thus the maximum of RHS is achieved when $f_1 = f_2$. Note that when $f_1 = f_2$, by Claim 2, we have $f_2 \le \frac{1}{2}$, which implies

$$
\begin{aligned}
\frac{c_i(X_i)}{\mathsf{MMS}_i} &\le \frac{2n}{k} + \frac{2k - 4n + 2i - 1}{k} \cdot f_2 \\
&\le \frac{4n}{2k} + \frac{2k - 4n + 2i - 1}{2k} = 1 + \frac{2i - 1}{2k} < 1 + \frac{2n}{1.5n} = \frac{5}{3}.
\end{aligned}
$$

If $2i < k$, then using the fact that $i > n - \frac{k-2}{4}$, we have

$$\begin{aligned}
\frac{c_i(X_i)}{\mathsf{MMS}_i} &\le \frac{2n}{k} + \frac{k-2i}{k} \cdot f_1 + \frac{k-2(2n-2i+1)}{k} \cdot f_2 \\
&< \frac{2n}{k} + \frac{k-2n+\frac{k-2}{2}}{k} \cdot f_1 + \frac{k-2(2n-k+1)}{k} \cdot f_2 \\
&= \frac{2n}{k} + \frac{3k-4n-2}{2k} \cdot f_1 + \frac{3k-4n-2}{k} \cdot f_2 \\
&\le \frac{4}{3} + \frac{1}{6} \cdot f_1 + \frac{1}{3} \cdot f_2 = \frac{4}{3} + \frac{1}{3} \cdot (\frac{f_1}{2} + f_2),
\end{aligned}$$

where the last inequality holds since $k \ge 1.5n$. It not difficult to check that by Claim 2, $\frac{f_1}{2} + f_2 \le 1$, which implies $\frac{c_i(X_i)}{\mathsf{MMS}_i} \le \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$. □

Combining Lemmas 4, 5 and 6, we have proved Theorem 1. It remains to prove Claim 2.

Proof of Claim 2 We call items $\{1, 2, \dots, i\}$ *heavy* items and items $\{i+1, i+2, \dots, 2n-i+1\}$ *light* items. Note that every heavy item must have cost at least $f_1 \cdot \mathsf{MMS}_i$ and every light item must have cost at least $f_2 \cdot \mathsf{MMS}_i$. Now consider the MMS allocation of agent i. If there exists a bundle containing both heavy and light items, or two heavy items, then we have

$$\mathsf{MMS}_i \ge f_1 \cdot \mathsf{MMS}_i + f_2 \cdot \mathsf{MMS}_i,$$

which implies $f_1 + f_2 \le 1$. Otherwise, we know that if a bundle contains a heavy item, then it is a singleton. Note that there are i heavy items, $2(n-i)+1$ light items and n bundles. Hence we must have $i < n$. Moreover, there must exist a bundle containing three light items, which implies $\mathsf{MMS}_i \ge 3f_2 \cdot \mathsf{MMS}_i$ and thus $f_2 \le \frac{1}{3}$. □

3.3 Lower Bounds

In the following, we give the lower-bound results showing that the approximation ratios we obtained for $n \le 3$ are optimal for deterministic ordinal algorithms.

Theorem 3 (Lower Bound for Deterministic Algorithms) *No deterministic ordinal algorithm has approximation ratio (w.r.t. MMS) smaller than*

- $4/3$ *for* $n = 2$*;*
- $7/5$ *for* $n = 3$.

Proof We first give a counter example for $n = 2$. Consider the instance in which the two agents have identical ranking on $m = 4$ items $\{1, 2, 3, 4\}$. Without loss of generality, assume the first item (with maximum cost) is given to agent 1. If agent 1 is allocated only one item, then for the case when $c_2 = (1, 1, 1, 1)$, the approximation ratio is $\frac{3}{2}$ since agent 2 has total cost 3 while $\mathsf{MMS}_2 = 2$.

Otherwise (agent 1 gets ≥ 2 items), for the case when $c_1 = (3, 1, 1, 1)$, the approximation ratio is at least $\frac{4}{3}$, as agent 1 has total cost at least $3 + 1 = 4$ while $\mathsf{MMS}_1 = 3$.

Next, we consider the case when $n = 3$. Suppose there exists an allocation that is strictly better than $7/5 = 1.4$-approximate. Let $1.4 - \epsilon$ be the approximation ratio of the algorithm, where $0 < \epsilon < 0.4$. In the following we consider a few instances with $m \geq \frac{2}{\epsilon}$ items, in which the three agents have identical ranking. For convenience of discussion, we fix m to be an odd number.

First, observe that the first three items must be allocated to three different agents, otherwise the approximation is at least 2. Without loss of generality, suppose item $i \in \{1, 2, 3\}$ is allocated to agent i. Then item 4 must be allocated to agent 3, as otherwise when all agents have cost function $(2, 2, 1, 1, 0, \ldots, 0)$, the approximation ratio is 1.5. Next, we consider how the items $M' = \{5, 6, \ldots, m\}$ are allocated. Let y_1, y_2 and y_3 be the number of items in M' allocated to agent $1, 2$ and 3, respectively.

Agent-1. Consider the instance in which the cost function of agent 1 is

$$c_1 = (1, \frac{2}{m-1}, \frac{2}{m-1}, \ldots, \frac{2}{m-1}).$$

Note that since m is odd, we have $\mathsf{MMS}_1 = 1$. To ensure an approximation ratio of $1.4 - \epsilon$, we have $c_1(X_1) = 1 + \frac{2 \cdot y_1}{m-1} \leq 1.4 - \epsilon$, which implies

$$y_1 \leq \frac{m-1}{2} \cdot (0.4 - \epsilon) < 0.2 \cdot m - 0.5 \cdot \epsilon.$$

Agent-2. Now consider the instance in which

$$c_2 = (1, 1, \frac{1}{m-2}, \frac{1}{m-2}, \ldots, \frac{1}{m-2}).$$

Note that $\mathsf{MMS}_2 = 1$. To ensure an approximation ratio of $1.4 - \epsilon$, we have $c_2(X_2) = 1 + \frac{y_2}{m-2} \leq 1.4 - \epsilon$, which implies

$$y_2 \leq (m-2) \cdot (0.4 - \epsilon) < 0.4 \cdot m - \epsilon \cdot m \leq 0.4 \cdot m - 2,$$

where the last inequality follows from $m \geq \frac{2}{\epsilon}$.

Agent-3. Finally, we consider the instance in which

$$c_3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{m-3}, \frac{1}{m-3}, \ldots, \frac{1}{m-3}).$$

Since there are $m - 4$ items with cost $\frac{1}{m-3}$, and m is odd, we can verify that $\mathsf{MMS}_3 = 1$. To ensure an approximation ratio of $1.4 - \epsilon$, we have $c_3(X_3) = 1 + \frac{y_3}{m-3} \leq 1.4 - \epsilon$, which implies

$$y_3 \leq (m-3) \cdot (0.4 - \epsilon) < 0.4 \cdot m - \epsilon \cdot m \leq 0.4 \cdot m - 2.$$

However, observe that now we have $y_1 + y_2 + y_3 < m - 4$, which is a contradiction since there are $m - 4$ items in M'. □

Combining Theorems 1 and 3, we have shown that our algorithm is optimal for $n = 2$ and $n = 3$. It would be natural to conjecture that the algorithm achieves optimal approximation ratios for larger n. Unfortunately, this is not true. We defer this discussion to Section 6.

4 Strategyproof Maximin Share Allocations

In this section, we take a mechanism design perspective and design strategyproof algorithms that approximately achieve MMS fairness. We first note that periodic sequential picking algorithms, e.g., the round robin algorithm, are not necessarily strategyproof. The following example shows that round-robin is not strategyproof even for two agents.

Example 1 Suppose there are two agents and four items. The first agent has ranking $c_{11} < c_{12} < c_{13} < c_{14}$ on the items, in the ascending order of costs. The second agent has ranking $c_{24} < c_{22} < c_{21} < c_{23}$. Suppose that both agents report truthfully then the algorithm allocates items $\{1, 2\}$ to agent 1 and items $\{3, 4\}$ to agent 2. However, if the second agent reports differently as $c_{22} < c_{24} < c_{21} < c_{23}$, then the algorithm will allocate items $\{1, 3\}$ to agent 1 and items $\{2, 4\}$ to agent 2. In other words, agent 2 receives a strictly better allocation by misreporting, and hence the algorithm is not strategyproof.

4.1 Deterministic Algorithm

We present a deterministic sequential picking algorithm that is $O(\log \frac{m}{n})$-approximate and strategyproof. Recall that when items are goods, Amanatidis et al. [2016] gave a deterministic $O(\frac{1}{m-n})$-approximate strategyproof ordinal algorithm. In the following, we show that if all the items are chores, the approximation ratio is $O(\log \frac{m}{n})$. Observe that by Lemma 1, it is trivial to achieve approximation ratio $\frac{m}{n}$ by assigning $\frac{m}{n}$ arbitrary items to each agent; it is trivial to achieve approximation ratio n by assigning all items to a single agent. Hence in the remaining part of the paper we assume that $\min\{n, \frac{m}{n}\} = \omega(1)$. Moreover, we assume $\log \frac{m}{n} < \frac{n}{4}$, as otherwise the n-approximation algorithm is also $O(\log \frac{m}{n})$-approximate. Equivalently, we have $m < n \cdot 2^{n/4}$.

Theorem 4 *There exists a deterministic strategyproof ordinal algorithm with approximation ratio $O(\log \frac{m}{n})$.*

We first define a special class of sequential picking algorithms, where each agent has a single chance to select items.

ConsecutivePick. Fix a sequence of integers $a_1, \dots, a_n$ such that $\sum_{i \le n} a_i = m$. Order the agents arbitrarily. For $i = n, n-1, \dots, 1$, let agent i pick a_i items from the remaining items. We do not restrict which items each agent should pick, but of course strategic agents want to select items with smallest costs.

Algorithm 2: ConsecutivePick Algorithm.

Parameters: Integers $a_1, \ldots, a_n$ such that $\sum_{i \le n} a_i = m$.
Input: The ordinal preference σ of agents.
Initialize: $X_i = \emptyset$ for all $i \in N$.
for $i = n, n-1, \ldots, 1$ **do**
  for $j = 1, 2, \ldots, a_i$ **do**
    Let $e^* = \arg\max_{e \in M}\{\sigma_i^{-1}(e)\}$; Set $X_i = X_i \cup \{e^*\}$ and $M = M \setminus \{e^*\}$.
Output: Allocation $X = (X_1, \ldots, X_n)$.

The pseudocode is provided in Algorithm 2. Recall that $\sigma_i(1)$ is the least preferred item of agent i with largest cost, and $\sigma_i(m)$ is the most preferred.

We note that as long as the a_i's do not depend on the reported preferences of agents, the rule discussed above is the serial dictatorship rule for multi-unit demands. When it is agent i's turn to pick items, it is easy to see that her optimal strategy is to pick the top-a_i items with the smallest costs among the remaining items. Hence immediately, we have the following lemma.

Lemma 7 *For any $\{a_i\}_{i \le n}$,* ConsecutivePick *is strategyproof.*

It remains to prove the approximation ratio.

Lemma 8 *There exists a sequence $\{a_i\}_{i \le n}$ such that the approximation ratio of* ConsecutivePick *is $O(\log \frac{m}{n})$.*

Proof We first establish an upper bound on the approximation ratio in terms of $\{a_i\}_{i \le n}$. Then we show how to fix the numbers appropriately to get a small ratio. Denote by r the approximation ratio of the algorithm.

Consider the moment when agent i needs to pick a_i items. Recall that at this moment, there are $\sum_{j \le i} a_j$ items, and the a_i ones with the smallest cost will be chosen by agent i. Let δ be the average cost of items agent i picks, i.e., $c_i(X_i) = \delta \cdot a_i$. On the other hand, each of the $\sum_{j \le i-1} a_j$ items left has cost at least δ. Thus we have $\mathsf{MMS}_i \ge \delta \cdot \left\lceil \frac{a_1 + \ldots + a_{i-1}}{n} \right\rceil$ and

$$r = \max_{i \in N} \left\{ \frac{c_i(X_i)}{\mathsf{MMS}_i} \right\} \le \max_{i \in N} \left\{ \frac{a_i}{\left\lceil \frac{a_1 + \ldots + a_{i-1}}{n} \right\rceil} \right\}.$$

It suffices to compute a sequence of $a_1, \ldots, a_n$ that sum to m and minimize this ratio. Fix $K = \lceil 2 \log \frac{m}{n} \rceil$. Recall that we assume w.l.o.g. that $\log \frac{m}{n} < \frac{n}{4}$. Hence we have $1 \ll K < n$. Let

$$a_i = \begin{cases} 2, & i \le \frac{n}{2}, \\ \min\{m - \sum_{j<i} a_j, \lceil K \cdot (1 + \frac{K}{n})^{i - \frac{n}{2} - 1} \rceil\}, & i > \frac{n}{2}. \end{cases}$$

The first term in $\min\{\cdot, \cdot\}$ is to guarantee that the summation of a_i's does not exceed m. Note that truncating a_i is only helpful for minimizing the approximation ratio and thus it suffices to consider the case when a_i equals the second term of $\min\{\cdot, \cdot\}$. In the following, we show that

1. all items are picked: $\sum_{i\in N} a_i = m$;
2. for every $i > \frac{n}{2}$: $a_i \leq K \cdot \left\lceil \frac{a_1+\ldots+a_{i-1}}{n} \right\rceil$.

Note that for $i \leq \frac{n}{2}$, since agent i receives 2 items, the approximation ratio is trivially guaranteed. The first statement holds because

$$\sum_{i=1}^{\frac{n}{2}} 2 + \sum_{i=\frac{n}{2}+1}^{n} \left(K \cdot (1+\frac{K}{n})^{i-\frac{n}{2}-1} \right)$$
$$= \sum_{i=1}^{\frac{n}{2}} \left(K \cdot (1+\frac{K}{n})^{i-1} \right) + n = (1+\frac{K}{n})^{\frac{n}{2}} \cdot n - n + n > 2^{\frac{K}{2}} \cdot n > m,$$

where the first inequality is because $1+\frac{K}{n} > 2^{\frac{K}{n}}$ for $\frac{K}{n} < 1$ and the second inequality is by the definition of K. Thus a_i's will be truncated when their sum exceeds m.

For $i > \frac{n}{2}$, observe that (let $l = i - \frac{n}{2} - 1$)

$$\frac{1}{n}\sum_{j=1}^{i-1} a_j = 1 + \frac{1}{n}\sum_{j=1}^{l} K \cdot (1+\frac{K}{n})^{j-1} = 1 + (1+\frac{K}{n})^l - 1 = (1+\frac{K}{n})^l.$$

Thus we have

$$a_i \leq \lceil K \cdot (1+\tfrac{K}{n})^l \rceil \leq K \cdot \lceil (1+\tfrac{K}{n})^l \rceil \leq K \cdot \left\lceil \tfrac{a_1+\ldots+a_{i-1}}{n} \right\rceil,$$

as claimed. □

We conclude this section by showing that our approximation ratio is asymptotically optimal for all ConsecutivePick algorithms.

Lemma 9 (Limits of ConsecutivePick) *The ConsecutivePick algorithm (with any $\{a_i\}_{i\in N}$) has approximation ratio $\Omega(\log \frac{m}{n})$.*

Proof Fix $K = \frac{1}{4}\log\frac{m}{n} < \frac{n}{16}$. Suppose there exists a sequence of $\{a_i\}_{i\in N}$ such that the algorithm is K-approximate. Then the last agent to act must receive at most K items, i.e., $a_1 \leq K$. Next, we show by induction on $i = 2, 3, \ldots, n$ that $a_i \leq K(1+\frac{2K}{n})^{i-1}$ for all $i \in N$. Suppose the statement is true for $a_1, \ldots, a_i$. Then if $a_{i+1} > K(1+\frac{2K}{n})^i$, we have

$$\frac{a_{i+1}}{a_1+\ldots+a_{i+1}} > \frac{K(1+\frac{2K}{n})^i}{K\cdot\frac{n}{2K}((1+\frac{2K}{n})^{i+1}-1)} \geq \frac{K}{n},$$

which is a contradiction with the algorithm being K-approximate. Thus

$$\textstyle\sum_{i=1}^{n} a_i \leq K \cdot \left((1+\frac{2K}{n})^n - 1\right) \leq n \cdot \left(e^{2K}-1\right) < m,$$

which means not all items are allocated and completes the proof. □

4.2 Randomized Algorithm

Via a carefully designed ConsecutivePick algorithm, we obtained a logarithmic approximation for the problem. However, the algorithm may still have poor performance when the number of items is much larger than the number of agents, e.g., $m = 2^n$. In this section, we present a randomized $O(\sqrt{\log n})$-approximation ordinal algorithm, which is strategyproof in expectation.

Basically, if we randomly allocate all the items, one is able to show that the algorithm achieves an approximation of $O(\log n)$. Since items are allocated in a uniformly-at-random manner, the expectation for the cost of every agent $i \in N$ is $\frac{1}{n} \cdot c_i(M) \leq \mathsf{MMS}_i$. Using a standard measure concentration bound, e.g., the Chernoff Bound, one can show that $\mathbf{E}\left[\max_{i \in N}\{\frac{c_i(X_i)}{\mathsf{MMS}_i}\}\right] = O(\log n)$. The drawback of this naïve randomized algorithm is that it totally ignores the rankings of agents. In the following, we show that if the agents have opportunities to decline some "bad" items, the performance of this randomized algorithm improves to $O(\sqrt{\log n})$. Note that since we already have an $O(\log \frac{m}{n})$-approximate deterministic algorithm for the ordinal model, it suffices to consider the case when $m \geq n \log n$.

RandomDecline. Let $K = \lfloor n\sqrt{\log n} \rfloor$. Based on the ordering of items submitted by agents, for each agent i, we label the K items with the largest cost as "large", and the remaining to be "small". It can also be regarded as each agent reports a set M_i of large items with $|M_i| = K$. The algorithm operates in two phases.

- Phase 1: every item is allocated to a uniformly-at-random chosen agent, independently. After all allocations, gather all the large items assigned to every agent into set M_b. Note that M_b is also a random set.
- Phase 2: Redistribute the items in M_b evenly to all agents: every agent gets $|M_b|/n$ random items.

The pseudocode is provided in Algorithm 3.

Algorithm 3: RandomDecline Algorithm.

Input: The ordinal preference σ of agents.
Initialize: $X_i = \emptyset$ for all $i \in N$ and $M_b = \emptyset$.
For each $i \in N$: let $M_i = \{\sigma_i(1), \sigma_i(2), \ldots, \sigma_i(K)\}$, where $K = \lfloor n\sqrt{\log n} \rfloor$.
for $j = 1, 2, \ldots, m$ **do**
    Randomly and uniformly select an agent i and set $X_i = X_i \cup \{j\}$.
for $i = 1, 2, \ldots, n$ **do**
    Set $M_b = M_b \cup (M_i \cap X_i)$ and $X_i = X_i \setminus M_i$.
Randomly divide M_b into n bundles $(Y_1, \ldots, Y_n)$, each with size $|M_b|/n$.
for $i = 1, 2, \ldots, n$ **do**
    Set $X_i = X_i \cup Y_i$.
Output: Allocation $X = (X_1, \ldots, X_n)$.

Theorem 5 *There exists a randomized strategyproof ordinal algorithm with approximation ratio $O(\sqrt{\log n})$.*

We prove Theorem 5 by showing the following Lemmas 10 and 11.

Lemma 10 *In expectation,* $\mathsf{RandomDecline}$ *achieves $O(\sqrt{\log n})$-approximation.*

Proof We show that with probability at least $1 - \frac{2}{n}$, every agent i receives a collection of items of cost at most $O(\sqrt{\log n}) \cdot \mathsf{MMS}_i$. Fix any agent i. Without loss of generality, we order the items according to agent i's ranking, i.e., $\sigma_i(j) = j$ for any $j \in M$ and $c_{i1} \geq \ldots \geq c_{im}$.

For ease of analysis, we rescale the costs such that[3]

$$c_{i1} + c_{i2} + \ldots + c_{im} = n\sqrt{\log n} = K.$$

Note that after the scaling, agent i's maximin share is $\mathsf{MMS}_i \geq \sqrt{\log n}$. Let x_{ij} denote the random variable indicating the contribution of item j to the cost of agent i in Phase 1. Then for $j > K$, $x_{ij} = c_{ij}$ with probability $\frac{1}{n}$, and $x_{ij} = 0$ otherwise. For $j \leq K$, $x_{ij} = 0$ with probability 1. Note that

$$\mathbf{E}[\sum_{i=1}^{m} x_i] = \frac{1}{n} \cdot \sum_{i=K+1}^{m} c_{ij} \leq \frac{K}{n} = \sqrt{\log n}.$$

Moreover, we have $c_{ij} \leq 1$ for $j > K$, as otherwise we have the contradiction that $\sum_{j=1}^{K} c_{ij} > K$. Note that $\{x_{ij}\}_{j \leq m}$ are independent random variables taking value in $[0, 1]$. Hence by Chernoff bound we have

$$\Pr[\sum_{j=1}^{m} x_{ij} \geq 7\sqrt{\log n} \cdot \mathsf{MMS}_i] \leq \Pr[\sum_{j=1}^{m} x_{ij} \geq 7\log n]$$
$$\leq \exp\left(-\frac{1}{3} \cdot \left(\frac{7\log n}{\mathbf{E}[\sum_{i=1}^{m} x_i]} - 1\right) \cdot \mathbf{E}[\sum_{i=1}^{m} x_i]\right) < \frac{1}{n^2}.$$

Then by union bound over the n agents, we conclude that with probability at least $1-\frac{1}{n}$, every agent i receives a bundle of items of cost at most $O(\sqrt{\log n}) \cdot \mathsf{MMS}_i$ in Phase 1.

Now we consider the items received by an agent in the second phase. Recall that the items M_b will be reallocated evenly. By the second argument of Lemma 1, to show that every agent i receives a bundle of items of cost $O(\sqrt{\log n}) \cdot \mathsf{MMS}_i$ in the second phase, it suffices to prove that $|M_b| = O(n\sqrt{\log n})$ (with probability at least $1 - \frac{1}{n}$).

Let $y_j \in \{0, 1\}$ be the random variable indicating whether item j is contained in M_b. For every item j, let $b_j = |\{k : j \in M_k\}|$ be the number of

[3] Here we dropped the floor in the definition of $K = \lfloor n\sqrt{\log n} \rfloor$ for convenience of notation. This is without loss of generality as we are only interested in an asymptotic bound.

agents that label item j as "large". Then we have $y_j = 1$ with probability $\frac{b_j}{n}$. Since every agent labels exactly $n\sqrt{\log n}$ items, we have

$$\mathbf{E}[|M_b|] = \mathbf{E}[\sum_{i=1}^{m} y_i] = \frac{1}{n}\sum_{i=1}^{m} b_i = n\sqrt{\log n}.$$

Applying Chernoff bound we have

$$\Pr[\sum_{i=1}^{m} y_i \geq 2n\sqrt{\log n}] \leq \exp\left(-\frac{n\sqrt{\log n}}{3}\right) < \frac{1}{n}.$$

Thus, with probability at least $1 - \frac{2}{n}$, every agent i receives a bundle of items with cost $O(\sqrt{\log n} \cdot \mathsf{MMS}_i)$ in the two phases combined. Since in the worse case, i receives a total cost of at most $n \cdot \mathsf{MMS}_i$, in expectation, the approximation ratio is $(1 - \frac{2}{n}) \cdot O(\sqrt{\log n}) + \frac{2}{n} \cdot n = O(\sqrt{\log n})$. □

Lemma 11 RandomDecline *is strategyproof in expectation.*

Proof To prove that the algorithm is strategyproof in expectation, it suffices to show that for every agent, the expected cost she is assigned is minimized when she is truthful. Let $K = n\sqrt{\log n}$ and fix any agent i. Suppose $c_{i1}, \dots, c_{iK}$ are the costs of items labelled "large" by the agent; and $c_{i,K+1}, \dots, c_{im}$ are the remaining items. Then the expected cost assigned to the agent in the first phase is given by $\frac{1}{n}\sum_{j=K+1}^{m} c_{ij}$, as every item is assigned to her with probability $\frac{1}{n}$. Next, we consider the cost incurred to agents in the second phase.

Recall that the expected total cost of items to be reallocated in the second phase is $\mathbf{E}[\sum_{j\in M_b} c_{ij}] = \sum_{j=1}^{m} c_{ij} \cdot \frac{b_j}{n}$, where b_j is the number of agents that label item j "large". Let $\mathcal{E}$ be this expectation when agent i does not label any item "large". By labelling $c_{i1}, \dots, c_{iK}$ "large", agent i increases the probability of each item $j \leq K$ being included in M_b by $\frac{1}{n}$. Thus it contributes an $\frac{1}{n}\sum_{j=1}^{K} c_{ij}$ increase to the expectation of total cost of M_b. In other words,

$$\mathbf{E}[\sum_{j\in M_b} c_{ij}] = \mathcal{E} + \frac{1}{n}\sum_{j=1}^{K} c_{ij}.$$

Since a random subset of $\frac{|M_b|}{n}$ items from M_b will be assigned to agent i, the expected total cost of items assigned to her in the two phases is given by

$$\frac{1}{n}\sum_{j=K+1}^{m} c_{ij} + \frac{1}{n} \cdot \left(\mathcal{E} + \frac{1}{n}\sum_{j=1}^{K} c_{ij}\right).$$

Obviously, the expression is minimized when $c_{i1} + \ldots + c_{iK}$ is maximized. Hence every agent minimizes her expected cost by telling the true ranking. □

5 Related Works

The study of computing fair allocations of resources has a long history. Arguably, two of the most widely studied solution concepts are envy-freeness and proportionality, whose existence is guaranteed when the items are divisible, i.e., the *cake cutting problem* [Brams and Taylor, 1996, Stromquist, 2008, Aziz and Mackenzie, 2016]. The problem becomes tricky when the items are indivisible, because exact envy-free or proportional allocations barely exist and are hard to approximate. In order to characterize the extent to which fairness can be guaranteed in the indivisible setting, several relaxations have been proposed, such as envy-free up to one item (EF1) [Lipton et al., 2004], envy-free up to any item (EFx) [Caragiannis et al., 2019], and maximin share fair (MMS) [Budish, 2011], whose relations have been discussed by Amanatidis et al. [2018]. Among these relaxations, MMS is one of the most widely studied.

It has been conjectured that an MMS allocation is guaranteed to exist until Kurokawa et al. [2018] identified a counter-example. Recently, Feige et al. [2021] further proved that a better than 39/40-MMS allocation may not exist. There are rich works for the design of approximate MMS allocations. The first constant factor approximation algorithm was given by Kurokawa et al. [2018], whose approximation ratio is 2/3 but its running time can be exponential in the number of agents. Later, Amanatidis et al. [2017] refined the algorithm in [Kurokawa et al., 2018] and guaranteed the same approximation with a polynomial running time. The same approximation is also obtained in [Garg et al., 2019, Barman and Krishnamurthy, 2020]. Ghodsi et al. [2018] improved these results by giving a 3/4 approximation algorithm whose running time may be exponential. More recently, Garg and Taki [2021] designed a polynomial time algorithm to find a 3/4-approximate MMS allocation and proved the existence of $(3/4 + 1/12n)$-MMS allocation, breaking the barrier of 3/4.

Although most of the works on fair allocation of items are for the case of goods, recently, fair allocation of chores [Aziz et al., 2017c] or combinations of goods and chores [Aziz et al., 2019a, Kulkarni et al., 2021] have received much attention. Aziz et al. [2017c] proved that MMS allocations do not always exist but can be easily 2-approximated. Later, Barman and Krishnamurthy [2020] presented a 4/3-approximation algorithm for MMS allocation of chores, and Huang and Lu [2021] further improved this ratio to 11/9. Aziz et al. [2019b] extended the definition of MMS to the weighted version that deals with asymmetric agents.

Distortion. Our work is also inspired by the growing literature on the distortion in voting, where voters express ordinal preferences (instead of numerical utilities) over candidates [Procaccia and Rosenschein, 2006, Boutilier et al., 2015, Caragiannis et al., 2017, Mandal et al., 2020]; and matching, where only the edge ranking is known instead of the exact weights [Anshelevich and Sekar, 2016b, Anshelevich, 2016, Anshelevich and Sekar, 2016a]. The goal is to use partial information to find solutions that maximize the social welfare, and *distortion* is the measure to evaluate the worst-case multiplicative loss in social welfare due to this lack of information. A major focus of our work is identifying what approximation guarantees of fairness can be achieved by only using ordinal information, which is naturally connected to the work on distortion.

There has been a substantial amount of work on using ordinal preferences in fair allocation of indivisible goods. For example, Aziz et al. [2015] considered the question of checking the existence of allocations that possibly or necessarily satisfy certain fairness guarantees such as envy-freeness given only ordinal preferences of the agents over the goods. Bouveret et al. [2010] studied similar questions, but given partial ordinal preferences of the agents over bundles of goods. More closely related to ours are the papers that use ordinal allocation rules (such as picking sequence rules) in settings with cardinal valuations. For example, Aziz et al. [2016b] focused on the complexity of checking what social welfare such rules can possibly or necessarily achieve. Amanatidis et al. [2016] sought to use picking sequence rules to obtain an approximation of the MMS fairness, and Halpern and Shah [2021] showed that there is an algorithm using ordinal preferences to guarantee $O(1/\log n)$-MMS fairness when items are goods. Recently, following our work, Li et al. [2021] studied how to use ordinal preferences to allocate chores under the fairness notion of proportionality up to any item.

Mechanism Design without Money. Strategyproofness is a challenging property to satisfy for fair division algorithms. For the cake cutting problem, Chen et al. [2013] and Bei et al. [2017] studied the conditions under which there exist strategyproof algorithms to fairly allocate a cake to agents with piece-wise uniform or linear valuations. Maya and Nisan [2012] provided a characterization of strategyproof algorithms for the case of two agents. When items are indivisible, Caragiannis et al. [2009] and Lipton et al. [2004] have discussed how to elicit true information from the agents while ensuring some degree of envy-freeness. More recently, Amanatidis et al. [2016] initiated the work on strategyproof allocation of goods with respect to MMS fairness. One important algorithm class is sequential picking, which is a generalization of round-robin. The strategic aspect of sequential picking have been studied in [Kohler and Chandrasekaran, 1971, Bouveret and Lang, 2014, Aziz et al., 2017a,b]. There are also works on the approximation of social welfare that can be achieved by strategyproof algorithms for the allocation of *divisible* items (e.g., [Aziz et al., 2016a, Cole et al., 2013]).

6 Discussion and Conclusion

SesquiRR *is Not Optimal for Larger n.* As we have proved in Section 3, our algorithm SesquiRR achieves optimal approximation ratios for $n = 2$ and $n = 3$. However, it fails to return an optimal solution when $n = 4$. Actually, following similar analysis for $n = 2$ and $n = 3$, one can show that the approximation ratio of our algorithm is 1.5 for $n = 4$. However, we are aware of an algorithm that performs strictly better than 1.499-approximate. Furthermore, we are aware of an instance with $n = 4$, for which no ordinal algorithm performs better than 1.405-approximate. Therefore, we conjecture that the optimal approximation ratio $r^*(n)$ (with n agents) is an increasing function of n. In this paper we have shown that

$$r^*(2) = \frac{4}{3} \approx 1.333, \quad r^*(3) = \frac{7}{5} = 1.4, \text{ and } \forall n, \; r^*(n) \le \frac{5}{3} \approx 1.667.$$

We can also show that $1.405 < r^*(4) < 1.499$.[4] We leave it as future work to analyze the optimal ratio $r^*(n)$ for $n \ge 4$.

Constant Approximations for Our Strategyproof Algorithm. We have shown in Section 4.1 a deterministic strategyproof algorithm that is $O(\log(m/n))$-approximate MMS. However, in many applications, it is desirable to obtain constant approximation ratios. While our algorithm has constant approximation ratios when $m = O(n)$, it is not clear how large the constant is. In particular, if we need to guarantee an approximation ratio r, what is the maximum number of items we can handle? In the following, we show how to answer this question. Following the analysis of Section 4.1, in order to guarantee an approximation ratio of r, we can set $a_1 = r$, and for each $i = 2, \ldots, n$, we set $a_i = r \cdot \left\lceil \frac{a_1 + \ldots + a_{i-1}}{n} \right\rceil$. To guarantee that all items are allocated, we have $m \le \sum_{i=1}^n a_i$. For example, if $r = 2$, we have

$$a_1 = \ldots = a_{\frac{n}{2}} = 2, \qquad a_{\frac{n}{2}+1} = \ldots = a_{\frac{3n}{4}} = 4,$$
$$a_{\frac{3n}{4}+1} = \ldots = a_{\frac{11n}{12}} = 6, \qquad a_{\frac{11n}{12}+1} = \ldots = a_n = 8.$$

Hence we have $m \le \sum_{i=1}^n a_i = \frac{11}{3} n \approx 3.67n$. Similarly, to guarantee an approximation of $r = 3$, we can let the first $\frac{n}{3}$ values of a_i be 3; the next $\frac{n}{6}$ values of a_i be 6; then the next $\frac{n}{9}$ values of a_i be 9, etc. Following similar calculations, one can verify that the maximum number of items the algorithm can handle to guarantee $r = 3$ is $m \approx 10.26n$; for $r = 4$, we have $m \approx 30.15n$. It would also be interesting to study lower bounds on the approximation ratio. For example, it remains unknown whether super-constant lower bounds exist for the approximation ratio of ordinal strategyproof mechanisms.

Conclusion. In this paper, we initiated the study of approximate and strategyproof maximin fair algorithms for chore allocation using ordinal preferences. Our study leads to several new questions. Two of the most obvious research questions are to find the optimal ordinal algorithm for an arbitrary number of agents, and to improve the approximation or study the lower bounds of strategyproof algorithms. At present, we have two parallel lines of research for goods and chores. It is important to consider similar questions for combinations of goods and chores [Aziz et al., 2019a]. Finally, it is interesting to extend our work to the case of asymmetric agents [Aziz et al., 2019b], where agents possess different weights and a fair allocation should respect their weights.

[4] Since we are not able to obtain the exact ratio, we did not include the analysis here.

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