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Short communication

Testing substitutability of weak preferences

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ABSTRACT

In various models of matching markets, substitutable preferences constitute the largest domain for which stable matchings are guaranteed to exist. Recently, Hatfield et al. (2012) have proposed an efficient algorithm to test substitutability of strict preferences. In this note we show how the algorithm by Hatfield et al. can be adapted in such a way that it can test substitutability of weak preferences as well. When restricted to the domain of strict preferences, our algorithm is faster than Hatfield et al.'s original algorithm by a linear factor.

1. Introduction

In matching problems, the aim is to match agents in a stable manner to objects or to other agents while considering the preferences of the agents involved. Matching theory has significant applications in assigning residents to hospitals, students to schools, etc. and has received tremendous attention in mathematical economics, computer science, and operations research (see, e.g., Gusfield and Irving (1989), Roth (2008) and Roth and Sotomayor (1990)).

In various matching models, individual preferences are supposed to be *responsive*, i.e., for any two sets that differ only in one object, the agent prefers the set containing the more preferred object (Roth and Sotomayor, 1990, p. 128f). For example in the case in which a hospital can hire multiple doctors, the hospitals are commonly assumed to submit preferences that render the choice between a pair of doctors independent of other available outcomes (Hatfield et al., 2012). An alternative is to allow hospitals to submit substitutable preferences, which allows for considerably more flexibility in expressing preferences over groups of doctors. An agent's preferences are *substitutable* if whenever its most preferred set of objects from a set of objects S contains an object w, then so will its most preferred set of objects from any subset of S that still includes w (Roth and Sotomayor, 1990, p. 173f).

Substitutable preferences, as introduced by Kelso and Crawford (1982) and Roth (1984), constitute a natural class of preference restrictions for which stable matchings are guaranteed to exist. In various matching models, substitutability is in fact a necessary and sufficient condition for the existence of a stable matching (see Hatfield et al. (2012, footnote 4)).[1]

The significance of substitutability leads to the natural algorithmic problem of testing whether a given preference relation is substitutable or not. Recently, Hatfield et al. (2012) have presented a polynomial-time algorithm – henceforth referred to as the *HIK algorithm* – for this problem. Hatfield et al. (2012) point out in their conclusion that the HIK algorithm "*could be distributed to market participants for use in the preparation of their preference relations for submission*".

Like most results in the literature concerning substitutability, both the original definition of substitutability and the HIK algorithm assume that individuals can only express *strict* preferences, i.e., preferences without indifferences. In many settings, allowing indifferences is not only a natural relaxation but also a practical necessity. Allowing for indifferences, however, may significantly affect the properties and structure of stable matchings. For example, stable matchings may have different cardinalities (Manlove et al., 2002) and for marriage markets man-optimal or woman-optimal stable matchings are no longer guaranteed to exist (Roth and Sotomayor, 1990). Moreover, weak preferences may also be a source of complexity for many computational problems concerning stable matchings. For instance, checking whether a stable roommate matching exists is polynomial-time solvable for strict preferences (Irving, 1985), but becomes NP-complete when indifferences are allowed (Ronn, 1990).

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[1] These settings include *many-to-many matching with contracts*, *many-to-many matching* and *many-to-one matching* markets, all with strict preferences (Hatfield et al., 2012; Hatfield and Kominers, 2011). For a more nuanced discussion on the necessary and sufficient conditions for the existence of stable matchings, we refer the reader to the recent work of Aygün and Sönmez (2012a,b) and Hatfield and Kominers (2011).

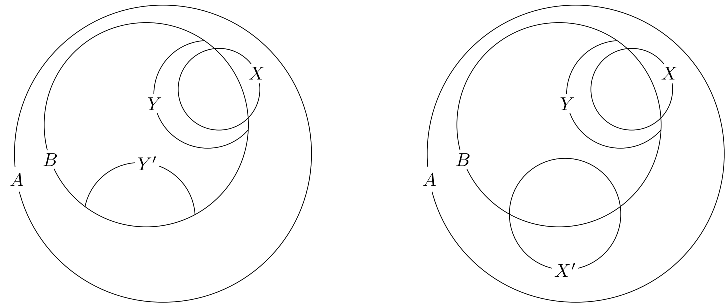

Fig. 1. Conditions S1 and S2. For the diagram on the left, we assume that $C(A) = \{X\}$ and $C(B) = \{Y, Y'\}$. Then, S1 is satisfied but S2 is not. In the diagram on right, we assume that $C(A) = \{X, X'\}$ and $C(B) = \{Y\}$. Then, S2 is satisfied but S1 is not.

For the more general domain which allows individuals to express indifferences (*weak* preferences), Sotomayor (1999) extended the concept of substitutability and provided an appropriate definition. Moreover, she showed that, in many-to-many matching markets with weak and substitutable preferences, a pairwise stable matching is still guaranteed to exist.

In this note, we examine the notion of substitutability for the general case of weak preferences from a computational point of view. We formulate conditions that characterize substitutable preferences. Using these conditions, we find that testing substitutability of weak preferences can be performed in polynomial time. When restricted to the domain of strict preferences, our algorithm is faster than the HIK algorithm by a linear factor.

2. Preliminaries

Let U be a finite set of alternatives. A *weak preference relation* is a transitive and complete relation R on 2^U. R is said to be *strict* if it is also anti-symmetric. Let P and I denote the strict and symmetric parts of R, respectively. A set $X \subseteq U$ is called *acceptable* if $X\,R\,\emptyset$. By an *equivalence class of* R we understand a family $\{Y \in U : X\,I\,Y\}$ for some subset X in U. Each preference relation R induces a choice correspondence C that returns, for each $X \subseteq U$, the set of all R-maximal subsets of X, i.e.,

$$C(X) = \{Y \subseteq X : Y\,R\,Z \text{ for all } Z \subseteq X\}.$$

Since R is transitive, $C(X)$ is never empty. Observe that $C(X)$ invariably contains a single set if R is strict but may contain more than one set if R allows for indifferences.

Example 1. Let $U = \{a, b, c, d\}$ and let the preference relation R restricted to the acceptable sets defined as follows.

$$\{a, b, d\}\,I\,\{b, c, d\}\,P\,\{a, b\}\,I\,\{b, c\}\,I\,\{a, c\}\,P\,\emptyset.$$

Then, $C(U) = \{\{a, b, d\}, \{b, c, d\}\}$ and $C(\{a\}) = \{\emptyset\}$.

Given a preference relation R, let s denote the maximal size of an indifference class consisting of acceptable sets. Observe that for each $X \subseteq U$ the size of $C(X)$ is bounded by s and that a preference relation is strict if and only if $s = 1$. Furthermore, let $u = |U|$ and let ℓ be the number of acceptable sets.

A very general and expressive way of representing R is via a preference list L, which contains all acceptable sets in descending order of preferability and using brackets to group sets in the same equivalence class.[2] For a preference relation R represented in list form and for $X \subseteq U$ it can be checked in time $O(\ell\,|X|)$ whether a given alternative is in $C(X)$.

The following lemma is a straightforward consequence of the definition of R-maximal sets (cf. Hatfield et al., 2012, Lemma 1).

Lemma 1. *For all* $A, B \subseteq U$ *with* $B \subseteq A$,

$$C(A) \cap 2^B \neq \emptyset \quad \textit{implies } C(B) = C(A) \cap 2^B.$$

Proof. Assume $C(A) \cap 2^B \neq \emptyset$. Then, $X \in C(A) \cap 2^B$ for some $X \subseteq U$. First, consider an arbitrary $Y \in C(B)$. Then $Y\,R\,X$. Hence, $Y \in C(A) \cap 2^B$ as well. Now consider an arbitrary $Y \notin C(B)$. If $Y \notin 2^B$, immediately $Y \notin C(A) \cap 2^B$. If $Y \in 2^B$, we have $X\,P\,Y$ and therefore $Y \notin C(A)$. Also then $Y \notin C(A) \cap 2^B$. □

3. Substitutability and weak preferences

In the restricted setting of strict preferences the choice correspondence invariably chooses a single set. For weak preferences the choice correspondence may select a family of any number of sets and the definition of substitutability for strict preferences has to be adapted accordingly. Sotomayor (1999) observed that substitutability for strict preferences can be characterized in two equivalent ways. Both of these definitions can naturally and conservatively be extended to the domain of weak preferences. In this more general setting, however, the conditions these generalizations give rise to (S1 and S2 in Definition 1 below) are no longer equivalent. Sotomayor (1999) has argued that both conditions capture an essential aspect of substitutability and suggested that, for weak preferences, substitutability be defined as their conjunction (see Fig. 1).

Definition 1. A preference relation R is *substitutable* if and only if the following two conditions hold:

(S1) for all non-empty $A, B \subseteq U$ with $B \subseteq A$ we have that for all $X \in C(A)$ there is some $Y \in C(B)$ such that $X \cap B \subseteq Y$, and
(S2) for all non-empty $A, B \subseteq U$ with $B \subseteq A$ we have that for all $Y \in C(B)$ there is some $X \in C(A)$ such that $X \cap B \subseteq Y$.

Example 2. Consider the preference relation R from Example 1. It can be verified that R satisfies S1 and violates S2. For the latter, take $A = U$ and $B = \{a, b, c\}$. Then,

$$C(B) = \{\{a, b\}, \{b, c\}, \{a, c\}\}.$$

Now $Y = \{a, c\}$ is in $C(B)$, but there exists no $X \in C(A)$ such that $X \cap B \subseteq Y$. Hence, R is not substitutable.

4. Testing substitutability

We now outline a way to test substitutability of weak preferences. The idea utilizes an insight of Hatfield et al. (2012) that instead of checking all violations of substitutability, one may restrict one's attention to violations of a specific type. Formally, by an *S1-violation for* R we understand a pair $(A, B) \in 2^U \times 2^U$ such that $B \subseteq A$ and for some $X \in C(A)$ it is the case that for all $Z \in C(B)$, $X \cap B \not\subseteq Z$. Obviously, a preference relation R satisfies S1 if and only if there are no S1-violations for R. We now have the following lemma.

[2] This list representation is reminiscent to the *representation by individually rational lists of coalitions* used in the context of hedonic coalition formation games (Ballester, 2004).

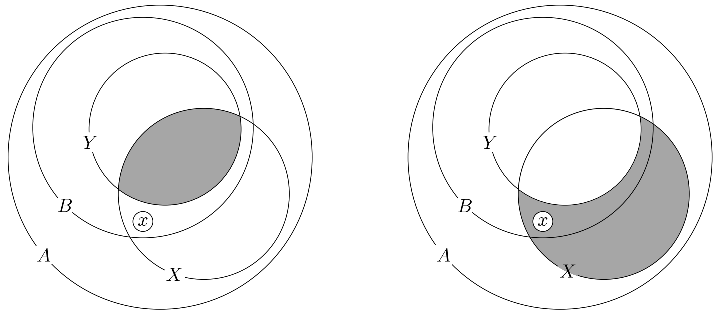

Fig. 2. The left diagram illustrates Lemma 2, the right one Lemma 3. In either case, Y is chosen so as to maximize and minimize the areas in gray, respectively.

Lemma 2. *Let R be a preference relation. If there exists an S1-violation for R, then there exist acceptable sets $X, Y \subseteq U$ and $x \in X$ such that $(X \cup Y, Y \cup \{x\})$ is also an S1-violation for R.*

Proof. Assume that (A, B) is an S1-violation for R. Then there is some $X \in C(A)$ such that $X \cap B \not\subseteq Z$ for all $Z \in C(B)$. As $C(B) \neq \emptyset$, there is some $Y \in C(B)$ such that $X \cap Y$ is maximal with respect to set inclusion, i.e., there is no $Z \in C(B)$ such that $X \cap Y \subsetneq X \cap Z$. Obviously, X and Y are acceptable. By our assumption, it is the case that $X \cap B \not\subseteq Y$ and we may therefore assume the existence of some $x \in (X \cap B) \setminus Y$ (see Fig. 2). We prove that $(X \cup Y, Y \cup \{x\})$ is an S1-violation for R, in particular we show that

(i) $Y \cup \{x\} \subseteq X \cup Y$,
(ii) $X \in C(X \cup Y)$, and
(iii) $X \cap (Y \cup \{x\}) \not\subseteq Z$ for all $Z \in C(Y \cup \{x\})$.

As $x \in X$, it is obvious that (i) holds. As for (ii), observe that $X \in C(A) \cap 2^{X\cup Y}$. Lemma 1 implies $C(X \cup Y) = C(A) \cap 2^{X\cup Y}$ and thus $X \in C(X \cup Y)$.

Finally, consider an arbitrary $Z \in C(Y \cup \{x\})$. Observe that $Y \in C(B) \cap 2^{Y\cup\{x\}}$. By another application of Lemma 1, we get $C(Y \cup \{x\}) = C(B) \cap 2^{Y\cup\{x\}}$ and, therefore, $Z \in C(B)$. Moreover, by the choice of Y, it is not the case that $X \cap Y \subsetneq X \cap Z$, i.e., either $X \cap Y \not\subseteq X \cap Z$ or $X \cap Y = X \cap Z$. If the former, then there is some $z \in X \cap Y$ with $z \notin X \cap Z$. If the latter, then $x \notin Z$. In either case, there is some $z \in X \cap (Y \cup \{x\})$ such that $z \notin Z$. Hence, $X \cap (Y \cup \{x\}) \not\subseteq Z$, which proves (iii). □

S2-violations are defined similarly to S1-violations. A pair $(A, B) \in 2^U \times 2^U$ is an *S2-violation for R* if $B \subseteq A$ and for some $Y \in C(B)$ it is the case that for all $Z \in C(A)$, $Z \cap B \not\subseteq Y$. Clearly, R satisfies S2 if and only if there are no S2-violations for R. Moreover, R is substitutable if and only if there are neither S1-violations nor S2-violations for R. We have the following lemma.

Lemma 3. *Let R be a preference relation. If there exists an S2-violation for R, then there exist acceptable sets $X, Y \subseteq U$ and $x \in X$ such that $(X \cup Y, Y \cup \{x\})$ is also an S2-violation for R.*

Proof. Assume that (A, B) is an S2-violation for R. Then there is some $Y \in C(B)$ such that $Z \cap B \not\subseteq Y$ for all $Z \in C(A)$. As $C(A) \neq \emptyset$, there is some $X \in C(A)$ such that $X \setminus Y$ is minimal with respect to set-inclusion, i.e., there is no $Z \in C(A)$ such that $Z \setminus Y \subsetneq X \setminus Y$. Obviously, X and Y are acceptable. By our assumption, it is the case that $X \cap B \not\subseteq Y$ and we may assume the existence of some $x \in (X \cap B) \setminus Y$ (see Fig. 2). We prove that $(X \cup Y, Y \cup \{x\})$ is also an S2-violation for R, in particular we show that

(i) $Y \cup \{x\} \subseteq X \cup Y$,
(ii) $Y \in C(Y \cup \{x\})$, and
(iii) $Z \cap (Y \cup \{x\}) \not\subseteq Y$ for all $Z \in C(X \cup Y)$.

As $x \in X$, (i) obviously holds. As for (ii), observe that $Y \in C(B) \cap 2^{Y\cup\{x\}}$. Lemma 1 implies that $C(Y \cup \{x\}) = C(B) \cap 2^{Y\cup\{x\}}$ and thus $Y \in C(Y \cup \{x\})$.

Finally, consider an arbitrary $Z \in C(X \cup Y)$. Observe that $X \in C(A) \cap 2^{X\cup Y}$. Another application of Lemma 1 yields $C(X \cup Y) = C(A) \cap 2^{X\cup Y}$ and, therefore, $Z \in C(A) \cap 2^{X\cup Y}$. Moreover, by choice of X, it is not the case that $Z \setminus Y \subsetneq X \setminus Y$. As $Z \setminus Y \subseteq X \setminus Y$, it then follows that $Z \setminus Y = X \setminus Y$. Hence, $x \in Z$ and, since $x \notin Y$, we obtain $Z \cap (Y \cup \{x\}) \not\subseteq Y$, which proves (iii). □

We can exploit Lemmas 2 and 3 to obtain a polynomial-time algorithm to check the substitutability of a preference relation. The algorithm works as follows. Instead of checking all potential violations of S1 and S2, due to Lemmas 2 and 3 we can restrict our attention to S1- and S2-violations of the form $(X \cup Y, Y \cup \{x\})$, where $X, Y \subseteq U$ are acceptable and $x \in U$. The number of these potential violations is polynomial in the number of acceptable subsets in R and a polynomial-time algorithm is obtained by exhaustively checking each of them. We note that the algorithm is not different from the HIK algorithm in that it exhaustively checks certain violations.

Theorem 1. *It can be checked in time $O(\ell^2 u^2(\ell + s^2))$ whether a given preference relation in list representation is substitutable.*

Proof. To test substitutability, we need to check whether both S1 and S2 hold for a preference relation R represented by list L. This is equivalent to verifying that neither S1-violations nor S2-violations exist for R.

Let us first consider the case of S1. To check S1, we know from Lemma 2 that we can restrict our attention to violations of the form $(X \cup Y, Y \cup \{x\})$ for some $X, Y \in L$ and $x \in X$. Therefore, the maximum number of pairs we need to check is upper-bounded by $\binom{\ell}{2} u$.

Verifying an S1-violation of type $(A, B) = (X \cup Y, Y \cup \{x\})$ requires us to perform the following three steps. First, compute $C(A)$. This takes time $O(\ell u)$. Then, compute $C(B)$, which also takes time $O(\ell u)$. Finally, test the main condition:

for all $X' \in C(A)$ there is some $Y' \in C(B)$ such that $X' \cap B \subseteq Y'$.

This can be performed in time $O(s^2 u)$. In total, verifying a violation of type $(A, B) = (X \cup Y, Y \cup \{x\})$ takes time

$$O(\ell u) + O(\ell u) + O(s^2 u) = O(\ell u + s^2 u).$$

The time needed to check whether an S1-violation exists is then equal to the maximum number of pairs we need to check multiplied by the time required to verify one S1-violation, which equals

$$O\left(\binom{\ell}{2} u\right) \times O(\ell u + s^2 u) = O(\ell^2 u (\ell u + s^2 u)).$$

The same analysis holds for checking whether an S2-violation exists. Therefore there exists an algorithm which runs in time $O(2\ell^2 u(\ell u + s^2 u)) = O(\ell^2 u(\ell u + s^2 u)) = O(\ell^2 u^2(\ell + s^2))$ and tests the substitutability of a preference relation. □

By letting $s = 1$, we get the following result for strict preferences as a corollary.

Corollary 1. *It can be checked in time $O(\ell^3 u^2)$ whether a given strict preference relation is substitutable.*

On the domain of strict preferences, the (worst case) asymptotic running time of the algorithm turns out to be slightly faster than the HIK algorithm: $O(\ell^3 u^2)$ as compared to $O(\ell^3 u^3)$. The reason for this is that the HIK algorithm considers violations of the form $(X \cup Y \cup \{x, z\}, X \cup Y \cup \{z\})$, involving two alternatives and two acceptable sets, whereas in this paper we found that one can restrict attention to S1- and S2-violations of the form $(X \cup Y, Y \cup \{x\})$, which involve one alternative less. In practice one might expect that s would be a polynomial function of ℓ. In that case, we can even obtain a polynomial bound entirely in terms of ℓ and u.

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